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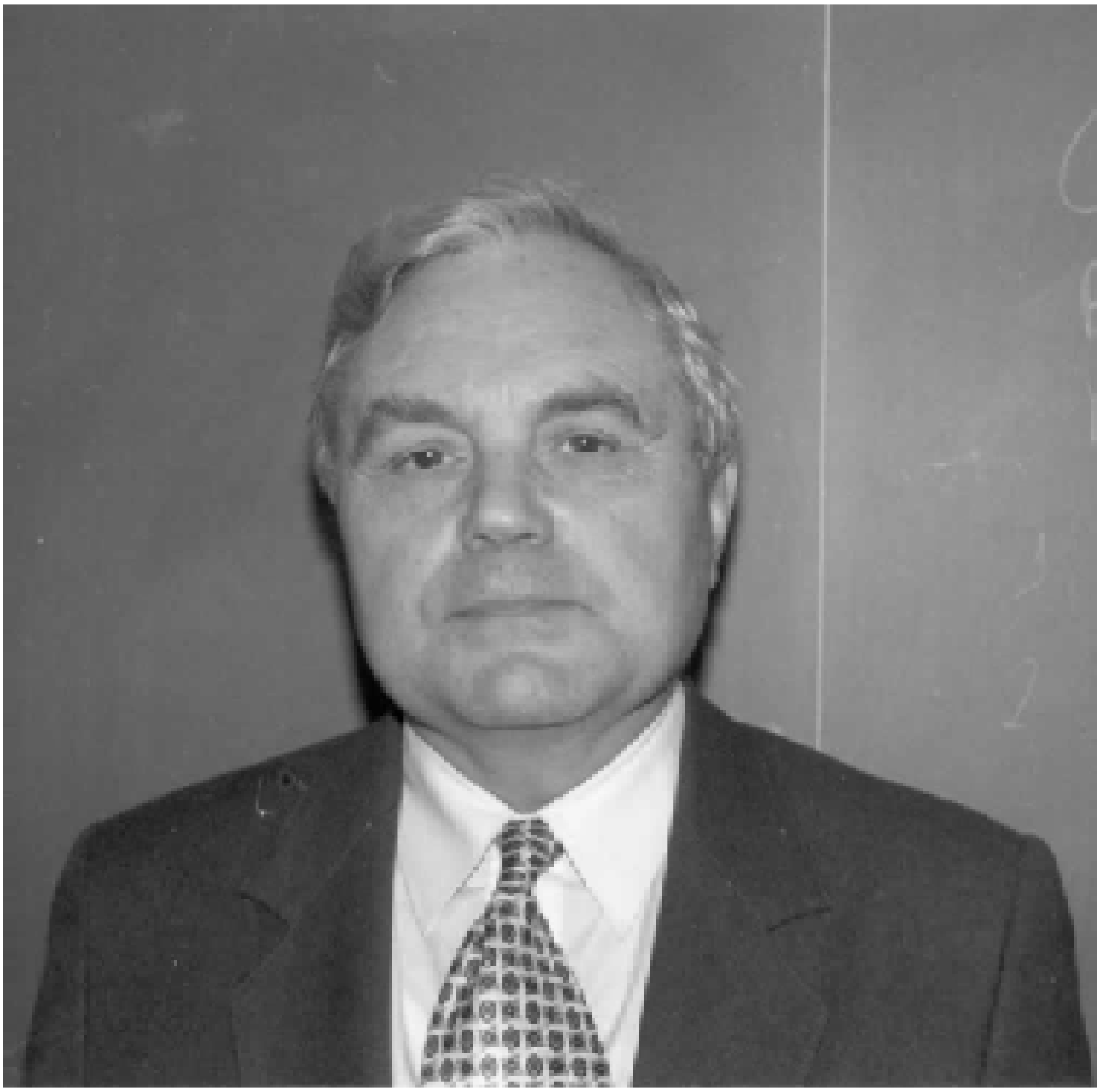

This volume is an outcome of XII International Conference Algebra and Number Theory: Modern Problems and Application, dedicated to 80-th anniversary of Professor V. N. Latyshev which was held in Leo Tolstoy Tula State Pedagogical University on 21-25 April 2014. The purpose of the conference is coordination of modern research in algebra and number theory. The program of the conference consisted of plenary talks and talks in eight thematic sections: Group Theory; Semigroup Theory and Universal Algebra; Ring and Module Theory; Applied and Computer Algebra, Cryptography and Discrete Mathematics; Analytical Number Theory; Diophantine Approximations and Transcendental Number Theory; Geometry of Numbers and Uniform Distribution; Number-theoretical Method in Approximate Analysis.

Plenary reports cover a wide range of modern achievements in algebra and number theory in the following areas:
— combinatorial group theory;
— finite groups and representation;
— Abelian groups;
— semigroups of transformations;
— rings and modules, homological methods;
— Hopf algebras;
— algebra and superalgebra Lie;
— varieties of algebras;
— Boolean algebras and functions;
— algebraic surface;
— cryptography and coding;
— computer algebra;
— analytical numbers theory;
— Diophantine approximations and the theory of transcendental numbers;
— geometry of numbers;
— number-theoretic method of approximate analysis.

Section 1 "Group Theory"
The reports of this section regarded to new results in the theory of groups such as finite groups and representations; Abelian groups; infinite groups of various classes (nilpotent, solvable); combinatorial group theory; theory of varieties of groups.

Section 2 "Semigroup Theory and Universal Algebra"
The reports of this section presented a series of new works, related to the modern theory of semigroups of transformations, semigroup of quotients, to the constructions of semigroups and the theory of representations of semigroups. Also the achievements of the Volga Algebraic school which was projected by L. A. Skornyakov were submitted in the program of this section.

Section 3 "Ring and Module Theory"
New results in the ring theory reflect the contemporary tendency to create a common theory of rings (associative and nonassociative) and corresponding representation theory. Significant attention is devoted to generalizations of the ring theory

such as semirings, nearring arising in applications, and rings with additional structures such as topological, ordered, graded and filtered rings. In the program section we notice the results concerning with Lie superalgebras and other classes of algebras close to ones, with combinatorial methods, and with the development of structural methods.

Section 4 "Applied and Computer Algebra, Cryptography and Discrete Mathematics"

The reports of this section deal with wide range of algebraic applications in Computer Science, Economics, Physics, Management and Cryptography.

Section 5 "Analytical Number Theory"

The reports of this section is devoted to the studies on the classic areas of the analytical number theory such as theories of Riemann zeta function and Dirichlet L-functio; additive objectives; the method of trigonometric sums.

Section 6 "Diophantine Approximations and Transcendental Number Theory"

The reports of this section presents the latest advances in algebraic, transcendental and p-adic numbers.

Section 7 "Geometry of Numbers and Uniform Distribution"

The achievements of the Vladimir-theoretic were presented in the program section

Section 8 "Number-theoretical Method in Approximate Analysis"

In the program section the achievements of the Tula theoretic-number school founded by V. D. Podsypanin were presented. All reports deal with modern development of theoretical and numerical method of approximate analysis. This method was based and developed in 1957–1963 years in the works of the participants of the seminar Steklov Mathematical Institute of Academy of Sciences of the USSR under direction of N. C. Bakhvalov, N. M. Korobov and N. N. Chentsov.

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BOUNDED REMAINDER SETS OF SMALL DIMENSIONS[1]

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We consider the set T, for this is set given the counting function $r(\alpha, i, T) = \sharp\{j : 0 \leq j < i, \{j\alpha\} \in T\}$. If the parameter α is irrational, then by using the criterion of Weyl, we obtain an asymptotic formula

$$r(\alpha, i, T) = i|T| + \delta(\alpha, i, T), \tag{1}$$

where $\delta(\alpha, i, T) = o(i)$ is the remainder of the formula (1) and $i \to \infty$.

Definition 1. *The set T is called the bounded remainder set (BR-sets), when we have the inequality $|\delta(\alpha, i, T)| \leqslant C$ for all values i, where C is constant.*

E. Hecke constructed the first example bounded remainder sets. It was intervals T^1 of length $0 < |b+a\alpha| < 1$, where $a, b \in \mathbb{Z}$. Hecke obtained the following remainder estimate

$$|\delta(\alpha, i, X)| \leqslant |a|.$$

The more difficult task to find a constructions for multidimensional bounded remainder sets. R. Szüsz constructed the first example of two-dimensional bounded remainder sets in 1954. They were a family of parallelograms. P. Liardet analyzed this construction and proved the possibility of reduction of sets with the dimension D to a similar sets with the dimension $D-1$. Rauzy G. (1982) and S. Ferenczi (1992), belonging to French mathematical school, have found a link between the property to be BR-sets with first return map. But none of the above has not received estimates were founded by the above authors in multidimensional case.

In 2011 V.G.Zhuravlev found a new way of constructing the bounded remainder sets, he used a stretching of the multidimensional unit cube. And he proved the Hecke's theorem for the multidimensional case [2].

The author of this paper constructs two-dimensional bounded remainder sets using hexagonal torus development and constructs three-dimensional sets using the Fedorov hexagonal prism.For this sets we found exact estimates of the remainder.

In the two-dimensional case parameter $c = (c_1, c_2)$, where $c \in C = \{c = (c_1, c_2) \in \mathbb{R}^2; c_i \geq 0,\ \mathtt{min}(c_1, c_2) \leqslant 1\}$, produces the hexagonal torus development $T^2(c)$. We denote $\sigma(x) = x_1 + x_2$, where $x = (x_1, x_2)$. If $\sigma(c) \leqslant 1$, we get a convex hexagon, otherwise we get nonconvex hexagon. Coordinates of the vertices of our hexagon are $(0,0)$, $(1-c_1, -c_2)$, $(1,0)$, $(1-c_1, 1-c_2)$, $(0,1)$, $(-c_1, 1-c_2)$.

Using the vector $\alpha = (\alpha_1, \alpha_2) = tc$, we get a tiling of development on the field $T^2_k, k = 0, 1, 2$. There $0 \leqslant t \leqslant 1$ for $\sigma(c) \leqslant 1$ и $0 \leqslant t \leqslant \frac{1}{c_1+c_2}$ for $\sigma(c) > 1$. In the

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article [3] author proved, that exchange transformation S_v of tiles T_k^2 is equal to the rotation S_α of the torus $\mathbb{T}^2$ on vector α.

For each tile, we can define the counting functions $r_k(i) = \sharp\{j : S_\alpha^j(0) \in T_k^2, 0 \le j < i\}$ and a corresponding functions $\delta_k(i) = r_k(i) - is_k$, $k = 0, 1, 2$. The following theorem is proved.

