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Essentially Baer modules

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Maeda (1958) defined Rickart rings (known as p.p. rings). Also, defined by Kaplansky, Hattori (1960): A ring is called right Rickart if the right annihilator of any single element is generated by an idempotent element. The notion of Rickart rings was generalized to a module theoretic version by Lee, Rizvi, and Roman (2010): A right R-module M is called a *Rickart module* if for every $\varphi \in S = \mathrm{End}_R(M)$ then $\mathrm{Ker}\varphi = eM$ for some $e^2 = e \in S$. Dually, a module M is called a *d-Rickart module* (or a *dual Rickart module*) if for every $\varphi \in S = \mathrm{End}_R(M)$ then $\mathrm{Im}\varphi = eM$ for some $e^2 = e \in S$.

A ring R is called a right *ACS ring* if the right annihilator of every element of R is an essential submodule of a direct summand of R_R. According to [1] and [2], a module M is called a *CS-Rickart module* (res., *strongly CS-Rickart module*) if $\mathrm{Ker}\varphi$ is essential in a direct summand (res., fully invariant direct summand) of M for every $\varphi \in S = \mathrm{End}_R(M)$. Dually, a module M is called a *d-CS-Rickart module* (res., *strongly d-CS-Rickart module*) if $\mathrm{Im}\varphi$ lies above a direct summand (res., fully invariant direct summand) of M for every $\varphi \in S = \mathrm{End}_R(M)$.

Kaplansky (1965) introduced a Baer ring as the right annihilator of every non empty subset of the ring is generated by an idempotent element. Clark (1967) introduced a quasi-Baer ring as the right annihilator of every two sided ideal of the ring is generated by an idempotent element. In 2004, Rizvi and Roman [4] introduced the notions of Baer and quasi-Baer modules in general module theoretic settings. A module M is called a *Baer module* if $r_M(I)$ is a direct summand of M for every left ideal I of S. A module M is quasi-Baer if the right annihilator in M of every ideal of the ring $S = \mathrm{End}_R(M)$ is generated by an idempotent of S. In 2010, Tütüncü and Tribak [5] introduced the notion of dual Baer modules. A module M is called a *dual Baer module* if $\sum_{\varphi\in I} \mathrm{Im}\varphi$ is a direct summand of M for every right ideal I of S.

Birkenmeier, Park, and Rizvi (2006) introduced a *right essentially Baer ring* as the right annihilator of any nonempty subset of the ring is essential in a right ideal generated by an idempotent. In this paper, we introduce the concept of essentially Baer modules, dual essentially Baer modules in general module theoretic settings. A module M is called a *essentially Baer module* (res., *strongly essentially Baer module*) if $r_M(I)$ is essential in a direct summand (res., fully invariant direct summand) of M for every left ideal I of S. A module M is called a *dual essentially Baer module* (res., *strongly dual essentially Baer module*) if $\sum_{\varphi\in I} \mathrm{Im}\varphi$ lies above a direct summand (res., fully invariant direct summand) of M for every right ideal I of S.

Theorem 1. *The following implications hold for a module M:*
(1) M is Baer if and only if M is K-nonsingular essentially Baer.
(2) M is dual Baer if and only if M is T-noncosingular dual essentially Baer.

Theorem 2. *Let M be a right R-module.*
(1) If M is CS-Rickart and M has the SSIP-CS then M is essentially Baer. The converse is true if $\mathrm{Soc}M \trianglelefteq M$.
(2) If M is d-CS-Rickart and M has the SSSP-d-CS then M is dual essentially Baer. The converse is true if $\mathrm{Rad}M \ll M$.

Recall that a ring R is called a *right semi-artinian ring* if every non-zero right R-module has non-zero socle. A ring R is called a *right max ring* if every nonzero right R-module has a maximal submodule.

Corollary 1. *Let M be a right R-module.*
(1) If R is a right semi-artinian ring, then M is essentially Baer if and only if M is CS-Rickart and M has the SSIP-CS.
(2) If R is a right max ring, then M is dual essentially Baer if and only if M is d-CS-Rickart and M has the SSSP-d-CS.

Theorem 3. *The following implications hold:*
(1) Every direct summand of a (dual) essentially Baer module is also a (dual) essentially Baer module.
(2) Every direct summand of a strongly essentially Baer module is also a strongly essentially Baer module.
(3) Every direct summand of a strongly dual essentially Baer module is also a strongly dual essentially Baer module.
(4) [2, Proposition 3.7] Every direct summand of a strongly CS-Rickart module is also a strongly CS-Rickart module.
(5) Every direct summand of a strongly d-CS-Rickart module is also a strongly d-CS-Rickart module.

Corollary 2. *([4, Theorem 2.17], [5, Corollary 2.5]) The following implications hold:*
(1) Every direct summand of a Baer module is also a Baer module.
(2) Every direct summand of a dual Baer module is also a dual Baer module.

According to Bican et al. ([3]), the product of submodules A and B of M is defined by: $A *_M B = \mathrm{Hom}_R(M,B)A = \sum\{f(A) \mid f \in \mathrm{Hom}_R(M,B)\}$. A proper submodule N of a module M is called *2-prime* if $A *_M B \leq N$ implies $A \leq N$ or $B \leq N$, whenever A, B are submodules of N. A module M is *prime* if 0 is a 2-prime submodule of M (see [3, Lemma 2.5]). Let $\mathcal{A}$ be the class of all prime modules. Denote by $\mathcal{P}(M)$ the set $\bigcap_{f\in \mathrm{Hom}(M,A), A\in\mathcal{A}} \mathrm{Ker} f$.

Theorem 4. *Let M be a projective module and $\mathcal{P}(M) = 0$. Then the following are equivalent.*

(1) M is a quasi-Baer module.
(2) Every fully invariant submodule of M is essential in a fully invariant direct summand of M.
(3) The right annihilator in M of every ideal of $S = \mathrm{End}_R(M)$ is essential in a fully invariant direct summand of M.

Corollary 3. *The following conditions are equivalent for a semiprime ring R:*
(1) R is a quasi-Baer ring.
(2) For every ideal I of the ring R, there exists an idempotent $e \in R$, such that I_R is essential in R_R and $(1-e)Re = 0$.
(3) For every ideal I of the ring R, there exists an idempotent $e \in R$, such that $r(I)$ is essential in R_R and $(1-e)Re = 0$.

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О новых свойствах некоторых многообразий алгебр Ли и Лейбница почти полиномиального роста

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