On some open problems in Diophantine approximation

by **Nikolay Moshchevitin**[1]

In memory of
Vladimir Igorevich Arnold
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Abstract. We discuss several open problems in Diophantine approximation. Among them there are famous Littlewood's and Zaremba's conjectures as well as some new and not so famous problems.

In the present paper I shall give a brief surwey on several problems in Diophantine aproximation which I was interested in and working on. Of course my briev survey is by no means complete. The main purpose of this paper is to make the problems under consideration more popular among a wider mathematical audience.

Basic facts concerning Diophantine approximation one can find in wonderful books [71, 29, 127]. As for the Geomerty of Numbers we refer to [30, 53]. Some topics related to the problems discussed below are considered in recent surveys [100, 145].

1 Littlewood conjecture and related problems

Every paper in Diohantine approximations should begin with the formulation of the Dirichlet theorem whitch states that for real numbers $\theta_1, ..., \theta_n, n \geqslant 1$ there exist infinitely many positive integers q such that

$$\max_{1\leqslant j\leqslant n} ||q\theta_j|| \leqslant \frac{1}{q^{1/n}}$$

(here and in the sequel $||\cdot||$ stands for the distance to the nearest integer), or

$$\liminf_{q\to+\infty} q^{1/n} \max_{1\leqslant j\leqslant n} ||q\theta_j|| \leqslant 1.$$

The famous Littlewood conjecture in Diophantine approximatios supposes that for any two real numbers θ_1, θ_2 one has

$$\liminf_{q\to+\infty} q\, ||q\theta_1||\, ||q\theta_2|| = 0. \tag{1}$$

The similar multidimensional problem is for given $n \geqslant 2$ to prove that for any reals $\theta_1, ..., \theta_n$ one has

$$\liminf_{q\to+\infty} q\, ||q\theta_1|| \cdots ||q\theta_n|| = 0. \tag{2}$$

This problem is not solved for any $n \geqslant 2$. Obviously the statement is false for $n = 1$: one can consider a badly approximable number θ_1 such that

$$\inf_{q\in\mathbb{Z}_+} q||q\theta_1|| > 0. \tag{3}$$

Any quadratic irrrationality satisfies (3); moreover as it was proved by V. Jarník [60] the set of all θ_1 satisfying (3) has zero Lebesgue measure but full Hausdorff dimension in $\mathbb{R}$.

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Of course Littlewood conjecture is true for almost all pairs $(\theta_1, \theta_2) \in \mathbb{R}^2$. Moreover, from Gallagher's [48] theorem we know that for a positive-valued decreasing to zero function $\psi(q)$ the inequality

$$||q\theta_1|| \, ||q\theta_2|| \leqslant \psi(q)$$

for almost all pairs (θ_1, θ_2) in the sense of Lebesgue measure has infinitely many solutions in integers q (respectively, finitely many solutions) if the series $\sum_q \psi(q) \log q$ diverges (respectively, converges). Thus for almost all (θ_1, θ_2) we have

$$\liminf_{q \to +\infty} q \log^2 q ||q\theta_1|| \, ||q\theta_2|| = 0.$$

Einsiedler, Katok and Lindenstrauss [42] proved that the set of pairs (θ_1, θ_2) for which (1) is not true is a set of zero Hausdorff dimension (see also a paper by Venkatesh [144] devoted to this result). In my opinion Littlewood conjecture is one of the most exiting open problems in Diophantine approximations. Some argument for Littlewood conjecture to be true are given recently by Tao [139].

At the beginning of our discussion we would like to formulate Peck's theorem [115] concerning approximations to algebraic numbers.

Theorem 1. *Suppose that $n \geqslant 2$ and $1, \theta_1, ..., \theta_n$ form a basis of a real algebraic field of degree $n+1$. Then there exists a positive constant $C = C(\theta_1, ..., \theta_n)$ such that there exist infinitely many $q \in \mathbb{Z}_+$ such that simultaneously*

$$\max_{1 \leqslant j \leqslant n} ||q\theta_j|| \leqslant \frac{C}{q^{1/n}}$$

and

$$\max_{1 \leqslant j \leqslant n-1} ||q\theta_j|| \leqslant \frac{C}{q^{1/n} (\log q)^{1/(n-1)}}.$$

Peck's theorem ia a quantitative generalization of a famous theorem by Cassels and Swinnwrton-Dyer [28]. We see that for a basis of a real algebraic field one has

$$\liminf_{q \to +\infty} q \log q \, ||q\theta_1|| \cdots ||q\theta_n|| < +\infty,$$

and so (2) is true for these numbers. In particular, Littlewood conjecture (1) is true for numbers θ_1, θ_2 which form togwther with 1 a basis of a real cubic field.

We should note here that the numbers $\theta_1, ..., \theta_n$ which together with 1 form a basis of an algebraic field are simultaneously badly approximable, that is

$$\inf_{q \in \mathbb{Z}_+} q^{1/n} \max_{1 \leqslant j \leqslant n} ||q\theta_j|| > 0 \qquad (4)$$

(see [29], Ch. V, §3).

A good inrtoduction to Littlewood conjecture one can find in [119]. Interesting discussion is in [18].

1.1 Lattices with positive minima

Suppose that $1, \theta_1, ..., \theta_n$ form a basis of a totally real algebraic field $\mathbb{K} = \mathbb{Q}(\theta)$ of degree $n+1$. This means that all algebraic conjugates $\theta = \theta^{(1)}, \theta^{(2)}, ..., \theta^{(n+1)}$ to θ are real algebraic numbers. So there exists a polynomial $g_j(\cdot)$ with rational coefficients of degree $\leqslant n$ such that $\theta_j = g_j(\theta)$. We consider conjugates $\theta_j^{(i)} = g_j(\theta^{(i)})$ and the matrix

$$\Omega = \begin{pmatrix} 1 & \theta_1^{(1)} & \cdots & \theta_n^{(1)} \\ \cdots & \cdots & \cdots & \\ 1 & \theta_1^{(n)} & \cdots & \theta_n^{(n)} \end{pmatrix}.$$

Let G be a diagonal matrix of dimension $(n+1)\times(n+1)$ with non-zero diagonal elements. We consider a lattice of the form

$$\Lambda = G\,\Omega\,\mathbb{Z}^{n+1}.$$

Lattices of such a type are known as algebraic lattices.

For an arbitrary lattice $\Lambda \subset \mathbb{R}^{n+1}$ we consider its homogeneous minima

$$\mathcal{N}(\Lambda) = \inf_{\mathbf{z}=(z_0,z_1,\dots,z_n)\in\Lambda\setminus\{\mathbf{0}\}} |z_0z_1\cdots z_n|.$$

One can easiily see that if Λ is an algebraic lattice then $\mathcal{N}(\Lambda) > 0$.

If $n = 1$ and ξ, η are arbitrary badly appoximable numbers (that is satisfying (4)) then for

$$\Xi = \begin{pmatrix} 1 & \xi \\ 1 & \eta \end{pmatrix}$$

the lattice $L = \Xi\mathbb{Z}^2 \subset \mathbb{R}^2$ will satisfy the property $\mathcal{N}(L) > 0$. Of course one can take ξ, η in such a way that L is not an algebraic lattice. So in the dimension $n + 1 = 2$ there exists a lattice $L \subset \mathbb{R}^2$ which is not an algebraic one, but $\mathcal{N}(L) > 0$.

A famous *Oppenheim conjecture* supposes that in the case $n \geqslant 2$ any lattice Λ satisfying $\mathcal{N}(L) > 0$ is an algebraic lattice. This conjecture is still open. Cassels and Swinnerton-Dyer [28] proved that from Oppenheim conjecture in dimension $n = 2$ Littlewood conjecture (1) follows.

Oppenheim conjecture can be reformulated in terms of *sails* of lattices (see [49, 50, 51]). A sail of a lattice is a very interesting geometric object whish generalizes Klein's geometric interpretation of the ordinary continued fractions algorithm. A connection between sails and Oppenheim conjecture was found by Skubenko [137, 138]. (However the main result of the papers [137, 138] is incorrect: Skubenko claimed the solution of Littlewood conjecture, however he had a mistake in Fundamental Lemma IV in [137].)

Oppenheim conjecture can be reformulated in terms of closure of orbits of lattices (see [137]) and in terms of behaviour of trajectories of certain dynamical systems. There is a lot of literature related to Littlewood-like problems in lattice theory and dynamical approach (see [42, 69, 43, 85, 88, 86, 133, 134, 144]). Some open problems in dynamics related to Diophantie approximations are discussed in [52].

1.2 W.M. Schmidt's conjecture and Badziahin-Pollington-Velani theorem

For $\alpha, \beta \in [0, 1]$ under the condition $\alpha + \beta = 1$ and $\delta > 0$ we consider the sets

$$\mathrm{BAD}(\alpha, \beta; \delta) = \left\{\xi = (\theta_1, \theta_2) \in [0, 1]^2 : \inf_{p\in\mathbb{N}} \max\{p^\alpha||p\theta_1||, p^\beta||p\theta_2||\} \geqslant \delta\right\}$$

and

$$\mathrm{BAD}(\alpha, \beta) = \bigcup_{\delta>0} \mathrm{BAD}(\alpha, \beta; \delta).$$

In [130] Schmidt conjectured that for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$, $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = 1$ the intersection

$$\mathrm{BAD}(\alpha_1, \beta_1) \bigcap \mathrm{BAD}(\alpha_2, \beta_2)$$

is not empty. Obviously if Schmidt's conjecture be wrong then Littlewood conjecture be true. But Schmidt's conjecture was recently proved in a breakthrough paper by Badziahin, Pollington and Velani [4]. They proved a more general result:

Theorem 2. *For any finite collection of pairs (α_j, β_j), $0 \leqslant \alpha_j, \beta_j \leqslant 1$, $\alpha_j + \beta_j = 1$, $1 \leqslant j \leqslant r$ and for any θ_1 under the condition*

$$\inf_{q \in \mathbb{Z}_+} q||q\theta_1|| > 0 \tag{5}$$

the intersection

$$\bigcap_{j=1}^{r} \{\theta_2 \in [0,1]: \ (\theta_1, \theta_2) \in \mathrm{BAD}(\alpha_j, \beta_j)\} \tag{6}$$

has full Hausdorff dimension.

Moreover one can take a certain infinite intersection in (6).

This result was obtained by an original method invented by Badziahin, Pollington and Velani. Author's preprint [103] is devoted to an exposition of the method in the simplest case. The only purpose of the paper [103] was to explain the mechanism of the method invented by Badziahin, Pollington and Velani. In this paper it is shown that for $0 < \delta \leqslant 2^{-1622}$ and θ_1 such that

$$\inf_{q \in \mathbb{N}} q^2 ||q\theta_1|| \geqslant \delta,$$

there exists θ_2 such that for all integers A, B with $\max(|A|, |B|) > 0$ one has

$$||A\theta_1 - B\theta_2|| \cdot \max(A^2, B^2) \geqslant \delta$$

and hence

$$\inf_{q \in \mathbb{Z}_+} q^{1/2} \max_{1 \leqslant j \leqslant 2} ||q\theta_j|| > 0.$$

This is a quantitative version of a corresponding results from [4].

However the method works with two-dimensional sets only. Of course we can consider multidimensional BAD-sets. For example for $\alpha, \beta, \gamma \in [0,1]$ under the condition $\alpha + \beta + \gamma = 1$ and $\delta > 0$ one may consider the sets

$$\mathrm{BAD}(\alpha, \beta, \gamma; \delta) = \left\{ \xi = (\theta_1, \theta_2, \theta_3) \in [0,1]^3 : \ \inf_{p \in \mathbb{N}} \ \max\{p^{\alpha}||p\theta_1||, \ p^{\beta}||p\theta_2||, \ p^{\beta}||p\theta_3||\} \geqslant \delta \right\}$$

and

$$\mathrm{BAD}(\alpha, \beta, \gamma) = \bigcup_{\delta > 0} \mathrm{BAD}(\alpha, \beta, \gamma; \delta).$$

The question if for given $(\alpha_1, \beta_1, \gamma_1) \neq (\alpha_1, \beta_1, \gamma_1)$ one has

$$\mathrm{BAD}(\alpha_1, \beta_1, \gamma_1) \bigcap \mathrm{BAD}(\alpha_2, \beta_2, \gamma_2) \neq \varnothing$$

remains open.

A positive answer is obtained in a very special cases (see, [1, 79, 117]).

Recently Badziahin [7] adopted the construction from [4] to Littlewood-like setting and proved the following result.

Theorem 3. *The set*

$$\{(\theta_1, \theta_2) \in \mathbb{R}^2 : \ \inf_{x \in \mathbb{Z}, x \geqslant 3} x \log x \ \log\log x \, ||\theta_1 x|| \, ||\theta_2 x|| > 0\}$$

has Hausdorff dimension equal to 2.