THEOREM 1. *Suppose we have a torus rotating S_α on irrational vector $\alpha = (\alpha_1, \alpha_2)$, i.e. numbers $\alpha_1, \alpha_2, 1$ are linearly independent over $\mathbb{Z}$. Let we have a tiling of the torus $\mathbb{T}^2 = \mathbb{T}_0^2 \sqcup \mathbb{T}_1^2 \sqcup \mathbb{T}_2^2$, and its development is $T^2(c)$. Then, for the remainders $\delta_k(i), k = 0, 1, 2,$ there are following inequalities:*

$$\begin{array}{llll} 0 & \leqslant \delta_0(i, x_0) \leqslant & 2 - \sigma(c) & \textit{for}\, \sigma(c) \leqslant 1; \\ \sigma(c) - 1 & \leqslant \delta_0(i, x_0) \leqslant & 1 & \textit{for}\, \sigma(c) > 1; \\ -1 & \leqslant \delta_1(i, x_0) \leqslant & c_1; & \\ -1 & \leqslant \delta_2(i, x_0) \leqslant & c_2. & \end{array}$$

In the three-dimensional case torus developments are the Fedorov hexagonal prisms T^3. We construct it by multiplying toric developments. This multiplication was first discussed in the article [2]. Now we use the product to unit segments and hexagonal toric developments [4]. We get a prism tiling $\mathbb{T}^3 = \mathbb{T}_0^3 \sqcup \mathbb{T}_1^3 \sqcup \mathbb{T}_2^3 \sqcup \mathbb{T}_3^3$. For each tile $T_n^3, n = 0, 1, 2, 3,$ we can define the counting functions $r_n(i)$ and the corresponding functions $\delta_n(i)$. In [4] for tiles T_n^3, the author has found the exact estimate of the remainder $\delta_n(i)$.

So the Hecke's theorem is proved for the two-dimensional and three-dimensional cases.

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Vladimir State University named after Alexander and Nikolay Stoletovs

ON PARTUALLY ORDERED SEMIGROUPS OF RELATIONS WITH DIOPHANTINE OPERATIONS

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A set of binary relation Φ closed with respect to some collection Ω of operations on relations forms an algebra (Φ, Ω) called *an algebra of relations*. As a rule, operations on relations are defined by formulas. These operations are called *logical*. One of the most important classes of logical operations on relations is the class of diophantine operations [1,2] (in other terminology – primitive-positive operations [3]). An operation on relations is called *diophantine* if it can be defined by a first order formula containing in its prenex normal form only existential quantifiers and conjunctions. Algebras of relations with diophantine operations can be considered to be partially ordered by the relation of set-theoretic inclusion $\subset$.

Note that the operation of relation product $\circ$ is a diophantine operation. An algebra of relations of the form $(\Phi, \circ)$ is a semigroup, and every semigroup isomorphic to some semigroup of relations. There are some other associative diophantine operations on relations. We concentrate our attention on the following associative diophantine operation of relation $*$ that is defined as follows. For any relations ρ and σ on U, put

$$\rho * \sigma = \{(u, v) \in U \times U : (\exists w, t)(u, w) \in \rho(u, t) \in \sigma\}.$$

For any set Ω of operations on relations, let $R\{\Omega\}$ ($R\{\Omega, \subset\}$) denote the class of all algebras (partially ordered algebras) isomorphic to ones whose elements are binary relations and whose operations are members of Ω. Let $Var\{\Omega\}$ ($Var\{\Omega, \subset\}$) be the variety generated by $R\{\Omega\}$ ($R\{\Omega, \subset\}$).

The main results are formulated in the following theorems.

Theorem 1. *An partially ordered semigroup $(A, \cdot, \leq)$ belongs to the class $R\{*, \subset\}$ if and only if it is commutative and the following identities hold*

$$x^2y = xy \ (1), \quad x \leq x^2 \ (2), \quad xy \leq x^2 \ (3).$$

Theorem 2. *An algebra $(A, \cdot, \wedge)$ of the type $(2, 2)$ belongs to the class $R\{*, \cap\}$ if and only if $(A, \cdot)$ is commutative semigroup that satisfies the identity (1), $(A, \wedge)$ is a semilattice, and the following identities hold*

$$x \wedge x^2 = x \ (4), \quad x(x \wedge y) = x \wedge y \ (5), \quad (x^2 \wedge y)z = xyz \ (6).$$

Theorem 3. *An algebra $(A, \cdot, \vee)$ of the $(2, 2)$ belongs to the variety $Var\{*, \cup\}$ if and only if $(A, \cdot)$ is commutative semigroup that satisfies the identity (1), $(A, \vee)$ is a semilattice, and the following identities hold*

$$x \vee x^2 = x^2 \ (7), \quad xy \vee x^2 = x^2 \ (8), \quad x(y \vee z) = xy \vee xz \ (9).$$

Theorem 4. *An algebra $(A, \cdot, \vee, \wedge)$ of the $(2, 2)$ belongs to the variety $Var\{*, \cup, \cap\}$ if and only if $(A, \cdot)$ is commutative semigroup that satisfies the identity (1), $(A, \vee, \wedge)$ is a distributive lattice, and the identities (4), (5), (6), (9) hold.*

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UDC 512.579

ABOUT GENERATING SETS OF DIAGONAL ACTS OVER SEMIGROUPS OF ISOTONE TRANSFORMATIONS

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A right act [1] over a semigroup S is defined as a set X with a mapping $X \times S \to X$, $(x, s) \mapsto xs$ satisfying the rule $(xs)s' = x(ss')$ for $x \in X$, $s, s' \in S$. *A left act* Y over a semigroup S is defined dually, that is a mapping $X \times S \to Y$, $(s, y) \mapsto sy$, if $s(s'y) = (ss')y$ for $y \in Y$, $s, s' \in S$. If a set X is a left act over a semigroup S and a right act over a semigroup T, we call it *bi-act* if there is the equality $(sx)t = s(xt)$ for $x \in X$, $s \in S$, $t \in T$ is held. If S is a semigroup, then the set $S \times S$ will be a right act relating to the rule $(x, y)s = (xs, ys)$ for any $x, y, s \in S$, and a left act if $s(x, y) = (sx, sy)$, and also a bi-act in case of two rules are held. We call them *right, left diagonal acts* and *diagonal bi-act*. Denote them $(S \times S)_S$, ${}_S(S \times S)$, ${}_S(S \times S)_S$ respectively. We will call a diagonal bi-act as *cyclic*, if it is generated by one element.

It was proved in [2] that diagonal right act $(S \times S)_S$, diagonal left act ${}_S(S \times S)$ and diagonal biact ${}_S(S \times S)_S$ are cyclic if $S = T_X$, P_X or B_X when X is an infinite set, T_X is the semigroup of all transformatios $X \to X$, P_X is the semigroup of all partial transformations, B_X is the semigroup of binary relations over the set X. There is a similar question for the semigroup O_X of all *isotone* (order-preserving) mappings $\alpha : X \to X$, where X is partially ordered set, i.e. such mappings α, that $x \leq y \Rightarrow x\alpha \leq y\alpha$ for any $x, y \in X$. In previous papers the author investigated the diagonal acts over the semigroup O_X and defined conditions for cycling and finite generation of those acts (see [3]). It was also proved that for any infinite chain X diagonal acts over the semigroup O_X cannot be cyclic. The main result of this paper summarizes the results mentioned in case where X is a set of natural numbers $\mathbb{N}$ with normal order. Namely, a diagonal bi-act over semigroup of isotone mappings $O_{\mathbb{N}}$ does not have countable generating system.

Definition 1. *Let* α, β: $\mathbb{N} \to \mathbb{N}$ *be isotone mappings. We will call* (α, β) *a strong pair if the following rules are held:*

(i) $i\alpha \neq j\alpha$, $i\beta \neq j\beta$ *for* $i \neq j$

(ii) $i\alpha \neq j\beta$ *for any* i, j *(i.e.* $\mathrm{im}\alpha \cap \mathrm{im}\beta = \emptyset$*)*

(iii) *for any* k *there is* l *such that* $(l\alpha = k \vee l\beta = k)$ *(i.e.* $\mathrm{im}\alpha \cup \mathrm{im}\beta = X$*).*

The proof of the Theorem is based on two statements of a technical nature.

Lemma 1. *Let* $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ *be isotone transformations such that* $\mathrm{im}\alpha$, $\mathrm{im}\beta$ *are infinite sets. Then there are isotone transformations* $\tilde{\alpha}, \tilde{\beta}$ *which form a strong pair such that* $(\tilde{\alpha}, \tilde{\beta})\gamma = (\alpha, \beta)$ *for a some isotone transformation* γ.

Each strong pair can be associated with a sequence of zeros and ones. We will call an *elementary decimation* simultaneous removal of i-th one and i-th zero (for any i). A decimation is a result of elementary decimations by finite or infinite number of times.

Lemma 2. *Let* ε, η *be sequences of ones and zeros that are corresponded to strong pairs* (α, β) *and* (α', β') *and* $\alpha' = \gamma\alpha\delta$, $\beta' = \gamma\beta\delta$ *for some* $\gamma, \delta \in O_{\mathbb{N}}$. *Then the set* η *can be obtained from a set* ε *by decimation.*

With these statements reasoning close to Cantor's diagonal method, we proved the following Theorem.

Theorem 1. *Diagonal bi-act* ${}_{O_{\mathbb{N}}}(O_{\mathbb{N}} \times O_{\mathbb{N}})_{O_{\mathbb{N}}}$ *does not have an countable set of generators.*

The Theorem has two corollaries with similar results for diagonal left and diagonal right acts.

Corollary 1. *Diagonal left act* ${}_{O_{\mathbb{N}}}(O_{\mathbb{N}} \times O_{\mathbb{N}})$ *does not have a finite or countable generating set.*

Corollary 2. *Diagonal right act* $(O_{\mathbb{N}} \times O_{\mathbb{N}})_{O_{\mathbb{N}}}$ *does not have a finite or countable generating set.*

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GRADED MORITA CONTEXT

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Morita contexts are actively researched in recent decades, they have numerous applications [1, 2, 3, 4].

All rings considered in this paper are associative with identity, graded by arbitrary multiplicative group G, all modules are unitary G-graded, gr.mod$-A$ (A–gr.mod) is the category of right (left) graded A-modules whose objects are right (left) G-graded A-modules and morphisms are graded-preserving homomorphisms.

If $M, N \in$ gr.mod$-A$ then $\mathrm{HOM}_A(M,N)_g$ is the set of *homogeneous homomorphisms of degree* g, i.e. A-linear maps such that $f(M_h) \subseteq N_{gh}$ $(h \in G)$. It is clear that $\mathrm{HOM}_A(M,N) = \oplus_{g\in G}\mathrm{HOM}_A(M,N)_g$ is a graded Abelian group, and $\mathrm{END}_A(M) = \mathrm{HOM}_A(M,M)$ is a graded ring, which is called the *graded endomorphism ring* of a graded A-module M. If we consider left A-modules then $f \in \mathrm{HOM}_A(M,N)_g$ means that $(M_h)f \subseteq N_{hg}$. Let $M = \oplus_{g\in G}M_g$ and $\sigma \in G$ then $M(\sigma)$ is the module M considered with the grading $M(\sigma)_g = M_{\sigma g}$ for right module (and $M(\sigma)_g = M_{g\sigma}$ for left module); the module $M(\sigma)$ is called the *σ-suspension* of M. The graded equivalence in the categories of graded modules is called the equivalence which commutes with all σ-suspension functor.

Graded Morita context (A, B, P, Q, μ, τ) consists of two graded rings A and B, two graded bimodules ${}_AP_B$ and ${}_BQ_A$ and two bimodule homomorphisms $\mu : P \otimes_B Q \to A$, $\tau : Q \otimes_A P \to B$ preserving grading and satisfying the conditions:

$$\begin{aligned} &(1) \quad \mu(p \otimes q)p' = p\tau(q \otimes p'), \\ &(2) \quad \tau(q \otimes p)q' = q\mu(p \otimes q') \qquad (p, p' \in P, q, q' \in Q). \end{aligned}$$

For convenience, we denote $\mu(p \otimes q) = pq$ and $\tau(q \otimes p) = qp$.

It is clear that if $P \in$ gr.mod–B then denoted by $A = \mathrm{END}_B(P)$, $Q = \mathrm{HOM}_B(P,B)$, $g_p(q \otimes p) = qp$, $f_p(p \otimes q) = pq \in B$ where $(pq)p' = p(qp')$ for all $p, p' \in P$, $q \in Q$ we get the graded Morita context (A, B, P, Q, f_P, g_P) which called *standard Morita context associated to the module P_B*.

A module $P \in$ gr.mod$-A$ is called *gr-generator* if $\oplus_{g\in G}P(g)$ is the generator for gr.mod$-A$ [5, лемма 2.1].

Theorem 1. *Let P be a graded B-module, $A = END_B(P)$, $Q = HOM_B(P,B)$ and (A, B, P, Q, f_P, g_P) be graded Morita context associated to the module P_B. Then:*

1) g_P is isomorphism if and only if P_B is gr-generator;

2) f_P isomorphism if and only if P is finitely generated projective B-module.

Theorem 2. *Let (A, B, P, Q, μ, τ) be a graded Morita context such that μ и τ are isomorphisms. Then:*

1) μ и τ induce the bimodule isomorphisms

$$P \cong \mathrm{HOM}_B(Q, B) \cong \mathrm{HOM}_A(Q, A),$$
$$Q \cong \mathrm{HOM}_B(P, B) \cong \mathrm{HOM}_A(P, A)\ ;$$

2) the homomorphisms of the graded rings

$$\mathrm{END}_B(P) \longleftarrow A \longrightarrow \mathrm{END}_B(Q)^\circ,$$
$$\mathrm{END}_A(P)^\circ \longleftarrow B \longrightarrow \mathrm{END}_B(P),$$

induced by the structures of the bimodules P and Q are the isomorphisms;

3) the structure of the right graded A-ideals is isomorphic the structure of the graded B-submodules of P (under correspondence $I \to IP \cong I \bigotimes_A P$), moreover the graded ideals in A correspond to the graded A–B-subbimodules in P. Similar statements concerning the other structural isomorphisms follow from symmetry; in particular, the structure of two-sides graded ideals of A and B are isomorphic;

4) functor $T = \mathrm{HOM}_A(P, -) \cong Q \bigotimes_A - : A-\mathrm{gr.mod} \longrightarrow B-\mathrm{gr.mod}$ is graded equivalence of this categories

As consequences we obtain [6, теорема 3].

The graded Morita context define the series of impotant and interesting classes modules in the categories of graded modules, it establishes a relationship between graded radicals of the rings and modules, which is also the subject of investigation.

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MELLIN TRANSFORMS OF DIRICHLET L-FUNCTIONS

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Let χ be a Dirichlet character modulo $q > 1$, and $L(s, \chi)$, $s = \sigma + it$, is the corresponding L-function. In the moment problem for L-functions, the asymptotic behavior of the quantity

$$\sum_{\chi=\chi \pmod q} \int_1^T |L(\sigma + it, \chi)|^{2k} dt, \qquad k \geq 0, \quad \sigma \geq \frac{1}{2},$$

usually is studied as $T \longrightarrow \infty$. A more difficult moment problem is the asymptotics or estimates for

$$\int_1^T |L(\sigma + it, \chi)|^{2k} dt, \qquad T \longrightarrow \infty.$$

By analogy with the Riemann zeta-function, for the latter problem the method of Mellin transforms can be applied.

The classical Mellin transform $M_f(s)$ of a function $f(x)$ is defined by

$$M_f(s) = \int_0^\infty f(x) x^{s-1} dx$$

provided that the integral exists. Sometimes the modified Mellin transform

$$M_f^*(s) = \int_1^\infty f(x) x^{-s} dx$$

is more convenient because a possible convergence problem at the point $x = 0$ does not exist for it. The functions $M_f(s)$ and $M_f^*(s)$ are related by a simple relation. Let

$$\hat{f}(x) = \begin{cases} f(\frac{1}{x}) & \text{if} \quad 0 < x \leqslant 1, \\ 0 & \text{otherwise} . \end{cases}$$

Then it is known [1] that

$$M_f^*(s) = M_{\hat{f}(x)/x}(s).$$

Modified Mellin transforms of powers of the Riemann zeta-function $\zeta(s)$

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s}dx$$

were introduced, studied and applied for investigation of the moments

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt, \quad T \to \infty,$$

in a series of works by A. Ivič, M. Jutila and Y. Motohashi. M.Lukkarinen gave meromorphic continuation of $\mathcal{Z}_1(s)$ to the whole complex plane.

Let B_k stand for the k th Bernoulli number, and let γ be the Euler constant. In [6], the following theorem has been proved.

THEOREM 1. *The function $\mathcal{Z}_1(s)$ has a meromorphic continuation to the whole complex plane. It has a double pole at point $s = 1$, and its Laurent expansion at this point is*

$$\mathcal{Z}_1(s) = \frac{1}{(s-1)^2} + \frac{2\gamma - \log 2\pi}{s-1} + \ldots \quad .$$

The other poles of $\mathcal{Z}_1(s)$ are the simple poles at the points $s = -(2k-1), k \in \mathbb{N}$, and

$$\operatorname*{Res}_{s=-(2k-1)} \mathcal{Z}_1(s) = \frac{i^{-2k}(1-2^{-(2k-1)})}{2k} B_{2k} \quad .$$

The papers [2]— [5] are devoted to the Mellin transform

$$\int_1^\infty \left|\zeta\left(\varrho+ix\right)\right|^{2k} x^{-s}dx$$

with a fixed $\frac{1}{2} < \varrho < 1$.

In the report, will discuss the modified Mellin transform $\mathcal{Z}_1(s, \frac{1}{2}, \chi)$ of $\left|L\left(\frac{1}{2}+ix, \chi\right)\right|^2$ defined, for $\sigma > 1$, by

$$\mathcal{Z}_1(s, \chi) = \int_1^\infty \left|L\left(\frac{1}{2}+ix, \chi\right)\right|^2 x^{-s}dx.$$

We consider meromorphic contiuanation of $\mathcal{Z}_1(s, \frac{1}{2}, \chi)$ to the whole complex plane seperatlly for principal character χ_0 and primitive character χ modulo q. Let

$$a(q) = \sum_{p|q} \frac{\log p}{p-1},$$

and $\varphi(q)$ denote the Euler totient function. Moreover let $G(\chi)$ denote the Gauss sum and

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if} \quad b = 0, \\ \epsilon_1(\chi) & \text{if} \quad b = 1, \end{cases}$$

where

$$\epsilon(\chi)=\frac{G(\chi)}{\sqrt{q}},\qquad \epsilon_1(\chi)=-\frac{G(\chi)}{\sqrt{q}},$$

and

$$b=\begin{cases}0 & \text{if} \quad \chi(-1)=1,\\ 1 & \text{if} \quad \chi(-1)=-1.\end{cases}$$

THEOREM 2. *The function $\mathcal{Z}_1(s,\chi_0)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi_0)=\frac{\varphi(q)}{q}\left(\frac{1}{(s-1)^2}+\frac{2\gamma+2a(q)-\log 2\pi}{s-1}\right)+\dots \quad .$$

Other singularities are the simple poles at the points $s=-(2k-1), k\in\mathbb{N}$, and

$$\operatorname*{Res}_{s=-(2k-1)}\mathcal{Z}_1(s,\chi_0)=\frac{\varphi(q)}{q}\frac{i^{-2k}(1-2^{-(2k-1)})}{2k}B_{2k} \quad .$$

THEOREM 3. *The function $\mathcal{Z}_1(s,\chi)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi)=$$

$$=\frac{i^b}{G(\chi)\sqrt{q}E(\chi)}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\frac{1}{(s-1)^2}+\frac{2\gamma-\log\frac{q}{(q,a-1)^2}-\log 2\pi}{s-1}\right)+\dots \quad .$$

Other singularities are the simple poles at the points $s=-(2k-1), k\in\mathbb{N}$.

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SEMIGROUPS OF MINIMAL DIAGONAL RANK

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A right act over a semigroup S is a set X, on which the semigroup acts via $\varphi : (X \times S) \to X$, $(x, s) \mapsto xs$ and $(xs)t = x(st)$ for any $x \in X$, $s, t \in S$. To emphasise that X is a right S-act it is denoted by X_S. A left diagonal act ${}_SX$ is defined analogously. If X is both a left act over S and a right act over T it is called *a biact* and denoted by ${}_SX_T$.

If S is a semigroup then the product $S \times S$ can be considered to be an S act [2] if φ is an element-wise right multiplication: $(a, b)s = (as, bs)$. This act is called *a right diagonal act over* S and denoted by $(S \times S)_S$. In a similar way we define a left diagonal act ${}_S(S \times S)$ and a biact ${}_S(S \times S)_S$.

A diagonal act may be considered as a unary algebra. Indeed, if $(S \times S)_S$ is a right diagonal act over S, then multiplication by $s \in S$ corresponds to the unary operation $\varphi_s : S \times S \to S \times S$, $(a, b) \mapsto (as, bs)$.