Moreover if θ_1 is a badly approximable number (that is (5) is valid) then the set

$$\{\theta_2 \in \mathbb{R} : \ \inf_{x \in \mathbb{Z}, x \geqslant 3} x \log x \ \log\log x \, ||\theta_1 x|| \, ||\theta_2 x|| > 0\}$$

has Hausdorff dimension equal to 1.

1.3 p-adic version of Littlewood conjecture

Let for a prime p consider the p-adic norm $|\cdot|_p$, that is if $n = p^\nu n_1, (n_1, p) = 1, \nu \in \mathbb{Z}_+$ then $|n|_p = p^{-\nu}$. De Mathan and Teulié conjectured [89] that for any $\theta \in \mathbb{R}$ one has

$$\liminf_{q \to +\infty} q|q|_p||q\theta|| = 0.$$

This conjecture is known as *p-adic* or *mixed* Littlewood conjecture. Of course the conjecture is true for almost all numbers Θ. The conjecture can be reformulated as follows: to prove or to disprove that for irrational θ one has

$$\inf_{n,q \in \mathbb{Z}_+} q||p^n q\theta|| = 0.$$

It worth noting that de Mathan and Teulié themselves proved [89] that their conjecture is valid for every quadratic irrational θ.

As it is shown in [43] from Furstenberg's result discussed behind in 4), Subsection 1.4 it follows that for distinct p_1, p_2 one has

$$\inf_{m,n,q \in \mathbb{Z}_+} q||p_1^n p_2^m q\theta|| = 0$$

and so

$$\liminf_{q \to +\infty} q|q|_{p_1}|q|_{p_2}||q\theta|| = 0 \qquad (7)$$

for all θ. Moreover from Bourgain-Lindenstrauss-Michel-Venkatesh's result [9] it follows that for some positive κ one has

$$\liminf_{q \to +\infty} q(\log\log\log q)^\kappa |q|_{p_1}|q|_{p_2}||q\theta|| = 0.$$

We should note that the inequality (7) is not the main result of the paper [43] by Einsiedler and Kleinbock. The main result from [43] establishes the zero Hausdorff dimension of the exceptional set in the problem under consideration.

Many interesting metric results and multidimensional conjectures are discussed in [22].

Badziahin and Velani [5] generalized proved an analog of Theorem 2 for mixed Littlewood conjecture:

Theorem 4. *The set of reals θ satisfying*

$$\liminf_{q \to +\infty} q\ \log q\ \log\log q\ |q|_p\ ||q\theta|| > 0$$

has Hausdorff dimension equal to one.

In this subsection we consider powers of primes p^n only. Instead of powers of a prime it is possible to consider other sequences of integers. This leads to various generalizations. Many interesting results and conjectures of such a kind are discussed in [15, 6] and [55].

1.4 Inhomogeneous problems

Shapira [133] proved recently two important theorems. We put them below.

Theorem 5. *Almost all (in the sense of Lebesgue measure) pairs $(\theta_1, \theta_2) \in \mathbb{R}^2$ satisfy the following property: for every pair $(\eta_1, \eta_2) \in \mathbb{R}^2$ one has*

$$\liminf_{q \to \infty} q\ ||q\theta_1 - \eta_1||\ ||q\theta_2 - \eta_2|| = 0.$$

Theorem 6. *The conclusion of Theorem 5 is true for numbers θ_1, θ_2 which form together with* 1 *a basis of a totally real algebraic field of degree* 3.

Here we should note that the third theorem from [133] follows from Khintchine's result (see [67]) immediately:

Theorem 7. *Suppose that reals θ_1 and θ_2 are linearly dependent over $\mathbb{Z}$ together with* 1*. Then there exist reals η_1, η_2 such that*

$$\inf_{x\in\mathbb{Z}_+} x \cdot ||x\theta_1 - \eta_1|| \cdot ||x\theta_2 - \eta_2|| > 0.$$

Theorem 7 is discussed in author's paper [106]. Moreover in this paper the author deduces from Khintchine's argument [67] the following

Theorem 8. *Let $\psi(t)$ be a function increasing to infinity as $t \to +\infty$. Suppose that for any $w \geqslant 1$ we have the inequality*

$$\sup_{x\geqslant 1} \frac{\psi(wx)}{\psi(x)} < +\infty.$$

Then there exist real numbers θ_1, θ_2 linearly independent over $\mathbb{Z}$ together with 1 *and real numbers η_1, η_2 such that*

$$\inf_{x\in\mathbb{Z}_+} x\psi(x) \cdot ||\alpha_1 x - \eta_1|| \cdot ||\alpha_2 x - \eta_2|| > 0.$$

The following problem is an open one: *is it possible that for a constant function $\psi(t) = \psi_0 > 0\ \forall t$ the conclusion of Theorem 8 remains true.* If not, it means that a stronger inhomogeneous version of Littlewood conjecture is valid.

We would like to formulate here one open problem in imhomogeneous approximations due to Harrap [54]. Harrap [54] proved that given $\alpha, \beta \in (0, 1),\ \alpha + \beta = 1$ for a fixed vector

$$(\theta_1, \theta_2) \in \mathrm{BAD}(\alpha, \beta) \tag{8}$$

the set

$$\{(\eta_1, \eta_2) \in \mathbb{R}^2 : \inf_q \max(q^\alpha ||q\theta_1 - \eta_1||, q^\beta ||q\theta_2 - \eta_2|| > 0\} \tag{9}$$

has full Hausdorff dimension. It is possible to prove that this set is an 1/2-winning set in $\mathbb{R}^2$ (we discuss winning properties in Subsection 1.6 below). The following question formulated by Harrap [54]: *to prove that the set (9) is a set of full Hausdorff dimension (and even a winning set) without the condition (8).* Of course in the case $\alpha = \beta = 1/2$ the positive answer follows from Khintchine's approach (see results from the paper [104] and the historical discussion there). However in the case $(\alpha, \beta) \neq (1/2.1/2)$ Harrap's question is still open [2].

1.5 Peres-Schlag's method

In [116] Peres and Schlag proved the following result.

Theorem 9. *Consider a sequence $t_n \in \mathbb{R}$, $n = 1, 2, 3, ..$ Suppose that for some $M \geqslant 2$ one has*

$$\frac{t_{j+1}}{t_j} \geqslant 1 + \frac{1}{M}, \ \ \forall j \in \mathbb{Z}_+. \tag{10}$$

Then with a certain absolute constant $\gamma > 0$ for any sequence $\{t_j\}$ under the condition (10) there exists real α such that

$$||\alpha t_j|| \geqslant \frac{\gamma}{M \log M}, \ \ \forall j \in \mathbb{Z}_+. \tag{11}$$

[2] A sketch of a proof for Harrap's conjecture is given in a recent preprint [110].

This result has an interesting history with starts from famous Khintchine's paper [67]. We do not want to go into details about this history and refer to pepers [100, 107].

As it was noted by Dubickas [38], an inhomogeneous version of Theorem 9 is valid: *with a certain absolute constant $\gamma' > 0$ for any sequence $\{t_j\}$ under the condition (10) and for any sequence of real numbers $\{\eta_j\}$ there exists real α such that*

$$||\alpha t_j - \eta_j|| \geqslant \frac{\gamma}{M \log M}, \quad \forall j \in \mathbb{Z}_+. \tag{12}$$

One can easily see that for any large integer M there exists an infinite sequence $\{t_j\}$ such that such that for any real α there exists infinitely many j with

$$||\alpha t_j|| \leqslant \frac{1}{M} \tag{13}$$

(one may start this sequence $\{t_j\}$ with a finite part $1, 2, 3, ..., M$ and then continue by $1, M, 2M, 3M, ..., M \cdot M$, e.t.c.). Of cource constant 1 in the numerator of the right hand side may be improved. The open question is to find the right *order* of approximation in the *homogeneous* version of this problem. In general Peres-Schlag's method does not give optimal bounds. The conjecture is that Theorem 9 may be improved on, and the optimal result should be stronger than the inequality (11).

There are several results and papers dealing with Peres-Schlag's construction (see [23, 24, 38, 97, 102, 106, 107, 116, 120]). Here we refer to four such results.

The results in 1) and 2) below are taken from [106]. The paper [106] contains some other results related to Peres-Schlag's method.

1) In Littlewood-like setting we got the following result.

Let $\eta_q, q = 1, 2, 3, ..$ be a sequence of reals. Given positive $\varepsilon \leqslant 2^{-14}$ and a badly approximable real θ_1 such that

$$||\theta_1 q|| \geqslant \frac{1}{\gamma q} \quad \forall q \in \mathbb{Z}_+, \quad \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ and a real θ_2 such that

$$\inf_{q \geqslant X_0} q \ln^2 q \cdot ||q\theta_1|| \cdot ||q\theta_2 - \eta_q|| \geqslant \varepsilon.$$

If one consider the sequence $\eta_q = 0$ the result behind was obtained in [23]. It is worse than Badziahin's Theorem 3.

2) In Schmidt-like settind we proved the following statement.

Suppose that $\alpha, \beta > 0$ satisfy $\alpha + \beta = 1$. Let $\eta_q, q = 1, 2, 3, ..$ be a sequence of reals. Let η be an arbitrary real number. Let $\gamma > 0$. Suppose that ε is small enough. Suppose that for a certain real θ_1 and for $q \geqslant X_1$ one has

$$||\theta_1 q|| \geqslant \frac{\gamma (\ln q)^\alpha}{q^{1/\alpha}}.$$

Then there exist $X_0 = X_0(\varepsilon, \gamma, X_1)$ and a real θ_2 such that

$$\inf_{q \geqslant X_0} \max((q \ln q)^\alpha \cdot ||q\theta_1 - \eta||, \ (q \ln q)^\beta \cdot ||q\theta_2 - \eta_q||) \geqslant \varepsilon.$$

Note that if we take $\eta = 0, \eta_q \equiv 0$ we get a result from [102] which is much worse that Badziahin-Pollilgton-Velani's Theorem 2.

3) For a real θ we deal with the sequence $||q^2\theta||$. Peres-Schlag's argument gives the following statement (see [97]) which solves the simplest problem due to Schmidt [129].

Given a sequence $\{\eta_q\}$ there exists θ such that for all positive integer q one has

$$||q^2\theta - \eta_q|| \geqslant \frac{\gamma}{q\log q} \tag{14}$$

(here γ is a positive absolute constant).

Zaharescu [147] proved that for any positive ε and irrational α one has

$$\liminf_{q\to+\infty} q^{2/3-\varepsilon}||q^2\theta|| = 0. \tag{15}$$

I do not know if this result may be generalized for the value

$$\liminf_{q\to+\infty} q^{2/3-\varepsilon}||q^2\theta - \eta||$$

with a real η.

Nevertheless even in the homogeneous case the lower bound (14) is the best known. So in the homogeneous case we have a gap between (14) and (15).

4) Fürstenberg's sequence. Consider integers of the form $2^m3^m, m, n \in \mathbb{Z}_+$ written in the increasing order:

$$s_0 = 1 < s_1 = 2 < s_2 = 3 < s_3 = 4 < s_5 = 6 < s_6 = 8 < s_7 = 9 < s_8 = 12 < ...$$

Fürstenberg [47] (simple proof is given in [8]) proved that the sequence of fractional parts $\{s_q\theta\}, q \in \mathbb{Z}_+$ is dense for any irrational θ. Hence for any η one has

$$\liminf_{q\to+\infty} ||s_q\theta - \eta|| = 0.$$

Bourgain, Lindenstrauss, Michel and Venkatesh [9] proved a quantitative version of this result. In particular they show that if θ saitisfies for some positive β the condition

$$\inf_{q\in\mathbb{Z}_+} q^\beta||q\theta|| > 0$$

then with some positive κ and for any η one has

$$\liminf_{q\to+\infty} (\log\log\log q)^\kappa ||s_q\theta - \gamma|| = 0. \tag{16}$$

Of course Fürstenberg had a more general result: instead of the sequence $\{s_q\}$ he considered an arbitrary non-lacunary multiplicative semigroup in $\mathbb{Z}_+$. The result (16) deals with this general setting also.

Peres-Schlag's method gives the following result: *for an arbitrary sequence $\eta_q, q = 1, 2, 3, ...$ there exists irrational θ such that*

$$\inf_{q\geqslant 2} \sqrt{q}\log q\, ||s_q\theta - \eta_q|| > 0.$$

One can see that the results from 1), 2) with $\eta_q \equiv 0$ are known to be not optimal. We do not know if the original Theorem 9 and the results from 3), 4) with $\eta_q \equiv 0$ are not optimal. However I am sure that in *homogemeous* setting the original Theorem 9 and the results from 3), 4) are not optimal and may be improved on. From the other hand it may happen that the order of approximation in the setting with arbitrary sequence $\{\eta_q\}$ is optimal for some (and even for *all*) results from 1), 2), 3), 4) and for lacunary sequences.