Subset $A \subseteq S \times S$ is called *a generating set* if $AS^1 = S \times S$. If no proper subset of A is a generating set then it is called *an irreducible generating set*. If A has the minimal cardinality among all generating sets then it is called *a minimal generating set*. As every diagonal act is a unary algebra then by the virtue of Theorem 1 from [3] we get that for every right (left, bi-) act an irreducible generating set is minimal.

The cardinality of the minimal generating set of an act $(S \times S)_S$ is called *the right diagonal rank of* S. The left diagonal rank and the bidiagonal rank are defined analogously. We denote them by $\operatorname{rdr} S$, $\operatorname{ldr} S$ and $\operatorname{bdr} S$ respectively.

It is easy to check that groups have the minimal possible diagonal rank equal to the cardinality of a group. The two-element right zero semigroup (which is obviously not a group) possesses the same property, i.e. its right diagonal rank is equal 2. That does not hold in general since the right diagonal rank of the n-element right zero semigroup is equal to n^2-n. It is only natural to ask a question about all semigroups which satisfy this property. It has turned out that the above examples almost exhaust all such semigroups. The full answer is given in the following theorem.

THEOREM 1. *Let S be a semigroup of n elements. The equality* $\operatorname{rdr} S = n$ *holds iff S is one of the following is true:*

1. *S is a group;*
2. *S is a two-element right zero semigroup;*
3. *S is a semigroup with the following Cayley table:*

$$\begin{array}{c|ccc} \cdot & e & a & b \\ \hline e & e & a & b \\ a & b & b & b \\ b & b & b & b \end{array}$$

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ON PROBLEM OF SIMULTANEOUS DIOPHANITE APPROXIMATIONS

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This work is information review on the problem of the best simaltaneous diophantine approximations.

The one-dimensional case

We will call the fraction $\frac{a}{b}$, $(b > 0)$ *best approximation of the first kind* to the number α, if any $c$and d such that $0 < d \leq b$ and $\frac{a}{b}\ neq\frac{b}{d}$, true

$$\left|\alpha - \frac{a}{b}\right| < \left|\alpha - \frac{c}{d}\right|. \tag{1}$$

We will call the fraction $fracab$, $(B > 0)$ it best approximation of the second kind to the number $alpha$, if any c and d such that $0 < d\ leqb$ and $fracab\ neq\ fracbd$, true

$$|b\alpha - a| < |d\alpha - c|. \tag{2}$$

In terms of the distance the last condition can be written as $\|b\alpha\|_s < \|d\alpha\|_s$.

THEOREM 1. *[6]. Every best approximation of the first kind to the number α has intermediate or convergent to the continued fraction representing this number.*

For the best approximation of the second kind we have the following theorem:

THEOREM 2. *[6].*
Every best approximation of the second kind is a convergent.

For this theorem , there is the reverse :

THEOREM 3. *[6].*
Every convergent is a best approximation of the second kind. The only exception is $\alpha = a_0 + \frac{1}{2}$ and $\frac{p_0}{q_0} = \; fraca_0$ 1.

All these theorems (p. 169) fully reveals the value chain unit fractions. Continued fractions are not just good approximations real numbers, and do it well.

The multidimensional case

Consider a real Euclidean space $\mathbb{R}^s$. Integer lattice called the design of the form [3, 4]

$$\Gamma = \left\{ m_1\gamma^{(1)} + m_2\gamma^{(2)} + \ldots + m_s\gamma^{(s)} | m_1, \ldots,\ m_s \in Z \right\}, \tag{3}$$

where $\overrightarrow{\gamma^{(1)}}$; $\overrightarrow{\gamma^{(2)}}, \ldots,\ \overrightarrow{\gamma^{(s)}}$ linearly independent points of $\mathbb{R}^s$. These points are called basis of the lattice Γ.

The following assertions

THEOREM 4. *[2]. Every vertex of Klein is relative minimum of unimodular lattice.*

THEOREM 5. *[1]. Any two adjacent relative minimum - $\gamma^{(1)} = (a_1,\ \ldots,\ a_s)$ and $\Gamma^{(2)} = (b_1,\ \ldots,\ b_s$ of lattice Γ can be extended to a basis Γ, if $\frac{\gamma^{(1)}+\gamma^{(2)}}{2} \notin \Gamma$.*

Then, by Theorem 4 (p. 18) every Diophantine approximation is a relative minimum of the lattice Γ.

Generalize the results early in order to apply them in the next part in the construction of an efficient algorithm for finding common approximations.

By Theorem 5 any two relative minimum lattice can be extended to a basis Γ. So , among all relative minima can choose a basis of Γ. Any Diophantine approximation, as shown earlier, there is a relative minimum . That is, peruse Diophantine approximations can always choose a basis of Γ. What are the advantages of such a unit basis before? For a linear combination in which we get a new Diophantine approximation coefficients are smaller in absolute value, i.e., the desired bust - decreases.

Thus obtained are considered to be pseudopolynomial and are more efficient than previously known search algorithms (mean estimates [8, 9]).

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SUCCESSIVELY ORTHOGONAL SYSTEMS OF K-ARY OPERATIONS

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Systems of k-ary operations generalizing orthogonal sets are considered.

These systems have the following property: every k successive k-ary operations, $k \geq 2$, of the system are orthogonal.

We call these systems successively orthogonal, establish some properties, give examples and methods of construction of these systems.

A k-ary operation A (briefly, a k-operation) on a set Q is a mapping $A : Q^k \to Q$ defined by $A(x_1^k) \to x_{k+1}$, and in this case write $A(x_1^k) = x_{k+1}$.

A k-groupoid (Q, A) is a set Q with one k-ary operation A, defined on Q.

The k-operation $E_i : E_i(x_1^k) = x_i$, $1 \leq i \leq k$, on Q is called *the i-th identity operation (or the i-th selector) of arity k.*

An i-invertible k-operation A, defined on Q, is a k-operation with the following property: the equation $A(a_1^{i-1}, x, a_{i+1}^k) = a_{k+1}$ has a unique solution for each fixed k-tuple $(a_1^{i-1}, a_{i+1}^k, a_{k+1})$ of Q^k.

A k-tuple $< A_1^k >$ of k-operations is orthogonal if and only if the mapping $\theta = (A_1^k) : Q^k \to Q^k$, $(x_1^k) \to (A_1(x_1^k), A_2(x_1^k), ..., A_k(x_1^k)) = (A_1^k)(x_1^k)$ is a permutation on Q^k [1].

All 2-invertible binary operations, given on a set Q, form the group $(\Lambda_2; \cdot)$ under the multiplication $(A \cdot B)(x, y) = A(x, B(x, y))$.

A k-ary quasigroup (or simply a k-quasigroup) is a k-groupoid (Q, A) such that the k-operation A is i-invertible for each $i = 1, 2, \ldots, k$.

Definition 1. *[1] A k-tuple* $< A_1, A_2, ..., A_k >=< A_1^k >$ *of k-operations, given on a set Q, is called orthogonal if the system* $\{A_i(x_1^k) = a_i\}_{i=1}^k$ *has a unique solution for all* $a_1^k \in Q^k$.

Definition 2. *[1] A set* $\{A_1, A_2, \ldots, A_t\}$, $t \geq k$, *of k-operations is called orthogonal if every k-tuple of these k-operations is orthogonal.*

Definition 3. *[1] A set* $\Sigma = \{A_1^t\}$, $t \geq 1$, *of k-ary operations, given on a set Q, is called strongly orthogonal if the set* $\overline{\Sigma} = \{A_1^t, E_1^k\}$ *is orthogonal.*

Definition 4. *A system* $\Sigma = \{A_1^t\}$, $t \geq k$, *of k-ary operations, given on a set Q,* $\mid Q \mid \geq 3$, *is called successively orthogonal system (briefly, a SOS) if any successive k operations are orthogonal.*

Every orthogonal set of k-operations is a successively orthogonal system.

Let (Q, A) be a quasigroup, $A^i(x, y) = A(x, A^{i-1}(x, y))$, $i = 2, \ldots$.

Theorem 1. *If* $A, A_1, A_2, ..., A_t, ...$ *are binary quasigroups of the order* $s_0, s_1, \ldots$ $\ldots, s_t, \ldots$, *respectively, in the group* $(\Lambda_2; \cdot)$ *of all 2-invertible binary operations, given on a set Q, then the sequence*

$$F, E, A, A^2, ..., A^{s_0-1},\ F, E, A_1, A_1^2, ..., A_1^{s_1-1},$$

$$F, E, A_2, A_2^2, ..., A_2^{s_2-1}, ...,\ F, E, A_t, A_t^2, ..., A_t^{s_t-1}, ...$$

is a SOS.

Proposition 1. *Let* $\Sigma_1 = \{A_1, A_2, ..., A_{s_1}\}$, $\Sigma_2 = \{B_1, B_2, ..., B_{s_2}\}$ *be strongly orthogonal sets of k-operations. Then the system*

$$\Sigma_3 = \{E_1, E_2, ..., E_k, A_1, A_2, ..., A_{s_1}, E_1, E_2, ..., E_k, B_1, B_2, ..., B_{s_2}\}$$

is a SOS.

THEOREM 2. *Let A be an 1-invertible k-operation on a set Q, $\theta = (E_2, E_3, ..., E_k, A) = (E_2^k, A)$, and s_0 be the order of the permutation θ in the group S_{Q^k}, then the system of k-operations*

$$E_1, E_2, ..., E_k, A, A\theta, A\theta^2, ..., A\theta^{k-1}, A\theta^k, ..., A\theta^{s_0-k-1},$$

$$E_1, E_2, ..., E_k, A, A\theta, A\theta^2, ..., A\theta^{k-1}, A\theta^k, ..., A\theta^{s_0-k-1}, ...$$

is successively orthogonal.

COROLLARY 1. *In the theorem 2 the k-operation $A\theta^{s_0-k-1}$ is k-invertible.*

In [2] for a function f: $Q^k \to Q$ it was defined a complete k-recursive code $K(n/f^{(0)}, f^{(1)}, ..., f^{(n-k-1)})$ with the check functions: $f^{(0)} = f, f^{(1)}, ..., f^{(n-k-1)}$. The function $f^{(i)}$ is called *the i-th recursive derivative* of a function f and is defined recursively as follows:

$f^{(0)}(x_1^k) = f(x_1^k),\ f^{(1)}(x_1^k) = f(x_2^k, f^{(0)}(x_1^k)), ...,$
$f^{(i)}(x_1^k) = f(x_{i+1}^k, f^{(0)}(x_1^k), ..., f^{i-1}(x_1^k))$ for $i < k$, and
$f^{(i)}(x_1^k) = f(f^{(i-k)}(x_1^k), f^{(i-k+1)}(x_1^k), ..., f^{(i-1)}(x_1^k))$ for $i \geq k$.

V. Izbash and P. Syrbu in [3, Proposition 2] proved that if a k-operation f is a k-quasigroup, then $f^{(i)} = f\theta^i, i = 1, 2, ...,$ where

$$\theta : Q^k \to Q^k, \theta(x_1^k) = (x_2, x_3, ..., x_k, f(x_1^k))$$

for all $(x_1^k) \in Q^k$.

A k-quasigroup operation f $(k \geq 2)$ is called *recursively r-differentiable* if all its k-recursive derivatives $f^{(0)}, f^{(1)}, ..., f^{(r)}$ are k-quasigroups [2].

A k-quasigroup we call *strongly recursively r-differentiable* if it is recursively r-differentiable and $r = s_0 - k - 1$, where s_0 is the order of the permutation $\theta = (E_2^k, A)$. In this case $A^{(r+1)} = E_1$. For the binary case this notion was introduced in [4].

From Theorem 2 we obtain the following corollary for any 1-invertible k-function f.

COROLLARY 2. *If f is an 1-invertible k-function, then*
$f^{(i)} = f\theta^{(i)}, i = 1, 2, ...,$ where $\theta = (E_2^k, f)$.
The sequence of the recursive derivatives has the form
$E_1, E_2, ..., E_k, f, f\theta, f\theta^2, ..., f\theta^{k-1}, f\theta^k, ..., f\theta^{s_0-k-1},$
$E_1, E_2, ..., E_k, f, f\theta, f\theta^2, ..., f\theta^{k-1}, f\theta^k, ..., f\theta^{s_0-k-1}, ...,$
where s_0 is the order of the permutation θ.
If f is an r-differentiable k-quasigroup, then $r \leq s_0 - k - 1$.
If a k-quasigroup is strongly recursively r-differentiable, then $r = s_0 - k - 1$.
For an 1-differentiable k-quasigroup $s_0 \geq k + 2$.

THEOREM 3. *Let a permutation (E_2^k, A) have the order s_0, then a successively orthogonal system of Theorem 2 contains s_0 different k-operations.*
If $s_0 = k + 1$, then the k-operation A is a quasigroup k-operation.
For any 1-invertible k-operation $s_0 \geq k + 1$.

THEOREM 4. *Let $A, A_1, ..., A_t, ...$ be 1-invertible k-operations and the permutations $\theta = (E_2^k, A)$, $\theta_1 = (E_2^k, A_1)$,..., $\theta_t = (E_2^k, A_t)$,... have the order $s_0, s_1, ..., s_t, ...$ respectively, then the system*

$$E_1, E_2, ..., E_k, A, A\theta, A\theta^2, ..., A\theta^{k-1}, A\theta^k, ..., A\theta^{s_0-k-1},$$

$$E_1, E_2, ..., E_k, A_1, A_1\theta_1, A_1\theta_1^2, ..., A_1\theta_1^{k-1}, A_1\theta_1^k, ..., A_1\theta_1^{s_1-k-1}, ...,$$

$$E_1, E_2, ..., E_k, A_t, A_t\theta_t, A_t\theta_t^2, ..., A_t\theta_t^{k-1}, A_t\theta_t^k, ..., A_t\theta_t^{s_t-k-1}, ...$$

is a SOS.

PROPOSITION 2. *Any orthogonal set of k-operations can be continued to a SOS.*

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ON DETERMINATION OF ABELIAN GROUPS OF SOME CLASSES BY THEIR ENDOMORPHISM RINGS

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Abelian group epimorphisms, which nearly preserve some properties of the group classes, are of considerable interest. The reason for that is the opportunity to extend the known results to the new classes of group epimorphic images. In this sense the classification and non-isomorphic direct decomposition constructions deserve to be generalized from the class of almost completely decomposable groups to some groups of countable ranks and their special epimorphic images.

We consider class $\mathcal{C}^0$ of *block-rigid local almost completely decomposable groups with generally cyclic regulator quotient* of countable rank.

Definition 1. *A torsion-free abelian group X belongs to the class $\mathcal{C}^0$, if it contains a block-rigid completely decomposable subgroup $R(X)$ of countable rank such that its canonical decomposition $R(X) = \bigoplus\limits_{\tau\in T_{cr}(R(X))} A_\tau$ satisfies the following conditions:*

1. *A_τ is a finite rank pure subgroup of X for each $\tau \in T_{cr}(R(X))$;*
2. *$X/R(X) = \bigoplus\limits_{p\in P_X} T_p^X$ for some set P_X of primes and cyclic p-primary groups T_p^X with $\exp(T_p^X) = p^{n_p(X)}$;*
3. *for every $p \in P_X$ the set $\{q \in P_X : [T_p^X] \cap [T_q^X] \neq \emptyset\}$ is finite; here $[T_p^X]$ coincides with the minimal subset $\mathfrak{T}_p \subset T_{cr}(R(X))$, such that*
$$T_p^X \subseteq ((\bigoplus_{\tau\in\mathfrak{T}_p} C_\tau)_*^X + R(X))/R(X).$$

From now on, $V_*^X = \{g \in X : \text{ there is } n \in \mathbb{N}, \text{with } ng \in V\}$ denotes the *purification* of V in X, $(W)_p$ is a p-primary component of torsion group W, direct sum of l summands isomorphic to A is denoted by A^l.

Definition 2. *([1] - [3]). Let X and Y be torsion-free abelian groups. Then X and Y are said to be* **nearly isomorphic**, *$X \cong_{nr} Y$, if for any prime number p there exist monomorphisms $\Phi_p : X \to Y$ and $\Psi_p : Y \to X$ such that*

1. *$Y/X\Phi_p$ and $X/Y\Psi_p$ are torsion groups;*
2. *$(Y/X\Phi_p)_p = 0 = (X/Y\Psi_p)_p$;*
3. *for any finite rank pure subgroups $X' \subseteq X$ and $Y' \subseteq Y$ the quotients $(X'\Phi_p)_*^Y/X'\Phi_p$ and $(Y'\Psi_p)_*^X/Y'\Psi_p$ are finite groups.*

Theorem 1. *([1]). Let $X, Y \in \mathcal{C}^0$. Then groups X and Y are nearly isomorphic if and only if their endomorphism rings are isomorphic,* $\mathrm{End}(X) \cong \mathrm{End}(Y)$.

Let us consider a class of groups, which are special epimorphic images of groups from $\mathcal{C}^0$:

Definition 3. *([2, Definition 5.1]).*
Let X be a group from the class $\mathcal{C}^0$ and $(\alpha_\tau : \tau \in \mathfrak{T})$ be integers such that the following conditions hold:

(1.) *$R(X) = \bigoplus_{i\in\omega} A_i^{l_i}$ ($l_i \in \mathbb{N}$) with $\mathfrak{T} = \bigcup_{i\in\omega} T_i$ the disjoint union of finite subsets $T_{cr}(A_i) = T_i \subset \mathfrak{T}$ and $\mathrm{rk} A_i = |T_i| \neq 2$ for any $i \in \omega$;*

(2.) *for any $i \in \omega$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$, $\sigma \in T_i$;*

(3.) *if* $i \in \omega$ *and* $|T_i| \neq 1$ *then* $\bigcap_{\tau \neq \sigma, \tau \in T_i} \tau = \mathbb{Z}$ *for any* $\sigma \in T_i$*;*

(4.) *if* $i \in \omega$ *and* $|T_i| \neq 1$ *then each* α_τ $(\tau \in T_i)$ *is not p-divisible if* $\sigma(p) = \infty$ *for some* $\sigma \in T_i$*; if* $|T_i| = 1$ *then* $\alpha_\tau = 0$ *with* $T_i = \{\tau\}$*;*

(5.) *if* $i \in \omega$ *and* $|T_i| \neq 1$ *then* $\gcd(\{\alpha_\tau | \, \tau \neq \sigma, \tau \in T_i\}) = 1$ *for any* $\sigma \in T_i$*;*

(6.) *if there exists* $i \in \omega$ *with* $\tau \in T_i$ *and* $\sigma \in T_i$ *then* $\gcd(m_\tau(X), m_\sigma(X)) = 1$*;*

(7.) *if there exists* $i \in \omega$ *with* $\tau \in T_i$ *and* $\sigma \in T_i$ *then* α_τ *is relatively prime to* $m_\sigma(X)$*;*

(8.) *if there exists* i *with* $\tau \in T_i$ *and* $\sigma \in T_i$ *then* $\tau(p) \neq \infty$ *for any prime divisor* p *of* $m_\sigma(X)$*.*

Let $K = \bigoplus_{i \in \omega} K^{l_i}(A_i) \subset X$ with $K(A_i) = \langle \sum_{\tau \in T_i} \alpha_\tau a_\tau \rangle \subset A_i = \bigoplus_{\tau \in T_i} \tau a_\tau$. Then $B = X/K$ will be called a **proper** $\mathfrak{B}^{(1)}$**alr-group**.

The combinatorial (graphical) theory constructed in [3] for the so-called "almost rigid groups" from class $\mathcal{C}^0$ describes their direct decompositions up to near isomorphism and serves as the basis of an analogous theory for the class of proper $\mathfrak{B}^{(1)}$alr-groups, see [2, Theorem 5.13]. However, endomorphism rings of the latter are so poor that there is no possibility of determining the groups themselves. Analyzing these facts we come to the combinatorial non-isomorphic direct decomposition theory of torsion-free abelian groups with relatively simple endomorphism ring structures. Note that these endomorphism rings also admit non-isomorphic decompositions as their properties are tightly connected with those of groups themselves.

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UDC 512.5

ON SUBVARIETY OF VARIETY GENERATED BY A SIMPLE INFINITE LIE ALGEBRA OF CARTAN TYPE GENERAL SERIES W_2

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The necessary background was stated in monographs [1], [2]. Characteristic of main field Φ is equal to zero.