Of course Peres-Schlag's method gives a thick set (a set of full Hausdorff dimension) of θ's for which the discussed conclusions hold.

1.6 Winning sets

We give the definition of Schmidt's (α,β)-games and winning sets. Consider $\alpha,\beta\in(0,1)$, and a set $S\subseteq\mathbb{R}^d$. Whites an Blacks are playing the following game. Blacks take a closed ball $B_1\subset\mathbb{R}^d$ with diameter $l(B_1)=2\rho$. Then Whites choose a ball $W_1\subset B_1$ with diameter $l(W_1)=\alpha l(B_1)$. Then Blacks choose a ball $B_2\subset W_1$ with diamater $l(B_2)=\beta l(W_1)$, and so on... In such a way we get a sequence of nested balls $B_1\supset W_1\supset B_2\supset W_2\supset\cdots$ with diameters $l(B_i)=2(\alpha\beta)^{i-1}\rho$ and $l(W_i)=2\alpha(\alpha\beta)^{i-1}\rho$ $(i=1,2,\dots)$. The set $\bigcap\limits_{i=1}^{\infty}B_i=\bigcap\limits_{i=1}^{\infty}W_i$ consists of just one point. We say that Whites win the game if $\bigcap\limits_{i=1}^{\infty}B_i\in S$. A set S is defined to be an (α,β)-winning set if Whites can win the game for any Black's way of playing. A set S is defined to be an α-winning set if it is (α,β)-winning for every $\beta\in(0,1)$.

Schmidt [125, 126, 127] proved that for any $\alpha>0$ an α-winning set is a set of full Hausdorff dimension and that the intersection of a countable family of α-winning sets is an α-winning set also.

For example the set BAD$(1/2,1/2)$ is an $1/2$-winning set (more generally, form Schmidt [126] we know that the set of badly approximable linear forms is a $1/2$-winning set in any dimension). Given $\theta\in\mathbb{R}$ the set

$$\{\eta\in\mathbb{R}:\ \inf_{q\in\mathbb{Z}_+}q||q\theta-\eta||>0\}$$

is an $1/2$-winning set [104] (more generally, in [104] there is a result for systems of linear forms).

In 1) - 4) in the previous subsection we discuss the existence of certain real numbers θ_2 and θ. In all of these settings it is possible to show that the sets of corresponding θ_2 or θ have full Hausdorff Dimension. Badziahin-Pollington-Velani's Theorem 2, Badziahin-Velani's Theorem 4 for mixed Littlewood setting and Badziahin's Theorem 3 gives the sets of full Hausdorff dimension also. However neither in any result from 1) - 4) from the previous subsection, nor in Badziahin-Pollington-Velani's Theorem 2[3] and Badziahin's Theorem 3 we do not know if the sets constructed are winning. Moreover we do not know if the set BAD(α,β) is a winning set in the case $(\alpha,\beta)\neq(1/2,1/2)$.

The reason is that the property to be a winning set is a "local" property, but Peres-Schlag's agrument and Badziahin-Pollington-Velani's argument are "non-local". The construction of badly approximable sets by Peres-Schlag and Badziahin-Pollington-Velani suppose that at a certain level we have a collection of subsegments of a given small segment and we must choose some good subsegments from the collection. The methods do not enable one to say something about the location of good segments form the collection under consideration. All the methods give lower bound for the number of good subsegments in the collection. This is enough to establidh the full Hausdorff dimension, but does not enough to prove the winning property.

I would be very interesting to study winning properties of the sets arising from Peres-Schlag's method and Badziahin-Pollington-Velani's method.

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[3] Very recently Jinpeng An in his wonderful paper [64] showed that under the condition

$$\inf_q q^{1/\alpha}||q\theta_1||>0$$

the set

$$\{\theta_2\in\mathbb{R}:\ \ (\theta_1,\theta_2)\in\mathrm{BAD}(\alpha,\beta)\}$$

is 1/2-winning. So he proved the winning property in Theorem 2. The construction from [64] of course gives a better quanitiative version of the statement from [103] formulated in Section 1.2.

In his next paper [65] Jinpeng An proved that the *two-dimensional* set BAD(α,β) is $(32\sqrt{2})^{-1}$-winning set. The construction due to Jinpeng An seems to be very elegant and important. Probably it can give a solution to the multi-dimensionsl problem. As for the constant $(32\sqrt{2})^{-1}$, it seems to me that it may be improved to 1/2.

I do not know any "natural" example of a set of badly approximable numbers in a "natural" Diophantine problem which has full Haudorff dimension but which is not a winning set in the sence of Schmidt games.

At the end of this subsection I would like to fomulate a result by Badziahin, Levesley and Velani [6]:

Theorem 10. *For $\alpha, \beta \in (0,1), \alpha + \beta = 1$ and prime p the set*

$$\{\theta \in \mathbb{R} : \inf_{q \in \mathbb{Z}_+} q \max(|q|_p^{1/\alpha}, ||q\theta||^{1/\beta}) > 0\} \tag{17}$$

is $1/4$-winning set.

The main result from [6] is more general than Theorem 10: it deals not with p-adic norm $|\cdot|_p$ only but with a norm associated with an arbitrary bounded sequence of integers $\mathcal{D}$. It is interesting to undestand if the set (17) is 1/2-winning.

2 Best approximations

For positive integers m, n we consider a real matrix

$$\Theta = \left(\begin{array}{ccc} \theta_1^1 & \cdots & \theta_1^m \\ \cdots & \cdots & \cdots \\ \theta_n^1 & \cdots & \theta_n^m \end{array} \right). \tag{18}$$

Suppose that

$$\Theta \mathbf{x} \notin \mathbb{Z}^n, \quad \forall \mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}. \tag{19}$$

We consider a norm $|\cdot|_*^n$ in $\mathbb{R}^n$ and a norm $|\cdot|_*^m$ in $\mathbb{R}^m$ We are interested mostly in the sup-norm

$$|\xi|_{\sup}^k = \max_{1 \leqslant j \leqslant k} |\xi_j|$$

or in the Euclidean norm

$$|\xi|_2^k = \sqrt{\sum_{j=1}^k |\xi_j|^2}$$

for a vector $\xi = (\xi_1, ..., \xi_k) \in \mathbb{R}^k$ where k is equal to n or m.

Let

$$\mathbf{z}_\nu = (\mathbf{x}_\nu, \mathbf{y}_\nu) \in \mathbb{Z}^{m+n}, \ \ \mathbf{x} \in \mathbb{Z}^m, \ \ \mathbf{y} \in \mathbb{Z}^n, \quad \nu = 1, 2, 3, ... \tag{20}$$

be the infinite sequence of best approximation vectors with respect to the norms $|\cdot|_*^m, |\cdot|_*^n$. We use the notation

$$M_\nu = |\mathbf{x}_\nu|_*^m, \quad \zeta_\nu = |\Theta \mathbf{x}_\nu - \mathbf{y}_\nu|_*^n.$$

Recall that the definition of the best approximation vector can be formulated as follows: in the set

$$\{\mathbf{z}_\nu = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n} : \ \ |\mathbf{x}|_*^m \leqslant M_\nu, \ |\Theta \mathbf{x} - \mathbf{y}|_*^n \leqslant \zeta_\nu\}$$

there is no integer points $\mathbf{z}_\nu = (\mathbf{x}, \mathbf{y})$ different from $\mathbf{0}, \pm \mathbf{z}_\nu$.

In this section we formulate some open problems related to the sequence of the best approximations.

2.1 Exponents of growth for M_ν

We are interested in the value

$$G(\Theta) = \liminf_{\nu\to\infty} M_\nu^{1/\nu}. \tag{21}$$

First of all we recall well-known general lower bounds. In [14] it is shown that for sup-norms in $\mathbb{R}^m$ and $\mathbb{R}^n$ for any matrix Θ under the condition (19) one has

$$G(\Theta) \geqslant 2^{\frac{1}{3^{m+n}-1}}.$$

This is a generalization of Lagarias' bound [81] for $m = 1$.

In fact Lagarias' result deals with an arbitrary norm: for any norm $|\cdot|_*^n$ on $\mathbb{R}^n$ and a vector $\Theta \in \mathbb{R}^n$ that has at least one irrational coordinate, the inequality

$$M_{\nu+2^{n+1}} \geqslant 2M_{\nu+1} + M_\nu$$

is true for all $\nu \geqslant 1$. So $G(\Theta) \geqslant \phi_{1,n}$ where $\phi_{1,n}$ is the maximal root of the equation $t^{2^{n+1}} = 2t + 1$.

There is another well known statement which is true in any norm. Given a norm $|\cdot|_*^n$ in $\mathbb{R}^n$, consider the contact number $K = K(|\cdot|_*^n)$ This number is defined as the maximal number of unit balls with respect to the norm $|\cdot|_*^n$ without interior common points that can touch another unit ball. Consider a vector $\Theta \in \mathbb{R}^n$ that has at least one irrational coordinate. Then the inequality

$$M_{\nu+K} \geqslant M_{\nu+1} + M_\nu$$

holds for all $\nu \geqslant 1$. So $G(\Theta) \geqslant \phi_{2,n}$ where $\phi_{2,n}$ is the maximal root of the equation $t^K = t + 1$.

The general problem is to find optimal bounds for the value

$$\inf_\Theta G(\Theta) \tag{22}$$

for fixed dimensions m, n and for fixed norms $|\cdot|_*^m, |\cdot|_*^n$ (the infimum here is taken over matrices Θ satisfying (19)). This problem seems to be difficult. The only case where we know the answer is the case $m = n = 1$ (of course in this case there is no dependence on the norms). For $m = n = 1$ the theory of continued fractions gives

$$\inf_\Theta G(\Theta) = G\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2}.$$

Here we consider the case $m = 1, n = 2$ when better bounds are known. In Subsections 2.1.1 and 2.1.2 we formulate the best known results for sup-norm and Euclidean norm. Any improvement of these bounds may be of interest. Of course any generalizations to larger values of n are of interest too. We write $g_{1,2;\infty}$ for the infimum (22) in the case of sup-norm and $m = 1, n = 2$ and $g_{1,2;2}$ for the infimum (22) in the case of the Euclidean norm and $m = 1, n = 2$

2.1.1 Case $m = 1, n = 2$, sup-norm

In [93] Moshchevitin improved on Lagarias' result form [81] by proving

$$g_{1,2;\infty} \geqslant \phi \cdot \left(\frac{8 + 13\phi_3}{\phi_3^{13}}\right)^{\frac{1}{11}} = 1.28040^+, \quad \phi_3 = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

2.1.2 Case $m = 1, n = 2$, the Euclidean norm

Improving on Romanov's result from [121], Ermakov [44] proved that

$$g_{1,2;2} \geqslant 1.228043.$$

Here we should note that this result involves numerical computer calculations.

2.1.3 Brentjes' example related to cubic irrationalities

Brentjes [13] considers the following example. Let $\phi_4 = 1.324^+$ be the unique real root of the equation

$$t^3 = t + 1.$$

Consider the lattice Λ consisting of all points of the form

$$\lambda(\alpha) = \begin{pmatrix} \alpha \\ \operatorname{Re}\alpha' \\ \operatorname{Im}\alpha' \end{pmatrix},$$

where α is an algebraic integer from the field $\mathbb{Q}(\phi_4)$ and α' is one of its algebraic conjugates. The triple

$$\lambda(1). \ \lambda(\phi_4), \ \lambda(\phi_4^2)$$

form a basis of the lattice Λ. Brentjes consideres the sequence of the best approximations $\mathbf{w}_\nu \in \Lambda, \nu = 1, 2, 3, ...$ But his definition differs from our definition behind. A vector $\mathbf{w} = (w_0, w_1, w_2) \in \Lambda$ is a best approximation (in Brentjes' sense) if the only points of the lattice Λ belonging to the cylinder

$$\{\xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3 : \ |\xi_0| \leqslant |w_0|, \ |\xi_1|^2 + |\xi_2|^2 \leqslant |w_1|^2 + |w_2|^2\}$$

are the points $\mathbf{0}, \pm\mathbf{w}$. Brentjes shows that these best approximations form a periodic sequence and that

$$\lim_{\nu\to\infty} |\mathbf{w}_\nu|^{1/\nu} = \phi_4 = 1,324^+$$

(here $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^3$). This Brentjes' result can be easily obtained by means of the Dirichlet theorem on algebraic units.

Cusick [33] studied the best approximations for linear form

$$\phi_4^2 x_1 + (\phi_4^2 - \phi_4)x_2 - y$$

in the case $m = 2, n = 1$ and in sup-norm.