Let $R_k = \Phi[t_1, t_2, \ldots, t_k]$ be the polynomial ring in variables $t_1, t_2, \ldots, t_k$. Denote by W_k simple infinite Lie algebra of Cartan type general series, where $k = 1, 2, \ldots$. Recall that W_k include first order differential operator of the form $\sum_{i=1}^{k} f_i \partial_i$, where ∂_i is operator of taking the partial derivative with respect to t_i, and $f_i \in R_k$, $i = 1, \ldots, k$. Let $\mathbf{W_k}$ denote variety generated by the corresponding algebra. It is well known that exponent of variety $\mathbf{W_1}$ is equal to 4. At the same time, variety $\mathbf{W_2}$ has fractional exponent. This result was proved in [3]:

$$13,1 < LEXP(\mathbf{W_2}) \leqslant HEXP(\mathbf{W_2}) < 13,5.$$

In paper [4] was constructed an infinite series of Lie algebras L_s, where $s = 3, 4, \ldots$, with different fractional exponents its codimensions growth. More precisely: For the varieties of Lie algebras L_s, $s = 3, 4, \ldots$, over field of zero characteristic the following strict inequalities hold

$$3 = EXP(L_3) < \ldots < EXP(L_s) < EXP(L_{s+1}) < \ldots < 4\,, \text{ where } s = 4, 5, \ldots\ .$$

Lie algebras L_s, where $s = 3, 4, \ldots$, do not belong to the variety $\mathbf{W_1}$, because standard Lie identity of five degree is not satisfied in L_s, but, as well known, it is holds in W_1. Appeared that considered series of algebras belong to variety $\mathbf{W_2}$.

Theorem 1. *Discrete series of Lie algebras L_s with different fractional exponents codimension growth belongs the variety generated by a simple infinite Lie algebra of Cartan type general series W_2.*

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ON ALGEBRAS OF RELATIONS WITH OPERATION OF DOUBLE CYLINDROFICATION

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A set of binary relation Φ closed with respect to some collection Ω of operations on relations forms an algebra (Φ, Ω) called *an algebra of relations*. Each such algebra can be considered to be partially ordered $(\Phi, \Omega, \subset)$ by the relation of set-theoretic inclusion $\subset$.

Tarski A. was the first to treat algebras of relations from the point of view of universal algebra [1]. He considered algebras of relations with the following operations: Boolean operations $\cup, \cap, ^{-}$, operations of relation product $\circ$, relation inverse $^{-1}$, and constant operations: $\emptyset$ (empty set), Δ (diagonal relation), $U \times U$ (universal relation). Now, the theory of algebras of relations is an essential part of algebraic logic [2] and modern algebra [3].

For any set Ω of operations on binary relations, let $R\{\Omega\}$ $(R\{\Omega, \subset\})$ denote the class of all algebras (partially ordered algebras) isomorphic to ones whose elements are binary relations and whose operations are members of Ω. Let $Q\{\Omega\}$ $(Q\{\Omega, \subset\})$ be the quasivariety and $Var\{\Omega\}$ $(Var\{\Omega, \subset\})$ be the variety generated by $R\{\Omega\}$ $(R\{\Omega, \subset\})$.

In the investigation of algebras of relations, the following problems naturally arise:

1. *Find a basis of identities for the variety $Var\{\Omega\}$. Find out whether this variety is finitely based.*

2. *Find a basis of quasiidentities for the quasivariety $Q\{\Omega\}$. Find out whether this quasivariety is finitely based. Find out whether this quasivariety forms a variety.*

3. *Find a system of elementary axioms (abstract characterization) for the class $R\{\Omega\}$. Find out whether this class is finitely axiomatizable. Find out whether this class forms a quasivariety (variety).*

Similar problems can be formulated for partially ordered algebras.

As a rule, operations on relations are defined by formulas. These operations are called *logical*. One of the most important classes of logical operations on relations is the class of Diophantine operations [4,5] (in other terminology – primitive-positive operations [6]). An operation on relations is called *Diophantine* if it can be defined by a first order formula containing in its prenex normal form only existential quantifiers and conjunctions. It is easy to see that the operation of relation product $\circ$, relation inverse $^{-1}$, and intersection $\cap$ are Diophantine operations. Equational and quasiequational theories of algebras of relations with Diophantine operations are described in [4, 5, 7].

A Diophantine operation is called atomic if it can be defined by a first order formula containing in its prenex normal form only existential quantifiers. It is clear that such formulas contain only one atomic subformula. Hence atomic operations are unary operations. There exist nine atomic operations (excepting identical).

We concentrate our attention on the Diophantine operation of relation product $\circ$ and on the atomic operation of double cylindrification ∇ that are defined as follows. For any relations ρ and σ on U, put

$$\rho \circ \sigma = \{(u, v) : (\exists w)(u, w) \in \rho(w, v) \in \sigma\},$$
$$\nabla(\rho) = \{(u, v) : (\exists w, z)(w, z) \in \rho\}.$$

Note that $\nabla(\emptyset) = \emptyset$, and $\nabla(\rho) = U \times U$ if $\rho \neq \emptyset$.

The main results are formulated in the following theorems.

Theorem 1. *An algebra $(A, \cdot, ^*)$ of the type $(2, 1)$ belongs to the variety $Var\{\circ, \nabla\}$ if and only if it satisfies the identities:*

$$(xy)z = x(yz), \quad x^{**} = x^*, \quad (x^*)^2 = x^*, \ \ x^*y^* = y^*x^*,$$
$$x^*(xy)^* = (xy)^*y^* = (xy)^*, \quad (xy^*z)^* = x^*y^*z^* = x^*yz^*,$$
$$xyz^* = xyx^*z^*, \quad x^*yz = x^*z^*yz.$$

Theorem 2. *The quasivariety $Q\{\circ, \nabla, \subset\}$ forms a variety in the class of all partially ordered algebras of the type $(2, 1)$. A partially ordered algebra $(A, \cdot, ^*, \leq)$ of the type $(2, 1)$ belongs to the quasivariety $Q\{\circ, \nabla, \subset\}$ if and only if it satisfies to the conditions of Theorem 1 and the identities:*

$$x \leq xx^*x, \quad xy \leq xx^*, \ \ xy \leq y^*y.$$

Theorem 3. *The class $R\{\circ, \nabla, \subset\}$ does not form a quasivariety. A partially ordered algebra $(A, \cdot, ^*, \leq)$ of the type $(2, 1)$ belongs to the class $R\{\circ, \nabla, \subset\}$ if and only if it satisfies to the conditions of Theorem 2 and the axioms:*

$$x^* = y^* \vee x^*y^*z = zx^*y^* = x^*y^*, \quad x^* = y^* \vee x^*y^* \leq z.$$

Theorem 4. *The quasivariety $Q\{\circ, \nabla, \cap\}$ forms a variety. An algebra $(A, \cdot, ^*, \wedge)$ of the type $(2, 1, 2)$ belongs to the quasivariety $Q\{\circ, \nabla, \cap\}$ if and only if it satisfies to the following conditions:*

a) $(A, \wedge)$ is a semilattice with the canonical partial order relation $\leq$ that is compatible with operations $\cdot$ and *.

b) $(A, \cdot, ^*, \leq)$ satisfies to the conditions of Theorem 2;

c) the following identities hold

$$x^*y^* = x^* \wedge y^*, \quad x(y \wedge z^*z) = xy \wedge z^*z, \quad x(y \wedge zz^*) = xy \wedge zz^*,$$
$$(xx^* \wedge y)z = xx^* \wedge yz, \quad (x^*x \wedge y)z = x^*x \wedge yz.$$

Theorem 5. *The class $R\{\circ,\nabla,\cap\}$ does not form a quasivariety. An algebra $(A,\cdot,{}^*,\cap)$ of the type $(2,1,2)$ belongs to the class $R\{\circ,\nabla,\cap\}$ if and only if it satisfies to the conditions of Theorem 4 and the axioms from the Theorem 3.*

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CORNER BOUNDARY LAYER IN NONLINEAR SINGULARLY PERTURBED ELLIPTIC PROBLEMS CONTAINING DERIVATIVES OF THE FIRST ORDER

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In the rectangular domain the first boundary value problem is considered for a singularly perturbed elliptic equation

$$\varepsilon^2\Delta u-\varepsilon^\alpha A(x,y)\frac{\partial u}{\partial y}=F(u,x,y,\varepsilon)$$

with a nonlinear on u function F. Uniform in a closed rectangle the complete asymptotic expansion of the solution is constructed for $\alpha>1$. If $0<\alpha<1$, then the uniform asymptotic approximation is constructed in the zero and first approximation.

Keywords: boundary layer, singularly perturbed equation, asymptotic expansion.
Bibliography: 8 titles.

In [1–3] in the rectangular domain $\Omega := \{(x,y)\,|\,0 < x < a,\, 0 < y < b\}$ it is studied the linear problem

$$\varepsilon^2 \Delta u - \varepsilon^\alpha A(x,y)\,\frac{\partial u}{\partial y} - k^2(x,y)u = f(x,y,\varepsilon), \quad (x,y) \in \Omega, \tag{1}$$

$$u(x,y,\varepsilon) = \phi(x,y), \quad (x,y) \in \partial\Omega, \tag{2}$$

where $\partial\Omega$ is the boundary of the rectangle Ω, ε is a small positive parameter, Δ is the Laplace operator.

In [1] we considered the case $A(x,y) \equiv 0$ for which it is shown that a complete asymptotic expansion of the solution, uniform in a closed rectangle $\bar{\Omega}$, consists of a regular part, of four boundary parts and four corner boundary layer parts (in accordance with the four sides and four vertices of the rectangle).

In [2] a more complex case $A(x,y) > 0$ in $\bar{\Omega}$ and $\alpha = 0$ is considered for which it is also shown that the asymptotic expansion of the solution consists of regular and boundary layer parts. However, there is a significant factor associated with the construction of the asymptotic behavior of arbitrary order. In particular, we must consider parabolic equations (parabolic boundary layer) to describe the boundary layer in the neighborhood of the vertical sides of the rectangle $x = 0$ and $x = b$. For example, in the neighborhood of the side $x = 0$ boundary layer operator has the form

$$\frac{\partial^2}{\partial \xi^2} - A(0,y)\,\frac{\partial u}{\partial y} - k^2(0,y), \quad \xi = \frac{x}{\varepsilon}\,.$$

The emergence of a parabolic boundary layer is related to the fact that the vertical sides of the rectangle are the characteristics of a degenerate operator

$$-A(x,y)\,\frac{\partial}{\partial y} - k^2(x,y).$$

In determining the parabolic boundary of ε^2 and higher order functions, unlimited members appear in the right-hand sides of equations. To construct a uniform in $\bar{\Omega}$ asymptotic solution it is necessary to impose certain restrictive conditions on the f and ϕ functions.

In [3] a more complicated case $A(x,y) > 0$ in $\bar{\Omega}$ and $\alpha > 0$ is considered. Depending on the order of magnitude of ε^α the boundary layer part of asymptotic is constructed differently. If $\alpha \geq 1$, the boundary layer structure of the solution is the same as in [1]. If $0 < \alpha < 1$, the boundary layer structure of the solution changes significantly. Uniform asymptotic approximation in the rectangle $\bar{\Omega}$ can be obtained only in the zero and first approximation.

Developed in [4–8] method allowed to justify the uniform asymptotic solution of the problem (1), (2) in case contrast to [1] function f is nonlinear on u. It was viewed the equation

$$\varepsilon^2 \Delta u = F(u, x, y, \varepsilon), \quad (x, y) \in \Omega, \tag{3}$$

with the boundary condition (2) and nonlinear $F(u, x, y, \varepsilon)$ function on u. The construction of regular and the boundary layer parts of the asymptotic solution did not cause additional difficulties in comparison with the linear case. However, while constructing a corner boundary layer, we had to consider nonlinear elliptic equations of the same type as (3).

In this paper the methods of [1–8] works are applied to the equation

$$\varepsilon^2 \Delta u - \varepsilon^\alpha A(x, y) \frac{\partial u}{\partial y} = F(u, x, y, \varepsilon) \tag{4}$$

with the boundary condition (2), nonlinear $F(u, x, y, \varepsilon)$ function on u and a rational number $\alpha = \gamma/\beta$. For the problem (4), (2) a solution in the form of a series in powers of $\varepsilon^{1/\beta}$ can be constructed. This solution consists of three parts:

$$u(x, y, \varepsilon) = \bar{u} + \Pi + P,$$

where $\bar{u}$ is a regular part, Π is a boundary functions that play a role near the sides of the Ω rectangle, P is corner boundary functions that play a role near the vertices of the Ω rectangle. For the case $\alpha > 1$, we can construct a complete asymptotic expansion of the solution, uniform in the $\bar{\Omega}$ rectangle. For the case $0 < \alpha < 1$, we can construct a uniform in $\bar{\Omega}$ asymptotic approximation only on the first order. In all cases the qualitative character of the asymptotic behavior is the same as in cases where the $F(u, x, y, \varepsilon)$ function is linear on u.

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THE MOMENTS OF THE PERIODIC ZETA-FUNCTION

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Let $s = \sigma + it$ be a complex variable, and $\lambda \in \mathbb{R}$ be a fixed parameter. The periodic zeta-function $\zeta_\lambda(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere. For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function $\zeta(s)$. Moreover,

$$\zeta_\lambda(s) = e^{2\pi i \lambda} L(\lambda, 1, s),$$

where, for $0 < \alpha \leq 1$,

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \ \sigma > 1,$$

is the Lerch zeta-function. Since the function $e^{2\pi i\lambda m}$ is periodic with minimal period 1, we may assume that $0<\lambda\le 1$.

We consider the mean square and the fourth power moment of the function $\zeta_\lambda(s)$. The first problem deals with Atkinson type formula for

$$E_\sigma(q,T)=\sum_{a=1}^{q}\int_0^T|\zeta_{\frac{a}{q}}(\sigma+it)|^2dt-q\zeta(2\sigma)T-\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma},$$

where a and q are integers, $1\le a\le q$. Let $c_1T<N<c_2T$, where $c_1<c_2$,

$$N_1=N_1(q,T,N)=q\left(\frac{T}{2\pi}+\frac{qN}{2}-\left(\left(\frac{qN}{2}\right)^2+\frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right)$$

and $\sigma_\alpha(m)=\sum_{d|m}d^\alpha$. Define

$$\begin{aligned}\sum\nolimits_1(q,T)&=2^{\sigma-1}q^{1-\sigma}\left(\frac{T}{\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N}\frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\\&\times\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}}\\&\times\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)+\sqrt{2\pi qmT+T^2q^2m^2}-\frac{\pi}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\sum\nolimits_2(q,T)&=-2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N_1}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1}\\&\times\cos\left(T\log\left(\frac{qT}{2\pi m}\right)-T+\frac{\pi}{4}\right).\end{aligned}$$

THEOREM 1. *Suppose that σ, $\frac{1}{2}<\sigma<\frac{3}{4}$, is fixed. Then, for $q\le T$,*

$$E_\sigma(q,T)=\sum\nolimits_1(q,T)+\sum\nolimits_2(q,T)+R(q,T),$$

where $R(q,T)=O(q^{\frac{7}{4}-\sigma}\log T)$, with the O - constant depending only on σ.

Next we study the mean square of $E_\sigma(q,T)$.

Theorem 2. *Suppose that σ, $\frac{1}{2} < \sigma < \frac{3}{4}$, is fixed. Then, for $T \to \infty$ and $q \le T^{1-\frac{4\sigma}{3}}$,*

$$\int_2^T E_\sigma^2(q,t)dt = 2(5-4\sigma)^{-1}(2\pi)^{2\sigma-\frac{3}{2}}q^{\frac{3}{2}-2\sigma}T^{\frac{5}{2}-2\sigma}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}$$

$$+ O(q^{\frac{11}{4}-2\sigma}T^{\frac{7}{4}-\sigma}\log T).$$

If $q=1$, then Theorems 1 and 2 give the results of [2].

An analogue of the Theorem 2 is also true for $\sigma = \frac{1}{2}$. Let

$$E(q,T) = \sum_{a=1}^{q}\int_0^T \left|\zeta_{\frac{a}{q}}\left(\frac{1}{2}+it\right)\right|^2 dt - qT\left(\log\frac{qT}{2\pi} - 2\gamma_0 - 1\right),$$

where γ_0 is the Euler constant, and $d(m) = \sum_{d|m} 1$.

Theorem 3. *For $T \to \infty$ and $q \le T^{\frac{1}{3}}$,*

$$\int_2^T E^2(q,t)dt = \frac{2\sqrt{q}T^{\frac{3}{2}}}{3\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}} + O(T^{\frac{5}{4}}q^{\frac{7}{4}}\log^4 T).$$

If $q=1$, then Theorem 3 contains the results of [1].

Now we will consider the fourth power moment of $\zeta_\lambda(s)$ in the critical strip. We have the following results.

Theorem 4. *Suppose that λ is irrational, $0<\lambda<1$, $\frac{1}{2}<\sigma<1$ and $T\to\infty$. Then, for every $\varepsilon>0$,*

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt$$
$$= T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}\right) + O(T^{\frac{3}{2}-\sigma+\varepsilon}).$$

The case of rational λ is more complicated, and we have only the following analogue of Theorem 4.

Theorem 5. *Suppose that the number λ is rational, $0<\lambda<1$, $\frac{3}{4}<\sigma<1$ and $T\to\infty$. Then, for every $\varepsilon>0$,*

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt$$
$$= T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}\right) + O(T^{\frac{7}{4}-\sigma+\varepsilon}).$$

Theorems 4 and 5 are generalizations of the known results for the Riemann zeta-function.

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UDC 511.34

ON POLYADIC NUMBERS AND POLYADIC EXPANSIONS

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The ring of polyadic integers is a direct product of the rings of p-adic integers over all primes p. The elements of it have canonical representation of the form

$$\sum_{n=1}^{\infty} a_n \cdot n!,\ 0 \leqslant a_n \leqslant n.$$

The ring of polyadic integers was introduced by H. Prufer and was investigated by A. G. Postnikov and E. V. Novoselov. The author considered E. Bombieri's notion of a global relation and studied the arithmetic properties of the series analogous to the given above.

The report presents the results of the author and some other mathematicians on the properties of such series and the properties of the so called polyadic expansions.

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GEOMETRIZATION OF GENERALISED FIBONACCI NUMERATION SYSTEM AND ITS NUMBER-THEORETIC APPLICATIONS[1]

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Define a sequence $\left\{F_i^{(g)}\right\}$ by the recurrence relation

$$F_{i+2}^{(g)} = gF_{i+1}^{(g)} + F_i^{(g)},$$

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where $i \geqslant 0$. The initial conditions are $F_0^{(g)} = 1$, $F_1^{(g)} = g$ for $g = 1, 2, 3, \ldots$. This sequence can be considered as a generalised Fibonacci sequence.

Any nonnegative integer n can be represented as a sum of different numbers from this sequence

$$n = \sum_{i=0}^{k} \varepsilon_i(n) F_i^{(g)},$$

where $\varepsilon_0(n)$ can equals $0, 1, \ldots, g'$. Here $g' = g-1$ if $i = 0$ and $g' = g$ if $i \geq 1$. Besides, for $0 \leqslant i \leqslant k-1$ from $\varepsilon_{i+1}(n) = g$ follows that $\varepsilon_i(n) = 0$. Thus representation is called the representation of n in generalised Fibonacci numeration system.

A tuple $(\varepsilon_0, \ldots, \varepsilon_l)$ such that $0 \leq \varepsilon_i \leq g'$ $\varepsilon_i = 0$ if $\varepsilon_{i+1} = g$ and $0 \leqslant i \leqslant l-1$, is called g-admissible. For any g-admissible tuple $(\varepsilon_0, \ldots, \varepsilon_l)$ consider a set

$$\mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l) = \{n \in \mathbb{Z} : n \geq 0, \varepsilon_0(n) = \varepsilon_0, \ldots, \varepsilon_l(n) = \varepsilon_l\}.$$

Sets $\mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l)$ consist of nonnegative integers with fixed $l+1$ last digits of their representation in generalised Fibonacci numeration system.

Sets $\mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l)$ are special cases of so called quasilattices. Recently there some woks devoted to number-theretic problems over quasylattices [3], [4].

Particularly, in [2] solutions of linear additive problem, Lagrange four squares problem, and ternary Goldbach problem over $\mathbb{F}(\varepsilon_1, \ldots, \varepsilon_l)$, are obtained.

Consider the function $\chi(n)$ defined by

$$\chi(n) = \{(n+1)\tau_g\},$$

where $\{x\}$ is a fractional part of x, and $\tau_g = \frac{\sqrt{g^2+4}-g}{2}$. Consider the set

$$X(\varepsilon_0, \ldots, \varepsilon_l) = \overline{\{\chi(n) : n \in \mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l)\}}.$$

The following geometrization theorem is holds.

Theorem 1. *For any g-admissible touple $(\varepsilon_0, \ldots, \varepsilon_l)$ the set $X(\varepsilon_0, \ldots, \varepsilon_l)$ is a segment $[\{-a\tau\}; \{-b\tau\}]$, where a, b are effectively determined integers with $0 \leqslant a, b < F_l^{(g)} + F_{l+1}^{(g)}$.*

As an application of the geometrization theorem we solve analogues of some classic problems of number theory over sets $\mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l)$.

Assume that

$$A_N(\varepsilon_0, \ldots, \varepsilon_l) = \sharp\{n : 1 \leqslant n \leqslant N, n \in \mathbb{F}^{(g)}(\varepsilon_0, \ldots, \varepsilon_l)\}.$$

In other words $A_N(\varepsilon_0, \ldots, \varepsilon_l)$ is a number of positive integers with fixed $l+1$ last digits of their representation in generalised Fibonacci numeration system.