In my opinion the following question remains open: *for the vector* $\Theta = \begin{pmatrix} \phi_4 \\ \phi_4^2 \end{pmatrix} \in \mathbb{R}^2$ *(we consider the case* $m = 1, n = 2$*) find the value of* $G(\Theta)$ *defined in (21). Is it equal to* ϕ_4 *or not?* Probably the solution should be easy.

Lagarias [81] conjectured that in the case $m = 1, n = 2$ for the value $G(\Theta)$ defined in (18) we have

$$\inf_{\text{over all norms on } \mathbb{R}^2} \ \inf_{\Theta} G(\Theta) = \phi_4.$$

2.2 Degeneracy of dimension: $m = n = 2$

The simplest facts concerning the degeneracy of dimension of subspaces generated by the best approximation vectors are discussed in [100].

If

$$\det \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \end{pmatrix} \neq 0$$

and (19) is satisfied then for any ν_0 the set of integer vectors $\{\mathbf{z}_\nu, \nu \geqslant \nu_0\}$ span the whole space $\mathbb{R}^4$. So

$$\dim \operatorname{span}\{\mathbf{z}_\nu, \nu \geqslant \nu_0\} = 4.$$

For a matrix under the condition (19) the equality

$$\dim \operatorname{span}\{\mathbf{z}_\nu, \nu \geqslant \nu_0\} = 3$$

never holds. These facts are proven by Moshchevitin (see Section 2.1 from [100]).

The following question is an opened one. *Does there exist a matrix Θ with zero determinant and satisfying (19) such that for all ν_0 (large enough) one has*

$$\dim \operatorname{span}\{\mathbf{z}_\nu, \nu \geqslant \nu_0\} = 2 \ ?$$

3 Jarník's Diophantine exponents

For a real matrix (18) satisfying (19) we consider function

$$\psi_\Theta(t) = \min_{\mathbf{x}=(x_1,\ldots,x_m)\in\mathbb{Z}^m : 0<|\mathbf{x}|_{\sup}^m \leqslant t} \ \min_{\mathbf{y}\in\mathbb{Z}^n} |\Theta\mathbf{x} - \mathbf{y}|_{\sup}^n.$$

Sometimes we need to consider the function $\psi_{\Theta^*}(t)$ for the transposed matrix Θ^*. In this case we suppose that Θ^* satisfies (19) also. Define ordinary Diophantine exponent $\omega = \omega(\Theta)$ and uniform Diophantine exponent $\hat{\omega} = \hat{\omega}(\Theta)$:

$$\omega = \omega(\Theta) = \sup\left\{\gamma : \ \liminf_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty\right\},$$

$$\hat{\omega} = \hat{\omega}(\Theta) = \sup\left\{\gamma : \ \limsup_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty\right\}.$$

In terms of the best approximations (with respect to sup-norm, however here is no dependence on a norm) we have

$$\omega = \omega(\Theta) = \sup\left\{\gamma : \ \liminf_{\nu\to+\infty} M_\nu^\gamma \zeta_\nu < +\infty\right\},$$

$$\hat{\omega} = \hat{\omega(\Theta)} = \sup\left\{\gamma : \ \limsup_{t\to+\infty} M_{\nu+1}^\gamma \zeta_\nu < +\infty\right\}.$$

Sometimes we shall use notation $\omega^*(\Theta)$ for $\omega(\Theta^*)$. There are trivial inequalities which are valid for all Θ:

$$\frac{m}{n} \leqslant \hat{\omega} \leqslant \omega \leqslant +\infty.$$

For $m = 1$ one has in addition

$$\frac{1}{n} \leqslant \hat{\omega} \leqslant 1.$$

For more details one can see our recent survey [100].

3.1 Jarník's theorems

In [62] Jarník proved the following theorem.

Theorem 11. *Suppose that θ satisfies (19).*

(i) Suppose that $m=1, n\geqslant 2$ and the column-matrix Θ consist at least of two lineqrly independent over $\mathbb{Z}$ together with 1 numbers θ_j^1, Then

$$\omega \geqslant \frac{\hat{\omega}^2}{1-\hat{\omega}}. \tag{23}$$

(ii) Suppose that $m=2$. Then

$$\omega \geqslant \hat{\omega}(\hat{\omega}-1). \tag{24}$$

(iii) Suppose that $m\geqslant 3$ and $\hat{\omega}\geqslant (5m^2)^{m-1}$ then

$$\omega \geqslant \hat{\omega}^{\frac{m}{m-1}} - 3\hat{\omega}. \tag{25}$$

From the other hand Jarník proved

Theorem 12. *(i) Let $m\geqslant 2$. Take real $T>2$. Then there exists Θ satisfying (19) such that*

$$\omega(\Theta)=T^m, \ \ \hat{\omega}(\Theta)=T^{m-1}.$$

(ii) Let $m=1, n\geqslant 2$. Take real $T>2$ satisfying

$$T^{n-1} > T^{n-2} + \sum_{k=0}^{n-2} T^k, \ \ T^n > 1 + 2\sum_{k=1}^{n-1} T^k.$$

Then there exists Θ satisfying (19) such that

$$\hat{\omega}(\Theta) = 1 - \frac{1}{T} - \frac{1}{T^2} - \ldots - \frac{1}{T^{n-1}}.$$

$$\omega(\Theta) = T\,\frac{T^{n-1}-T^{n-2}-\ldots-1}{T^{n-1}+T^{n-2}+\ldots+1}$$

We see that from (i) of Theorem 12 it follows that for $\alpha > 2^{m-1}$ there exists Θ such that

$$\hat{\omega}(\Theta)=\alpha, \ \ \omega(\Theta) = (\hat{\omega}(\Theta))^{\frac{m}{m-1}}.$$

From (ii) of Theorem 12 it follows that for $m=1$ and arbitrary n for any $\alpha<1$ close to 1 there exists a vector Θ such that

$$\hat{\omega}(\Theta)=\alpha, \ \ \omega(\Theta) < \frac{\alpha}{1-\alpha}.$$

3.2 Case $(m,n)=(1,2)$ or $(2,1)$

Laurent [84] proved the following

Theorem 13. *The following statemens are valid for the exponents of two-dimensional Diophantine approximations.*

(i) For a vector-row $\Theta = (\theta^1, \theta^2) \in \mathbb{R}^2$ *such that* θ^1, θ^1 *and 1 are linearly independent over* $\mathbb{Z}$ *for the values*

$$w = \hat{\omega}(\Theta), \qquad w^* = \hat{\omega}(\Theta^*), \qquad v = \omega(\Theta), \qquad v^* = \omega(\Theta^*) \tag{26}$$

the following statements are valid:

$$2 \leqslant w \leqslant +\infty, \qquad w = \frac{1}{1-w^*}, \qquad \frac{v(w-1)}{v+w} \leqslant v^* \leqslant \frac{v-w+1}{w}. \tag{27}$$

(ii) Given four real numbers (w, w^*, v, v^*)*, satisfying* (27) *there exists a vector-row* $\Theta = (\theta^1, \theta^2) \in \mathbb{R}^2$*, such that* (26) *holds.*

This theorem is known as "four exponent theorem". It combines together Khintchine's thansference inequalities [67] for ordinary exponents ω, ω^* and Jarník's equality [61] for uniform exponents $\hat{\omega}, \hat{\omega}^*$ as well as some new results [19].

From Theorem 13 it follows that in the case $m = 1, n = 2$ the inequality (23) is the best possible and cannot be improved. Also in the case $m = 2, n = 1$ the inequality (24) is the best possible. The cases $m = 1, n = 2$ and $m = 2, n = 1$ are the only cases when the optimal bounds for ω in terms of $\hat{\omega}$ are known. In the next two subsections we will formulate the best known improvements of Jarník's Theorem 11. However all these improvements are far from optimal. The only possible exception is Theorem 14 below. The bound of Theorem 14 may happen to be the optimal one, however I am not sure.

To find optimal bounds for ω in terms of $\hat{\omega}$ (even for specific values of dimensions m, n) is an interesting open problem.

3.3 Case $m + n = 4$

Moshchevitin proved the following results In the case $m = 1, n = 3$ he get [108]

Theorem 14. *Suppose that* $m = 1, n = 3$ *and the vector* $\Theta = (\theta_1, \theta_2, \theta_3)$ *consists of numbers linearly independent, together with 1, over* $\mathbb{Z}$*. Then*

$$\omega \geqslant \frac{\hat{\omega}}{2}\left(\frac{\hat{\omega}}{1-\hat{\omega}} + \sqrt{\left(\frac{\hat{\omega}}{1-\hat{\omega}}\right)^2 + \frac{4\hat{\omega}}{1-\hat{\omega}}}\right). \tag{28}$$

.

The inequality (28) is better than Jarník's inequality (23) for all values of $\hat{\omega}(\Theta)$.

In [98, 100] the following two theorems are proved (a proof of Theorem 16 was just sketched).

Theorem 15. *Suppose that* $m = 3, n = 1$ *and the matrix* $\Theta = (\theta^1, \theta^2, \theta^3)$ *consists of numbers linearly independent over* $\mathbb{Z}$ *together with* 1*. Then*

$$\omega \geqslant \hat{\omega} \cdot \left(\sqrt{\hat{\omega} + \frac{1}{\hat{\omega}^2} - \frac{7}{4}} + \frac{1}{\hat{\omega}} - \frac{1}{2}\right). \tag{29}$$

The inequality (29) is better than Jarník's inequality (25) for all values of $\hat{\omega}(\Theta)$.

Theorem 16. *Consider four real numbers $\theta_j^i,\ i,j = 1,2$ linearly independent over $\mathbb{Z}$ together with 1. Let $m = n = 2$ and consider the matrix*

$$\Theta = \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \end{pmatrix}$$

satisfying (19). Then

$$\omega \geqslant \frac{1-\hat{\omega}+\sqrt{(1-\hat{\omega})^2+4\hat{\omega}(2\hat{\omega}^2-2\hat{\omega}+1)}}{2}. \tag{30}$$

The inequality (30) improves on the inequality (24) for $\hat{\omega}(\Theta) \in \left(1, \left(\frac{1+\sqrt{5}}{2}\right)^2\right)$.

It may happen that Theorem 14 gives the optimal bound. I am sure that the inequality from Theorem 16 may be improved.

3.4 A result by W.M. Schmidt and L. Summerer (2011)

Very recently Schmidt and Summerer [132] improved on Jarník's Theorem 11 in the cases $m = 1$ and $n = 1$.

For $m = 1$ and arbitrary $n \geqslant 2$ they obtained the bound

$$\omega \geqslant \frac{\hat{\omega}^2+(n-2)\hat{\omega}}{(n-1)(1-\hat{\omega})}. \tag{31}$$

As for the dual setting with $n = 1$ and $m \geqslant 2$ they proved that

$$\omega \geqslant (m-1)\frac{\hat{\omega}^2-\hat{\omega}}{1+(m-2)\hat{\omega}}. \tag{32}$$

No analogous inequalities are known in the case when both n and m are greater than one.

The result by Schmidt and Summerer deals with successive minima for one-parametric families of lattices and relies on their earlier research [131] and Mahler's theory of compound and pseudocompound bodies [87]. We suppose to write a separate paper concerning Schmidt-Summerer's result and its possibble extensions, jointly with O. German.

Here we should note that in the cases $m = 1, n = 3$ and $m = 3, n = 1$ the inequalities (28) and (29) are better than (31) and (32), correspondingly.

3.5 Special matrices

Here we would like to formulate one open problem which seems to be not too difficult. Consider a special set $\mathfrak{W}$ of matrices Θ. A matrix Θ belongs to $\mathfrak{W}$ if (19) holds and moreover there exists infinitely many $(m+n)$-tuples of *consecutive* best approximation vectors

$$\mathbf{z}_\nu, \mathbf{z}_{\nu+1}, \mathbf{z}_{\nu+2}, \ldots, \mathbf{z}_{\nu+m+n-1}$$

consisting of *linearly independent* vectors in $\mathbb{R}^{m+n}$. The definitoion of $\mathbf{z}_j \in \mathbb{Z}^{m+n}$ (see (20)) is given in the very beginning of Section 2.

I think that it is not dificult to improve on all the inequalities from Jarník's Theorem 11 in the case $\Theta \in \mathfrak{W}$. Moreower I think that in this case it is possible to get optimal inequalities and to prove the optimality of these inequalities by constructing special matrices $\Theta \in \mathfrak{W}$.

4 Positive integers

In this section we consider collections of real numbers $\Theta = (\theta^1, ..., \theta^m)$, $m \geqslant 2$. (The index $n = 1$ is omitted here.) We are interested in small values of the linear form

$$||\theta^1 x_1 + ... + \theta^n x_n||$$

in positive integers $x_1, ..., x_n$. Put

$$\psi_+(t) = \psi_{+;\Theta}(t) = \min_{x_1,...,x_m \in \mathbb{Z}_+,\ 0<\max(x_1,...,x_m)\leqslant t} ||\theta^1 x_1 + ... + \theta^m x_m||.$$

We introduce Diophantine exponents

$$\omega_+ = \omega_+(\Theta) = \sup\{\gamma : \liminf_{t\to\infty} t^\gamma \psi_{+;\Theta}(t) < \infty\},$$

and

$$\hat{\omega}_+ = \hat{\omega}_+(\Theta) = \sup\{\gamma : \limsup_{t\to\infty} t^\gamma \psi_{+;\Theta}(t) < \infty\}.$$

4.1 The case $m = 2$: W.M. Schmidt's theorem and its extensions

Put

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618^+.$$

In 1976 W.M. Schmidt[128] proved the following theorem.

Theorem 17 (W.M. Schmidt)**.** *Let real numbers θ^1, θ^2 be linearly independent over $\mathbb{Z}$ together with 1. Then there exists a sequence of integer two-dimensional vectors $(x_1(i), x_2(i))$ such that*

1. $x_1(i), x_2(i) > 0$;
2. $||\theta^1 x_1(i) + \theta^2 x_2(i)|| \cdot (\max\{x_1(i), x_2(i)\})^\phi \to 0$ *as* $i \to +\infty$.

In fact W.M. Schmidt proved (see discussion in [17]) that for $n = 2$ for $\Theta = (\theta^1, \theta^2)$ under consideration one has the inequality

$$\omega_+ \geqslant \max\left(\frac{\hat{\omega}}{\hat{\omega}-1}; \hat{\omega} - 1 + \frac{\hat{\omega}}{\omega}\right) \tag{33}$$

from which we deduce

$$\omega_+(\Theta) \geqslant \phi. \tag{34}$$

From Schmidt's argument one can easily see that for θ^1, θ^2 linearly independent together with 1 one has

$$\hat{\omega}_+ \geqslant \frac{\omega}{\omega - 1}. \tag{35}$$

We would like to note here that Thurnheer (see Theorem 2 from [142]) showed that for

$$\frac{1}{2} \leqslant \omega^* = \omega^*(\Theta) \leqslant 1 \tag{36}$$

($\omega^*(\Theta)$ was defined in the beginning of Section 3, here it is the Diophantine exponent for simultaneous approximations for numbers θ^1, θ^2) one has

$$\omega_+ \geqslant \frac{\omega^*+1}{4\omega^*} + \sqrt{\left(\frac{\omega^*+1}{4\omega^*}\right)^2 + 1}. \tag{37}$$

(inequality (37) is a particular case of a general result obtained by Thurnheer).

A lower bound for ω_+ in terms of ω was obtained by the author in [99]. It was based on the original Schmidt's argument from [128]. However the choice of parameters in [99] was not optimal. Here we explain the optimal choice [109]. From Schmidt's proof and Jarník's result (24) one can easily see that

$$\omega_+ \geqslant \max\left\{g : \max_{y,z\geqslant 1:\ y^{\hat\omega-1}\leqslant z\leqslant y^{\omega/\hat\omega}}\ \max_{y^{-\omega}\leqslant x\leqslant z^{-\hat\omega}}\ \min\left(x^{1-g}z^{-g}; xy^{-1}z^{g+1}\right)\leqslant 1\right\}.$$

The right hand side here can be easily calculated. We divide the set

$$\mathfrak{A} = \left\{(\omega,\hat\omega)\in\mathbb{R}^2:\ \ \hat\omega\geqslant 2,\ \omega\geqslant\hat\omega(\hat\omega-1)\right\}$$

of all admissible values of $(\omega,\hat\omega)$ into two parts:

$$\mathfrak{A} = \mathfrak{A}_1\cup\mathfrak{A}_2,$$

$$\mathfrak{A}_1 = \left\{(\omega,\hat\omega)\in\mathbb{R}^2:\ \ 2\leqslant\hat\omega\leqslant\phi^2,\ \omega\geqslant\frac{\hat\omega(\hat\omega-1)}{3\hat\omega-\hat\omega^2-1}\right\},$$

$$\mathfrak{A}_2 = \mathfrak{A}\setminus\mathfrak{A}_1.$$

If $(\omega,\hat\omega)\in\mathfrak{A}_1$ then

$$\omega_+\geqslant G(\omega) = \frac{1}{2}\left(\frac{\omega+1}{\omega}+\sqrt{\left(\frac{\omega+1}{\omega}\right)^2+4}\right)$$

(the function $G(\omega)$ on the right hand side decreases from $G(2)=2$ to $G(+\infty)=\phi$). If $(\omega,\hat\omega)\in\mathfrak{A}_2$ then

$$\omega_+\geqslant\hat\omega-1+\frac{\hat\omega}{\omega} \tag{38}$$

So

$$\omega_+\geqslant\max\left(\frac{1}{2}\left(\frac{\omega+1}{\omega}+\sqrt{\left(\frac{\omega+1}{\omega}\right)^2+4}\right);\hat\omega-1+\frac{\hat\omega}{\omega}\right), \tag{39}$$

and this is the best bound in terms of $\omega,\hat\omega$ which one can deduce from Schmidt's argument from [128].

4.2 A counterexample to W.M. Schmidt's conjecture

In the paper [128] W.M. Schmidt wrote that he did not know if the exponent ϕ in Theorem 17 may be replaced by a lagrer constant. At that time he was not able even to rule a possibility that there exists an infinite sequence $(x_1(i),x_2(i))\in\mathbb{Z}^2$ with condition 1. and such that

$$||\theta^1x_1(i)+\theta^2x_2(i)||\cdot(\max\{x_1(i),x_2(i)\})^2\leqslant c(\Theta) \tag{40}$$

with some large positive $c(\Theta)$. Later in [130] he conjectured that the exponent ϕ may be replaced by any exponent of the form $2-\varepsilon,\varepsilon>0$ and wrote that probably such a result should be obtained by analytical tools. It happened that this conjecture is not true. In [105] the author proved the following result.

Theorem 18. *Let $\sigma = 1.94696^+$ be the largest real root of the equation $x^4-2x^2-4x+1=0$. There exist real numbers θ^1,θ^2 such that they are linearly independent over $\mathbb{Z}$ together with 1 and for every integer vector $(x_1,x_2)\in\mathbb{Z}^2$ with $x_1,x_2\geqslant 0$ and $\max(x_1,x_2)\geqslant 2^{200}$ one has*

$$||\theta^1x_1+\theta^2x_2||\geqslant\frac{1}{2^{300}(\max(x_1,x_2))^\sigma}.$$

Theorem 18 shows that W.M. Schmidt's conjectue discussed in previous subsection turned out to be false.

Here we should note that for the numbers constucted in Theorem 18 one has

$$\omega = \frac{(\sigma+1)^2(\sigma^2-1)}{4\sigma} = 3.1103^+, \ \ \hat{\omega} = \frac{(\sigma+1)^2}{2\sigma} = 2.2302^+.$$

So $(\omega, \hat{\omega}) \in \mathfrak{A}_2$ and the inequality (38) gives

$$\omega_+ \geqslant \frac{\sigma+2}{\sigma^2-1} = 1.413^+.$$

However from the proof of Theorem 18 (see [105]) it is clear that for the numbers constructed one has $\omega_+ = \sigma = 1.94696^+$.

4.3 $m = 2$: large domains

The original paper [128] contained 5 remarks related to Theorem 17. One of these remarks was as follows. The condition 1. in Theorem 17 may be replaced by a condition $|\alpha_{1,1,}x_1(i) + \alpha_{1,2}x_2(i)| < |\alpha_{2,1}x_1(i)+\alpha_{2,2}x_2(i)|$ where $\alpha_{1,1}\alpha_{2,2}-\alpha_{1,2}\alpha_{2,1} \neq 0$. In this new setting we deal with good approximations from an "angular domain". Later Thurnheer [141] got a result dealing with even larger domain. For positive parameters ρ, τ he considered the domain

$$\Phi_0(\rho,\tau) = \{(x_1,x_2) \in \mathbb{R}^2 : \ |x_2| \leqslant |x_1|^\rho\} \cup \{(x_1,x_2) \in \mathbb{R}^2 : \ |x_1| \leqslant |x_2|^\tau\} \qquad (41)$$

and its image $\Phi(\rho,\tau)$ under a non-degenerate linear transform. Thurnheer [141] proved the following

Theorem 19. *Suppose that parameters $\rho > 1, \tau \geqslant 0$ and $1 < t \leqslant r \leqslant 2$ satisfy the condition*

$$(1-\tau)(\rho(t^2r - tr - t - r - 1) + t^2) + (1-\rho)(t^2-1) \leqslant 0. \qquad (42)$$

Then there exist infinitely many integer points (x_1, x_2) such that

$$(x_1,x_2) \in \Phi(\rho,0) \quad \text{and} \quad ||\theta^1 x_1 + \theta^2 x_2|| \leqslant c_1(\max(|x_1|,|x_2|))^{-r},$$

or

$$(x_1,x_2) \in \Phi(1,\tau) \quad \text{and} \quad ||\theta^1 x_1 + \theta^2 x_2|| \leqslant c_2(\max(|x_1|,|x_2|))^{-t}.$$

Here $c_{1,2}$ are positive constants depending on Θ and ρ,τ.

Thurnheer [141] considered three special cases of his Theorem 19:

1. by putting

$$\rho = 7/4, \tau = 0 \qquad (43)$$

and $t = r = 2$ one can see that there exist infinitely many integer vectors (x_1, x_2) such that

$$(x_1,x_2) \in \Phi(7/4,0) \quad \text{and} \quad ||\theta^1 x_1 + \theta^2 x_2|| \leqslant c_3(\max(|x_1|,|x_2|))^{-2} \qquad (44)$$

(with a certain value of $c_3 > 0$);

2. by putting

$$1 < \rho \leqslant 7/4, \quad \tau = \frac{7-4\rho}{4-\rho} \qquad (45)$$

and $t = r = 2$ one can see that there exist infinitely many integer vectors (x_1, x_2) such that

$$(x_1, x_2) \in \Phi(\rho, \tau) \quad \text{and} \quad ||\theta^1 x_1 + \theta^2 x_2|| \leqslant c_4(\max(|x_1|, |x_2|))^{-2} \tag{46}$$

(with a certain value of $c_4 > 0$);

3. for any $\rho \in (1, 7/4]$ and $\tau = 0$ one can consider the largest root $s(\rho)$ of the equation

$$\rho x^3 - 2(\rho - 1)x^2 - 2\rho x - 1 = 0. \tag{47}$$

Then one can see that there exist infinitely many integer vectors (x_1, x_2) such that

$$(x_1, x_2) \in \Phi(\rho, 0) \quad \text{and} \quad ||\theta^1 x_1 + \theta^2 x_2|| \leqslant c_5(\max(|x_1|, |x_2|))^{-s(\rho)} \tag{48}$$

(with a certain value of $c_5 > 0$); note that $s(1) = \phi = \frac{1+\sqrt{5}}{2}$, and this gives Schmidt's bound (34).

4.4 Large dimension ($m > 2$)

4.4.1 A remark related to Davenport-Schmidt's result

Another remark to Theorem 1 from [128] tells us that "no great improvement is affected by allowing a large number of variables". W.M. Schmidt showed the following result to be true.

Theorem 20. *There exists a vector $\Theta = (\theta^1, \dots, \theta^m)$, $m \geqslant 3$ such that:*
– *$1, \theta^1, ..., \theta^m$ are linearly independent over $\mathbb{Z}$;*
– *for any positive ε there exists a positive $c(\varepsilon)$ such that*

$$||\theta^1 x_1 + \dots + \theta^m x_m|| > c(\varepsilon)\Big(\max_{1\leqslant i\leqslant m} |x_i|\Big)^{-2-\varepsilon}$$

for all integers $x_1, \dots, x_k$ under the condition $x_i > 0,\ i = 1, \dots, k$.

To prove Theorem 20 one should use a result by H. Davenport and W.M. Schmidt from the paper [34]. This result is based on existence of very singular vectors. Theorem 20 shows that for $m \geqslant 3$ for linearly independent collection Θ it may happen that

$$\omega_+(\Theta) \leqslant 2. \tag{49}$$

4.4.2 General Thurnheer's lower bounds

Here we formulate three general results by Thurnheer from [142]. Its particular case (inequality (37)) was discussed above. Thurnheer used the Euclidean norm to formulate his result. Of course it is not of importance and we may use sup-norm.

Given $\varepsilon > 0$ consider the domain

$$\Psi = \Psi_\varepsilon = \left\{(x_1, ..., x_m) \in \mathbb{R}^m :\ |x_m| \leqslant \varepsilon \max_{1\leqslant j\leqslant m-1} |x_j|\right\}.$$

Theorem 21. *Suppose that $1, \theta^1, ..., \theta^m$ are linearly independent over $\mathbb{Z}$. Put*

$$v(m) = \frac{1}{2}\left(m - 1 + \sqrt{m^2 + 2m - 3}\right) > m - \frac{1}{m}.$$

Then there exists infinitely many integer vectors $(x_1, .., x_m) \in \Psi$ such that

$$||\theta^1 x_1 + ... + \theta^m x_m|| \leqslant \delta(\max_{1\leqslant j\leqslant m-1} |x_j|)^{-v(m)}$$

(here δ is an arbitrary small fixed positive number).

Another Thurnheer's result deals with aproximations from a larger domain. For $w > 0$ put

$$\Phi(w) = \left\{ (x_1, ..., x_m) \in \mathbb{R}^m : \ |x_m| \leqslant (1+\varepsilon) \left(\sum_{j=1}^{m-1} |x_j|^2 \right)^{w/2} \right\} \cup$$

$$\cup \left\{ (x_1, ..., x_m) \in \mathbb{R}^m : \ \sum_{j=1}^{m-1} |x_j|^2 \leqslant 1 \right\}, \quad \varepsilon > 0.$$

Theorem 22. *Put*

$$w = w(m) = 1 + \frac{1}{m} + \frac{1}{m^2}.$$

Then for any real Θ *and and for any positive* δ *there exists infinitely many integer vectors* $(x_1, .., x_m) \in \Phi(w)$ *such that*

$$||\theta^1 x_1 + ... + \theta^m x_m|| \leqslant (1+\delta)(\max_{1 \leqslant j \leqslant m-1} |x_j|)^{-m}.$$

Another result deals with a lower bound in terms of ω^*.

Theorem 23. *Suppose that* $1, \theta^1, ..., \theta^m$ *are linearly independent over* $\mathbb{Z}$. *Suppose that*

$$\frac{1}{m} < \omega^* = \omega^*(\Theta) \leqslant \frac{1}{m-1}. \tag{50}$$

Put

$$u_0(m, \omega^*) = \frac{1}{2m\omega^*} \left(\omega^*(m-1)^2 + 1 + \sqrt{(\omega^*(m-1)^2+1)^2 + 4m^2(m-1)(\omega^*)^2} \right)$$

Then for any $u < u_0(m, \omega^*)$ *there exists infinitely many integer vectors* $(x_1, .., x_m) \in \Psi$ *such that*

$$||\theta^1 x_1 + ... + \theta^m x_m|| \leqslant (\max_{1 \leqslant j \leqslant m-1} |x_j|)^{-u}.$$

4.4.3 From Thurnheer to Bugeaud and Kristensen

Bugeaud and Kristensen [17] considered the following Diophantine exponents. Let $1 \leqslant l \leqslant m$. Consider the set

$$\Psi(m, l) = \Psi_\varepsilon(m, l) = \left\{ (x_1, ..., x_m \in \mathbb{R}^m : \ \max_{l+1 \leqslant j \leqslant m} |x_j| \ \leqslant \max_{1 \leqslant j \leqslant l} |x_j| \right\}.$$

Diophantine exponent $\mu_{m,l} = \mu_{m,l}(\Theta)$ is defined as the supremum over all μ such that the inequality

$$||\theta^1 x_1 + ... + \theta^m x_m|| \leqslant (\max_{1 \leqslant j \leqslant m} |x_j|)^{-\mu}$$

has infinitely many solutions in

$$(x_1, ..., x_m) \in \Psi(m, l) \cap \mathbb{Z}^m.$$

So Diophantine exponent $\mu_{m,1}$ corresponds just to ω_+. Thurnheer's Theorems 21 and 23 have the following interpretation in terms of $\mu_{m,m-1}$.

Suppose that $1, \theta^1, ..., \theta^m$ are linearly independent over $\mathbb{Z}$. Then

$$\mu_{m,m-1} \geqslant v(m). \tag{51}$$

If in addition (50) holds then

$$\mu_{m,m-1} \geqslant u_0(m, \omega^*). \tag{52}$$

Bugeaud and Kristensen formulate the following result.

Theorem 24. *Suppose that* $1, \theta^1, ..., \theta^m$ *are linearly independent over* $\mathbb{Z}$. *Then*

$$\mu_{m,l} \geqslant \frac{l\hat{\omega}}{\hat{\omega} - m + l}$$

and

$$\mu_{m,m-1} \geqslant \hat{\omega} - 1 + \frac{\hat{\omega}}{\omega}.$$

In fact linearly indenendency condition here is necessary.

Of course from this theorem the bound (51) follows immediatelly.

Here we should note that the main results of the paper [17] deal with metric prorerties of exponents $\mu_{m,l}$. Also in [17] several interesting problems are formulated.

4.5 Open questions

1. What is the optimal exponent $\inf_{\theta^1,\theta^2-\text{independent}} \omega_+(\Theta)$ in the problem for a linear form in two positive variables? Is it ϕ or σ or something else between ϕ and σ?

2. What are the best possible lower bounds for ω_+ and $\hat{\omega}_+$ in terms of ω, ω^* and $\hat{\omega}$? Any improvement of any of the lower bounds (35, 37, 39) will be of interest, in my opinion. Of course any improvement of lower bounds for $\mu_{m.l}$ given in (51, 52) as well as of the bounds from Theorem 24 will be of interest.

3. As it was shown behind for $\theta^1, ..., \theta^m, m \geqslant 3$ linearly independent over $\mathbb{Z}$ together with 1 it may happen (49). But in view of Theorem 18 I may conjecture that for $m = 3$ (or even for an arbitrary m) there exist a collection of linearly independent numbers $1, \theta^1, ..., \theta^m$ such that $\omega_+(\Theta) < 2$.

4. What are optimal exponents in the Thurnheer's setting for large domains Φ, Ψ? In particular, are the values of parameters ρ, τ from (43) and (45) optimal to get (44) and (46) or not? What is the optimal values of $s(\rho)$ to conclude that (48) has infinitely many solutions in integers (x_1, x_2)? Any improvements of the discussed results is of interest. Similar questions may be formulated for multi-dimensional results.

In view of our Theorem 18 I think that it is possible to solve some of the problems formulated in this subsection.

5 Zaremba conjecture

For an irreducible rational fraction $\frac{a}{q} \in \mathbb{Q}$ we consider its continued fraction expansion

$$\frac{a}{q} = [b_0; b_1, \dots, b_s] =$$

$$= b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots + \cfrac{1}{b_s}}}}, \quad b_j = b_j(a) \in \mathbb{Z}_+, \ j \geqslant 1. \tag{53}$$

The famous Zaremba's conjecture [148] supposes that there exists an absolute constant $\mathfrak{k}$ with the following property: for any positive integer q there exists a coprime to q such that in the continued fraction expansion (53) all partial quotients are bounded:

$$b_j(a) \leqslant \mathfrak{k}, \ 1 \leqslant j \leqslant s = s(a).$$

In fact Zaremba conjectured that $\mathfrak{k} = 5$. Probably for large prime q even $\mathfrak{k} = 2$ sould be enough, as it was conjectured by Hensley .

5.1 What happens for almost all $a \pmod q$?

N.M. Korobov [77] showed that for prime q there exists a, $(a,q)=1$ such that

$$\max_{\nu} b_\nu(a) \ll \log q.$$

Such a result is true for composite q also. Moreover Rukavishnikova [122] proved

Theorem 25.

$$\frac{1}{\varphi(q)}\#\left\{a\in\mathbb{Z}:\ 1\leqslant a\leqslant q,\ (a,q)=1,\ \max_{1\leqslant j\leqslant s(a)} b_j(a)\geqslant T\right\}\ll\frac{\log q}{T}.$$

Here we would like to note that the main results of Rukavishnikova's papers [122, 123] deal with the typical values of the sum of partial quotients of fractions with a given denominator: she proves an analog of the law of large numbers.

5.2 Exploring folding lemma

Niederreiter [112] proved that Zaremba's conjecture is true for $q=2^\alpha,3^\alpha,\ \alpha\in\mathbb{Z}_+$ with $\mathfrak{k}=4$, and for $q=5^\alpha$ with $\mathfrak{k}=5$. His main argument was as follows. If the conjecture is true for q then it is true for Bq^2 with bounded integer B. The construction is very simple. Consider continued fraction (53) with $b_0(a)=0$ and its denominator written as a continuant:

$$q=\langle b_1,b_2,\dots,b_s\rangle.$$

Define a^* by

$$aa^*\equiv\pm1 \pmod q$$

(the sign $\pm$ should be chosen here with respect to the parity of s). Then

$$\frac{a^*}{q}=[0;b_s(a),\dots,b_2(a),b_1(a)]=\frac{\langle b_1,\dots,b_{s-1}\rangle}{\langle b_1,\dots,b_s\rangle}$$

and

$$q=\langle b_1,\dots,b_{s-1}\rangle=\langle c_1,c_2,\dots,c_s\rangle,\ \ c_j=b_{s-j}.$$

At the same time if $c_1\geqslant 2$ then

$$\frac{q-a^*}{q}=[0;1,c_1-1,\dots,c_s]=\frac{\langle c_1-1,c_2,\dots,c_s\rangle}{\langle 1,c_1-1,c_2,\dots,c_s\rangle}$$

So we see that

$$\langle b_1,\dots,b_{s-1},b_s,X,1,c_1-1,c_2,\dots,c_s\rangle=$$
$$=\langle b_1,\dots,b_{s-1}b_s\rangle\langle X,1,c_1-1,c_2,\dots,c_s\rangle+\langle b_1,\dots,b_{s-1}\rangle\langle 1,c_1-1,c_2\dots,c_s\rangle=$$
$$=\langle b_1\dots,b_s\rangle\langle 1,c_1-1,c_2,\dots,c_s\rangle\left(X+\frac{\langle c_1-1,c_2,\dots,c_s\rangle}{\langle 1,c_1-1,c_2,\dots,c_s\rangle}+\frac{\langle b_1,\dots,b_{s-1}\rangle}{\langle b_1,\dots,b_s\rangle}\right)=$$
$$=\langle b_1\dots,b_s\rangle\langle 1,c_1-1,c_2\dots,c_s\rangle\,(X+1)\,.$$

This procedure is known as folding lemma.

By means of folding lemma Yodphotong and Laohakosol showed [146] that Zaremba's conjecture is true for $q = 6$ and $\mathfrak{k} = 6$. Komatsu [72] proved that Zaremba's conjecture is true for $q = 7^{r2^r}, r = 1, 3, 5, 7, 9, 11$ and $\mathfrak{k} = 4$. Kan and Krotkova [66] obtained different lower bounds for the number

$$f = \#\{a \pmod{p^m} : \ a/p^m = [0; b_1, \dots . b_s], \ b_j \leqslant p^n\}$$

of fractions with bounded partial quotients and the denominator of the form p^n. In particular they proved a bound of the form

$$f \geqslant C(n)m^\lambda, \ \ C(n), \lambda > 0.$$

Another applications of folding lemma one can find for example in [16, 32, 74, 75] and in the papers refered there. I think that A.N. Korobov proved Niederreiter's result concernind powers of 2 and 3 independently in his PhD thesis [74].

5.3 A result by J. Bourgain and A. Kontorovich (2011)

Recently J. Bourgain and A. Kontorovich [10, 11] achieved essential progress in Zaremba's conjecture. Consider the set

$$\mathcal{Z}_k(N) := \{q \leqslant N : \ \exists a \ \text{such that} \ (a, q) = 1, \ \ a/q = [0; b_1, ..., b_s], \ \ b_j \leqslant k\}$$

(so Zaremba's conjecture means that $\mathcal{Z}_k(N) = \{1, 2, ..., N\}$). In a wonderful paper [10] they proved

Theorem 26. *For k large enough there exists positive $c = c(k)$ such that for N lagre enough one has*

$$\#\mathcal{Z}_k(N) = N - O(N^{1-c/\log\log N}).$$

For example it follows from Theorem 26 that for k large enough the set $\cup_n \mathcal{Z}_k(N)$ contains infinitely many prime numbers.

Another result from [10] is as follows.

Theorem 27. *For $k = 50$ the set $\cup_N \mathcal{Z}_{50}(N)$ has positive proportion in $\mathbb{Z}_+$, that is*

$$\#\mathcal{Z}_{50}(N) \gg N.$$

These wondeful results follow from right order upper bound for the integral

$$I_N = \int_0^1 |S_N(\theta)|^2 d\theta, \tag{54}$$

where

$$S_N(\theta) = \sum_{q \leqslant N} N(q) e^{2\pi i q\theta},$$

and $N(q) = N_k(q)$ is the number of integers $a, (a, q) = 1$ such that all the partial quotients in the continued fraction expansion for $\frac{a}{q}$ are bounded by k

For example positive proportion result follows from the bound

$$I_N \ll \frac{S_N(0)^2}{N}$$

by the Cauchy-Schwarz inequality.

The procedure of estimating of the integral comes from Vinorgadov's method on estimating of exponential sums with polynomials (Weyl sums). The main ingredient of the proof (Lemma 7.1 from [10]) needs spectral theory of automorphic forms and follow from a result by Bourgain, Kontorovich and Sarnak from [12].

I think that it is possible to simplify the proof given by Bourgain and Kontorovich and to avoid the application of a difficult result from [12]. Probably for a certain positive proportion result A. Weil's estimates on Kloosterman sums should be enough.

5.4 Real numbers with bounded partial quotients

In this subsection we formulate some well-known results concerning real numbers with bounded partial quotients. We deal with Cantor type sets

$$F_k = \{\alpha \in [0,1] : \alpha = [0; b_1, b_2, \dots],\ b_j \leqslant k\}.$$

For the Hausdorff dimension $\dim F_k$ Hensley [58] proved

$$r_k = \dim F_k = 1 - \frac{6}{\pi^2}\frac{1}{k} - \frac{72}{\pi^4}\frac{\log k}{k^2} + O\left(\frac{1}{k^2}\right),\ \ k \to \infty$$

Exlicit estimates for $\dim F_k$ for certain values of k one can find in[63]. Another result by Hensley [56, 57] is as follows. For the sums of the values

$$N_k(q) = \#\{a \pmod q :\ a/q = [0; b_1, \dots . b_s],\ b_j \leqslant k\}$$

considered in the previous subsection Hensley proved

$$\sum_{q \leqslant Q} N_k(q) \sim \text{constant} \times Q^{2r_k},\ \ Q \to +\infty. \tag{55}$$

We need a corollary to (55). Consider the set

$$B(k,T) = \{a \pmod q :\ a/q = [0; b_1, \dots . b_\nu, \dots, b_s],\ \langle b_1, ..., b_\nu\rangle \leqslant T \implies b_j \leqslant k\}.$$

Then for $T \ll \sqrt{q}$ one has

$$\#B(k,T) \asymp_k q^{r_k}.$$

5.5 Zaremba's conjecture and points on modular hyperbola

When we are speaking about "modular hyperbola" we are interested in the distribution of the points from the set

$$\{(x_1, x_2) \in \mathbb{Z}_q^2 :\ x_1 x_2 \equiv \lambda \pmod q\}.$$

For a wonderful survey we would like to refer to Shparlinski [135].

Proposition. *Suppose that for* $T \geqslant C\sqrt{q}$ *there exist* $x_1, x_2 \pmod q$ *such that*

$$x_1, x_2 \in B(k,T),$$

and

$$x_1 x_2 \equiv 1 \pmod q.$$

Then Zaremba's conjecture is is true with a certain $\mathfrak{k}$ *depending on* k *and* C.

From A. Weil's bound for complete Kloosterman sums we know that the points on modular hypebola are uniformily distributed in boxes of the form

$$I_1 \times I_2,\ \ I_\nu = [X_\nu, X_\nu + Y_\nu]$$

provided

$$Y_1 \times Y_2 \geqslant q^{3/2+\varepsilon}.$$

So we have the following

Corollary. *Suppose that*

$$\beta_1 + \beta_2 < \frac{1}{4}. \tag{56}$$

Then there exist $x_1, x_2 \pmod q$ *such that*

$$x_1 x_2 \equiv 1 \pmod q, \quad x_1 \in B(k, T_1),\ x_2 \in B(k, T_2)$$

and

$$T_1 \asymp q^{\beta_1},\ T_2 \asymp q^{\beta_2},$$

By means of application of bounds for incomplete Kloosterman sums Moshchevitin [96] proved the following result.

Theorem 28. *Put*

$$\omega_1 = \omega_1(\beta_1) = \frac{(4r_k\beta_1 - 1)(1 - 2\beta_1)}{8}.$$

Let $q = p$ *be prime and*

$$\frac{1}{4r_k} < \beta_1 < \frac{1}{2},\ 0 < \beta_2 < \omega_1.$$

Put $T_j = p^{\beta_j}, j = 1, 2$. *Then there exist*

$$x_1 \in B(k, T_1),\ x_2 \in B(k, T_2)$$

such that

$$x_1 x_2 \equiv 1 \pmod p.$$

Theorem 28 improves on the Corollary above in the case of prime $q = p$, as we can take $\beta_,\beta_2$ in the range

$$\frac{1}{2} - \varepsilon < \beta_1 + \beta_2 < \frac{1}{2},$$

which is not considered in (56). However in Theorem 28 there is no symmetry between β_1 and β_2. It gives no good result for $\beta_1 = \beta_2$.

5.6 Numbers with missing digits

In this section we will show that if one considers instead of "non-linear" fractal-like sets $B(k, T)$ a more simple fractal-like set, the corresponding problem becomes much more easier.

For positive integers s, k we consider sets

$$D = \{d_0, ..., d_k\},\ 0 = d_0 < d_1 < ... < d_k < s,\ 1 \leqslant k \leqslant s - 2,\ (d_1, ..., d_k) = 1,$$

$$K_s^D(N) = \{x \in \mathbb{Z}_+ :\ \ x < N,\ x = \sum_{j=0}^{h} \delta_j s^j,\ \delta_j \in D\}.$$

We are interested in properties of elements of $K_s^D(N)$ modulo q.

In [94] by means of A. Weil's bounds for Kloosterman sums the following result was proven. Let p be prime. Under certain natural conditions on s, D for any $\lambda \pmod p$ there exist $x_1, x_2 \in K_s^D(p)$ such that

$$x_1 x_2 \equiv \lambda \pmod p. \tag{57}$$

One can compare this result with proposition from Subsection 5.5.

We conclude this subsection by mentioning an interesting open problem formulated by Konyagin [73]. The question is as follows.

Suppose that $(s,q) = (d_0, ..., d_k) = 1, k \geqslant 2$. *Is it true that for some large* $\sigma > 0$ *for*

$$N \geqslant q^{\sigma} \tag{58}$$

for any $\lambda \pmod q$ *there exists* $x \in K_s^D(N)$ *such that* $x \equiv \lambda \pmod q$?

Konyagin [73] showed that the answer is "yes" for *almost all* q (a simple variant of large sieve argument). In the same paper he showed that the conclusion is true if we replace the condition (58) by $N \geqslant \exp(\sigma \log q \log\log q)$. Some related topics were considered by the author in [91, 92, 95]. In [94] it is shown that under the condition (58) with σ large enough and $q = p$ prime for any λ there exist $x_1, x_2 \in K_s^D(N)$ satisfying (57).

5.7 Discrepancy bounds

For $(a,q) = 1$ consider the discrepansy $D(a,q)$ of the finite sequence of points

$$\xi_k = \left(\frac{k}{q}, \left\{\frac{ak}{q}\right\}\right), \quad 0 \leqslant k \leqslant q-1. \tag{59}$$

It is defined as

$$D(a,q) = \sup_{\gamma_1,\gamma_2 \in (0,1)} \left| \#\{k : \xi_k \in [0,\gamma_1) \times [0,\gamma_2)\} - q\gamma_1\gamma_2 \right|.$$

Here we do not want to discuss the foundations an major results of the theory of uniformly distributed sequences; we refer to books [80] and [36].

It is a well-known fact that

$$D(a,q) \ll \sum_{j=1}^{s(a)} b_j(a) \tag{60}$$

where b_j are partial quotients from (53).

If Zaremba's conjecture is true then for any q there exists a coprime to q such that

$$D(a,q) \ll \log q. \tag{61}$$

Larcher [83] proved that for any q there exists a coprime to q such that

$$D(a,q) \ll \frac{q}{\varphi(q)} \log q \log\log q. \tag{62}$$

This bound is optimal up to the factor $\log\log q$. In fact from Rukavishnikova's results [122, 123] we see that (62) holds for almost all a coprime to q.

However Zaremba's conjecture is still open, and we do not know if for a given q one can get (61) instead of (62), for some a.

Ushanov and Moshchevitin [101] proved the following result.

Theorem 29. *Let* p *be prime,* U *be a multiplicative subgroup in* $\mathbb{Z}_p^*$. *For* $v \neq 0$ *we consider the set* $R = v \cdot U$ *and let*

$$\#R \geq 10^8 p^{7/8} \log^{5/2} p. \tag{63}$$

Then there exists an element $a \in R$, $a/p = [b_1, b_2, \cdots, b_l]$, $b_i = b_i(a)$, $l = l(a)$ *with*

$$\sum_{i=1}^{l} b_i \le 500 \log p \log\log p,$$

and hence

$$D(a,p) \ll \log p \log\log p.$$

The proof uses Burgess' inequality for character sums.

An open problem here is as follows. *Is it possible to replace exponent* $7/8$ *in the condition (63) by a smaller one*?

Recently Professor Shparlinski informed me that Chang [31] essentially repeated the result of Theorem 29. Moreover in [31] an analog of Rukavishnikova's Theorem 25 is proved for multiplicative subgroups $\pmod p$ of cardinality $\gg p^{7/8+\varepsilon}$.

A multidimensional version of Theorem 29 was obtained by Ushanov [143]. It is related to Theorem 30 below.

5.8 Discrepancy bounds: multidimensional case

We would like to conclude this section by mentioning a wonderful recent result due to Bykovskii [26, 27] dealing with multidimensional analog of the sequence (59). We consider positive integer s and integers $q \geqslant 1; a_1 = 1, a_1, ..., a_s$. The study of the distribution of the sequence

$$\xi_k = \left(\frac{a_1 k}{q}, \left\{ \frac{a_2 k}{q} \right\}, ..., \left\{ \frac{a_s k}{q} \right\} \right), \quad 0 \leqslant k \leqslant q-1$$

started with the works of Korobov [76] and Hlawka [59]. We are interested in upper bounds for the discrepancy

$$D(a_1, ..., a_s; q) = \sup_{\gamma_1, ..., \gamma_s \in (0,1)} \left| \#\{k : \xi_k \in [0, \gamma_1) \times \cdots \times [0, \gamma_s)\} - q \gamma_1 \cdots \gamma_s \right|.$$

The upper bound

$$\min_{(a_1, ..., a_s) \in \mathbb{Z}^s} D(a_1, ..., a_s; q) \ll_s (\log q)^s$$

was proved by Korobov [77, 78] for prime q and by Niederreiter [111] for composite q.

Bykovskii [26, 27] proved the following result.

Theorem 30. *For* $s \geqslant 2$ *one has and for any positive integer* q *one has*

$$\min_{(a_1, ..., a_s) \in \mathbb{Z}^s} D(a_1, ..., a_s; q) \ll_s (\log q)^{s-1} \log\log q. \tag{64}$$

For $s = 2$ and prime q this result coincides with Larcher's inequality (62) discussed in the previous subsection. However Larcher's proof is based on the inequality (60) while Bykovskii's proof is related to analytic argument (this argument is quite similar for the case $s = 2$ and the general case $s \geqslant 2$) and to consideration of relative minima of lattices. It happened that the proof of Bykovskii's result is not extremely difficult. It is related to a paper by Skriganov [136] and the previous paper by Bykovskii [25]. The method developed by Bykovskii may find applications in other problems (see for example [46]).

A famous well-known conjecture is that

$$\min_{(a_1, ..., a_s) \in \mathbb{Z}^s} D(a_1, ..., a_s; q) \ll_s (\log q)^{s-1},$$

that is that the factor $\log\log q$ in (64) may be thrown away. This conjecture seems to be very difficult.

6 Minkowski question mark function

6.1 Definition of Minkowski function

If real $x = [0; a_1, ..., a_t, ...] \in [0,1]$ is represented as a regular continued fraction with natural partial quotients, then Minkowski question mark function $?(x)$ is defined as follows:

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + ... + \frac{(-1)^{n+1}}{2^{a_1+...+a_n-1}} + ...$$

(in the case of rational x this sum is finite). It is a well known fact that $?(x)$ is a continuous strictly increasing function. By the Lebesgue theorem it has finite derivative almost everywhere in $[0,1]$. Moreower the derivative $?'(x)$, if exists (in finite or infine sense) can have only two values - 0 or $+\infty$. For more results we refer to papers [2, 3, 35, 39, 90, 68, 113, 114, 124].

It will be important for us to recall the definition of Stern-Brocot sequences F_n, $n = 0, 1, 2, \ldots$. For $n = 0$ one has

$$F_0 = \{0, 1\} = \left\{\frac{0}{1}, \frac{1}{1}\right\}.$$

Suppose that the sequence F_n is written in the increasing order

$$0 = \xi_{0,n} < \xi_{1,n} < \cdots < \xi_{N(n),n} = 1, N(n) = 2^n, \quad \xi_{j,n} = \frac{p_{j,n}}{q_{j,n}}, \quad (p_{j,n}, q_{j,n}) = 1.$$

Then the sequence F_{n+1} is defined as

$$F_{n+1} = F_n \cup Q_{n+1}$$

where

$$Q_{n+1} = \left\{\frac{p_{j,n} + p_{j+1,n}}{q_{j,n} + q_{j+1,n}}, \ j = 0, \ldots, N(n) - 1\right\}.$$

Note that for the number of elements in F_n one has

$$\#F_n = 2^n + 1.$$

The Minkowski question mark function $?(x)$ is the limit distribution function for the Stern-Brocot sequences:

$$?(x) = \lim_{n\to\infty} \frac{\#\{\xi \in F_n : \ \xi \leqslant x\}}{2^n + 1}.$$

6.2 Fourier-Stieltjes coefficients

Here we would like to mention a famous open problem by R. Salem [124]: *to prove or to disprove that for Fourier-Stieltjes coefficients $d_n, n \in \mathbb{N}$ of $?(x)$ one has*

$$d_n = \int_0^1 \cos(2\pi n x)\, \mathrm{d}?(x) \to 0, \quad n \to \infty.$$

Certain results related to this problem were obtained by G. Alkauskas [2, 3].

6.3 Two simple questions

Here we formulate two open questions.

1. One can see that

$$?(0)=0, \quad ?\left(\frac{1}{2}\right)=\frac{1}{2}, \quad ?(1)=1.$$

Moreover

$$?'(0)=?'\left(\frac{1}{2}\right)=?'(1)=0,$$

as in any rational point the question mark function has zero derivative. By continuouity agrument we see that there exist two points

$$x_1\in\left(0,\frac{1}{2}\right), \quad x_2\in\left(\frac{1}{2},1\right)$$

such that

$$?(x_i)=x_i, \ \ i=1,2.$$

So we see that the equation

$$?(x)=x, \quad x\in[0,1] \tag{65}$$

has at least five solutions. The question is if equation (65) has *exactly* five solutions.

2. Consider the function $m(x)$ inverse to $?(x)$. As $?(\xi_{j,n})=\frac{j}{2^n}$ we see that $m\left(\frac{j}{2^n}\right)=\xi_{j,n}$. Then by the Koksma inequality (see [80]) we have

$$\left|\frac{1}{2^n}\sum_{j=1}^{2^n}\left(\xi_{j,n}-\frac{j}{2^n}\right)^2-\int_0^1(m(x)-x)^2\mathrm{d}x\right|\leqslant\frac{D_n\cdot V}{2^n},$$

where D_n is the discrepancy of the sequence

$$\frac{j}{2^n}, \ \ 1\leqslant j\leqslant 2^n,$$

$0<D_n\leqslant 1$ and $V\leqslant 4$ is the variation of the function $x\mapsto(m(x)-x)^2$. One can easily see that

$$\int_0^1(m(x)-x)^2\mathrm{d}x=\int_0^1(?(x)-x)^2\mathrm{d}x>0. \tag{66}$$

That is why we have

$$\sum_{j=1}^{2^n}\left(\xi_{j,n}-\frac{j}{2^n}\right)^2=2^n\int_0^1(?(x)-x)^2\mathrm{d}x+R_n, \ \ |R_n|\leqslant 4. \tag{67}$$

The question is as follows. *Is it true that for the remainder in (67) one has $R_n\to 0$ as $n\to\infty$?*

This question is motivated by the famous Franel theorem. Instead of Stern-Brocot sequence F_n consider Farey series $\mathcal{F}_Q$ which consist of all rational numbers $p/q\in[0,1], (p,q)=1$ with denominators $\leqslant Q$. Suppose that $\mathcal{F}_Q$ form an increasing sequence

$$1=r_{o,Q}<r_{1,Q}<...<r_{j,Q}<r_{j+1,Q}<...<r_{\Phi(Q),Q}=1, \quad \Phi(Q)=\sum_{q\leqslant Q}\varphi(q)$$

(here $\varphi(\cdot)$ is the Euler totient function). Then for the limit distribution function one has

$$\lim_{Q\to\infty}\frac{\#\{r\in\mathcal{F}_Q: \ \ r\leqslant x\}}{\Phi(Q)+1}=x$$

and the integral similar to (66) is equal to zero. Franel's theorem (see [82]) states that the asymptotic formula

$$\sum_{j=1}^{\Phi(Q)}\left(r_{j,Q}-\frac{j}{\Phi(Q)}\right)^2=O_\varepsilon(Q^{-1+\varepsilon}),\ \ Q\to\infty.$$

for all positive ε is equivalent to Riemann Hypothesis. In fact the well-known asymptotic equality

$$\sum_{n\leqslant Q}\mu(n)=o(Q),\ \ Q\to\infty$$

leads to

$$\sum_{j=1}^{\Phi(Q)}\left(r_{j,Q}-\frac{j}{\Phi(Q)}\right)^2=o(1),\ \ Q\to\infty.$$

A analogous formula for R_n from (67) is unknown, probably.

6.4 Values of derivative

It is a well-known fact that if for $x\in[0,1]$ the derivative $?'(x)$ exists then $?'(x)=0$ or $?'(x)=+\infty$.

For a real irrational x represented as a condinued fraction expansion $x=[a_0;a_1,...,a_n,..]$ we consider the sum of its first partial quotients $S_x(t)=a_1+...+a_t$. Define

$$\kappa_1=\frac{2\log\frac{1+\sqrt{5}}{2}}{\log 2}=1.388^+,\ \ \kappa_2=\frac{4L_5-5L_4}{L_5-L_4}=4.401^+,\ \ L_j=\log\frac{j+\sqrt{j^2+4}}{2}-j\frac{\log 2}{2}.$$

Improving on results by Paradis, Viader and Bibiloni [114], Kan, Dushistova and Moshchevitin [40] proved the following four theorems.

Theorem 31. *(i) Assume for an irrational number x there exists such a constant C that for all natural t one has*

$$S_x(t)\leqslant\kappa_1 t+\frac{\log t}{\log 2}+C.$$

Then $?'(x)$ exists and $?'(x)=+\infty$.

(ii) Let $\psi(t)$ be an increasing function such that $\lim_{t\to+\infty}\psi(t)=+\infty$. Then there exists such an irrational number $x\in(0,1)$ that $?'(x)$ does not exist and for any t one has

$$S_x(t)\leqslant\kappa_1 t+\frac{\log t}{\log 2}+\psi(t).$$

Theorem 32. *Let for an irrational number $x\in(0,1)$ the derivative $?'(x)$ exists and $?'(x)=0$. Then for any real function $\psi=\psi(t)$ under conditions*

$$\psi(t)\geqslant 0,\ \ \psi(t)=o\left(\frac{\log\log t}{\log t}\right),\ t\to\infty$$

there exists T depending on ψ such that for all $t\geqslant T$ one has

$$\max_{u\leqslant t}\left(S_x(u)-\kappa_1 u\right)\geqslant\frac{\sqrt{2\log\lambda_1-\log 2}}{\log 2}\cdot\sqrt{t\log t}\cdot(1-t^{-\psi(t)}).$$

(ii) There exists such an irrational $x\in(0,1)$ that $?'(x)=0$ and for all large enough t

$$S_x(t)-\kappa_1 t\leqslant\frac{\sqrt{16\log\lambda_1-8\log 2}}{\log 2}\cdot\sqrt{t\log t}\cdot\left(1+2^5\left(\frac{\log\log t}{\log t}\right)\right).$$

Theorem 33. *(i) Assume for an irrational number x there exists such a constant C that for all natural t one has*

$$S_x(t) \geqslant \kappa_2 t - C.$$

Then $?'(x)$ exists and $?'(x) = 0$.

(ii) Let $\psi(t)$ be an increasing function such that $\lim_{t\to+\infty} \psi(t) = +\infty$. Then there exists such an irrational number $x \in (0,1)$ that $?'(x)$ does not exist and for any t we have

$$S_x(t) \geqslant \kappa_2 t - \psi(t).$$

Theorem 34. *(i) Assume for an irrational number $x \in (0,1)$ the derivative $?'(x)$ exists and $?'(x) = +\infty$. Then for any large enough t one has*

$$\max_{u\leqslant t} (\kappa_2 u - S_x(u)) \geqslant \frac{\sqrt{t}}{10^8}.$$

(ii) There exists such an irrational $x \in (0,1)$ that $?'(x) = +\infty$ and for large enough t we have

$$\kappa_2 t - S_x(t) \leqslant 200\sqrt{t}.$$

A weaker result is due to Dushistova and Moshchevitin [39]. All these results are related to deep analysis of sets of values of continuants. Theorems 31 and 33 are optimal, but Theorms 32 and 34 are not optimal. It should be interesting to prove optimal bounds related to this two last theorems.

Here we should note that the paper [40] contains some other results related to sets of real numbers with bounded partial quotients. Probably some of these results may be improved.

It is possible to prove that for any λ from the interval

$$\kappa_1 \leqslant \lambda \leqslant \kappa_2$$

there exist irrationals $x, y, z \in [0,1]$ such that

$$\lim_{t\to\infty} \frac{S_x(t)}{t} = \lim_{t\to\infty} \frac{S_y(t)}{t} = \lim_{t\to\infty} \frac{S_z(t)}{t} = \lambda$$

and $?'(x) = 0, ?'(y) = +\infty$, but $?'(z)$ does not exist.

Theorems 31 - 34 show that

$$\kappa_1 = \sup \left\{ \kappa \in \mathbb{R} : \limsup_{t\to\infty} \frac{S_t(x)}{t} < \kappa \implies ?'(x) = +\infty \right\}, \tag{68}$$

$$\kappa_2 = \inf \left\{ \kappa \in \mathbb{R} : \liminf_{t\to\infty} \frac{S_t(x)}{t} > \kappa \implies ?'(x) = 0 \right\}. \tag{69}$$

6.5 Denjoy-Tichy-Uitz family of functions

There are various generalizations of the Minkowski question mark function $?(x)$. One of them was considered by Denjoy [35] and rediscovered by Tichy and Uitz [140].

For $\lambda \in (0,1)$ we define for $x \in [0,1]$ a function $g_\lambda(x)$ in the following way. Put

$$g_\lambda(0) = 0, \quad g_\lambda(1) = 1.$$

Then if g_λ is defined for two neighbooring Farey fractions $\frac{a}{b} < \frac{c}{d}$, we put

$$g_\lambda\left(\frac{a+c}{b+d}\right) = (1-\lambda) g_\lambda\left(\frac{a}{b}\right) + \lambda g_\lambda\left(\frac{c}{d}\right).$$

So we define $g_\lambda(x)$ for all rational $x \in [0,1]$. For irrational x we define $g_\lambda(x)$ by continuouty.

The family $\{g_\lambda\}$ constructed consist of sungular functions. One can easily see that $g_{1/2}(x) = ?(x)$, that is the Minkowski question mark function is a member of this family. Hence $g_{1/2}$ has a clear arithmetic nature: it is the limit distribution function for Stern-Brocot sequences F_n. Recall that

$$F_n = \{x = [0; a_1, ..., a_t] : \ a_1 + ... + a_t = n+1\}.$$

Zhabitskaya [149] find out that for $\lambda = \frac{3-\sqrt{5}}{2}$ the function $g_\lambda(x)$ is the limit distribution function for the sequences

$$\Xi_n = \{x = [[1; b_1, ..., b_l]] : \ \ b_1 + ... + b_l = n+1\}$$

associated with "semiregular" continued fractions

$$[[1; b_1, b_2, b_3, ..., b_l]] = 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots - \cfrac{1}{b_l}}}}, \quad b_j \in \mathbb{Z}, \ \ b_j \geqslant 2.$$

It happens that disrtibution functions of some other sequences associated with special continued fractions do not belong to the family $\{g_\lambda\}$ (see [150, 151]).

It is interesting to find other values of λ for which the function $g_\lambda(x)$ is associated with an explicit and natural object.

Another open problem related to the family $\{g_\lambda\}$ is as follows. Analogously to (68,69) we define

$$\kappa_1(\lambda) = \sup \left\{ \kappa \in \mathbb{R} : \ \limsup_{t\to\infty} \frac{S_t(x)}{t} < \kappa \ \ \Longrightarrow \ \ g'_\lambda(x) = +\infty \right\},$$

$$\kappa_2(\lambda) = \inf \left\{ \kappa \in \mathbb{R} : \ \liminf_{t\to\infty} \frac{S_t(x)}{t} > \kappa \ \ \Longrightarrow \ \ g'_\lambda(x) = 0 \right\}.$$

The study of the functions

$$\kappa_j(\lambda), \ \ 0 < \lambda < 1$$

has never been made. Recently D. Gayfulin calculated the values $\kappa_j\left(\frac{3-\sqrt{5}}{2}\right)$, $j = 1, 2$.

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