THEOREM 2. *For any positive integer N we have*

$$A_N(\varepsilon_0,\ldots,\varepsilon_l)=\rho(\varepsilon_0,\ldots,\varepsilon_l)N+O(1),$$

where

$$\rho(\varepsilon_0,\ldots,\varepsilon_l)=\begin{cases}\tau^l, & \text{if} \ \ \varepsilon_l=0,\\ \tau^{l+1}+\tau^l, & \text{if} \ \ \varepsilon_l=1.\end{cases}$$

Denote by $A_N^{s,q}(\varepsilon_0,\ldots,\varepsilon_l)$ a number of positive integers, that less or equal N, belong to $\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l)$, and members of the arythmetic progression $st+q$, with $s,t\in\mathbb{N}$, $q\in\mathbb{Z}$, $0\leqslant q\leqslant s-1$, i.e.

$$A_N^{s,q}(\varepsilon_0,\ldots,\varepsilon_l)=\sharp\{n:1\leqslant n\leqslant N, n\in\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l), n=st+q\}.$$

THEOREM 3. *For fixed s and any natural N we have an asymptotic formula*

$$A_N^{s,q}(\varepsilon_0,\ldots,\varepsilon_l)=\rho(\varepsilon_0,\ldots,\varepsilon_l)\frac{N}{s}+O\left(\ln\frac{N}{s}\right).$$

Let $\pi(\varepsilon_0,\ldots,\varepsilon_l;n)$be a number of primes not exceeding n from $\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l)$, and $\pi(n)$ be a number of primes not exceeding n.

THEOREM 4. *The set $\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l)$ contains infinitely many prime numbers. Moreover*

$$\pi(\varepsilon_0,\ldots,\varepsilon_l;n)=\rho(\varepsilon_0,\ldots,\varepsilon_l)\pi(n)(1+o(1)).$$

Denote by $s_m(\varepsilon_0,\ldots,\varepsilon_l;n)$ a number of solutions of the equation

$$n_1+n_2+\ldots+n_m=n,$$

where $n_i\in\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l)$, $i=1,2,\ldots,m$.

THEOREM 5.

$$s_m(\varepsilon_0,\ldots,\varepsilon_l;n)=c_m\left(\varepsilon_0,\ldots,\varepsilon_l,\{n\tau\}\right)n^{m-1}+O\left(n^{m-2}\ln n\right),$$

where $c_m\left(\varepsilon_0,\ldots,\varepsilon_l,\delta\right)$ is some effectively computable continuous function that are piecewise polynom of degree $m-1$ in δ.

Ternary Goldbach problem asks to solve an equation

$$p_1+p_2+p_3=n,$$

where n is an odd natural number and p_1, p_2, p_3 are prime numbers.

Let $\nu(\varepsilon_0,\ldots,\varepsilon_l;n)$ be a number of solutionsof this equation with supplimentary condition $p_i\in\mathbb{F}^{(g)}(\varepsilon_0,\ldots,\varepsilon_l)$, $i=1,2,3$.

THEOREM 6.

$$\nu(\varepsilon_0,\dots,\varepsilon_l;n) = Q_{3,1}(n)\sigma(n,a,b) + O\left(n^2\ln^C n\right),$$

where

$$Q_{3,1}(n) = \frac{n^2}{2\ln^3 n}\prod_p\left(1+\frac{1}{(p-1)^3}\right)\prod_{p|n}\left(1-\frac{1}{p^2-3p+3}\right),$$

$$\sigma(n,a,b) = \sum_{|m|<\infty} e^{2\pi i m(\tau n - 1{,}5(a+b))}\frac{\sin^3\pi m(b-a)}{\pi^3 m^3},$$

and a, b are some effectively computable numbers from $\mathbb{Z}[\tau]$.

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THE ADDITIVE GROUP OF A UNIVERSAL LIE NILPOTENT ASSOCIATIVE RING OF CLASS 3

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This talk is based on the paper [5] and on the preprint [2].

Let $\mathbb{Z}\langle X\rangle$ be the free unital associative ring freely generated by an infinite countable set $X = \{x_i \mid i \in \mathbb{N}\}$. We define a left-normed commutator $[x_1, x_2, \dots, x_n]$ by $[a,b] = ab - ba$, $[a,b,c] = [[a,b],c]$. For $n \geqslant 2$, let $T^{(n)}$ be the two-sided ideal in $\mathbb{Z}\langle X\rangle$ generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in \mathbb{Z}\langle X\rangle$). Note that the quotient ring $\mathbb{Z}\langle X\rangle/T^{(n)}$ is the universal Lie nilpotent associative ring of class $n-1$ generated by X.

It is clear that the quotient ring $\mathbb{Z}\langle X\rangle/T^{(2)}$ is isomorphic to the ring $\mathbb{Z}[X]$ of commutative polynomials in $x_1, x_2, \ldots$. Hence, the additive group of $\mathbb{Z}\langle X\rangle/T^{(2)}$ is free abelian and its basis is formed by the (commutative) monomials. Recently Bhupatiraju, Etingof, Jordan, Kuszmaul and Li [1] have proved that the additive group of $\mathbb{Z}\langle X\rangle/T^{(3)}$ is also free abelian and found explicitly its basis [1, Prop. 3.2].

The aim of the present talk is to describe the additive group A of the ring $\mathbb{Z}\langle X\rangle/T^{(4)}$. Our principal results are as folows.

Theorem 1. *Let A be the additive group of $\mathbb{Z}\langle X\rangle/T^{(4)}$. Then $A = B \oplus C$ where B is a free abelian group and C is an elementary abelian 3-group.*

More precisely, let $T^{(3,2)}$ be the two-sided ideal of the ring $\mathbb{Z}\langle X\rangle$ generated by all elements $[a_1, a_2, a_3, a_4]$ and $[a_1, a_2][a_3, a_4, a_5]$ where $a_i \in \mathbb{Z}\langle X\rangle$. Clearly, $T^{(4)} \subset T^{(3,2)}$.

Theorem 2. *The additive group of $T^{(3,2)}/T^{(4)}$ is an elementary abelian 3-group and the additive group of the quotient $\mathbb{Z}\langle X\rangle/T^{(3,2)}$ is free abelian.*

It follows from Theorems 1 and 2 that the additive group of $\mathbb{Z}\langle X\rangle/T^{(4)}$ is a direct sum $B \oplus C$ where $C = T^{(3,2)}/T^{(4)}$ is an elementary abelian 3-group and B is a free abelian group isomorphic to the additive group of $\mathbb{Z}\langle X\rangle/T^{(3,2)}$.

We describe explicitly a $\mathbb{Z}/3\,\mathbb{Z}$-basis of C. Let

$$\mathcal{E} = \Big\{ x_{j_1} \ldots x_{j_l} [x_{i_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}][x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}] \mid \\ l \geqslant 0,\ k \geqslant 1,\ j_1 \leqslant j_2 \leqslant \ldots \leqslant j_l;\ i_1 < i_2 < \cdots < i_{2k+3} \Big\}.$$

Theorem 3. *The set $\{e + T^{(4)} \mid e \in \mathcal{E}\}$ is a basis of the elementary abelian 3-group $C = T^{(3,2)}/T^{(4)}$ over $\mathbb{Z}/3\,\mathbb{Z}$.*

Note that a description of a $\mathbb{Z}$-basis of the free abelian group $B \simeq \mathbb{Z}\langle X\rangle/T^{(3,2)}$ can be deduced from the results of either [3] or [4] or [6] or [7]; this basis is explicitly written in [5, Lemma 5.6].

Let

$$\mathcal{D}'_0 = \{1\}, \quad \mathcal{D}'_1 = \{[x_{i_1}, x_{i_2}] \mid i_1 < i_2\}, \quad \mathcal{D}'_2 = \{[x_{i_1}, x_{i_2}, x_{i_3}] \mid i_1 < i_2, i_1 \leqslant i_3\},$$
$$\mathcal{D}'_3 = \{[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \mid i_1 < i_2, i_3 < i_4, i_1 \leqslant i_3;\ \text{if } i_1 = i_3 \text{ then } i_2 \leqslant i_4\},$$
$$\mathcal{D}'_4 = \{[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}] \mid k \geqslant 3, i_1 < i_2 < \cdots < i_{2k}\}.$$

Let $\mathcal{D}' = \mathcal{D}'_0 \cup \mathcal{D}'_1 \cup \mathcal{D}'_2 \cup \mathcal{D}'_3 \cup \mathcal{D}'_4$. Let

$$\mathcal{D} = \{x_{i_1} x_{i_2} \ldots x_{i_k} d' \mid k \geqslant 0, i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k, d' \in \mathcal{D}'\}.$$

Theorem 4 (see [3, 4, 6, 7]). *The set $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$ is a basis of $\mathbb{Z}\langle X\rangle/T^{(3,2)} \simeq B$ over $\mathbb{Z}$.*

In the proof we use the following result that may be of independent interest.

Theorem 5. *Let K be an arbitrary unital associative and commutative ring and let $K\langle Y\rangle$ be the free associative K-algebra on a non-empty set Y of free generators. Let $T^{(4)}$ be the two-sided ideal in $K\langle Y\rangle$ generated by all commutators $[a_1, a_2, a_3, a_4]$ ($a_i \in K\langle Y\rangle$). Then the ideal $T^{(4)}$ is generated by the polynomials*

$$[y_1, y_2, y_3, y_4] \qquad (y_i \in Y), \tag{1}$$

$$[y_1, y_2, y_3][y_4, y_5, y_6] \qquad (y_i \in Y), \tag{2}$$

$$[y_1, y_2][y_3, y_4, y_5] + [y_1, y_5][y_3, y_4, y_2] \qquad (y_i \in Y), \tag{3}$$

$$[y_1, y_2][y_3, y_4, y_5] + [y_1, y_4][y_3, y_2, y_5] \qquad (y_i \in Y), \tag{4}$$

$$([y_1, y_2][y_3, y_4] + [y_1, y_3][y_2, y_4])[y_5, y_6] \qquad (y_i \in Y). \tag{5}$$

Note that $3\,[y_1, y_2][y_3, y_4, y_5] \in T^{(4)}$ for all $y_i \in Y$; one can deduce this from the results of [3, 4, 7], see also [5, Corollary 2.4]. Hence, if $\frac{1}{3} \in K$ then all polynomials

$$[y_1, y_2][y_3, y_4, y_5] \qquad (y_i \in Y) \tag{6}$$

belong to $T^{(4)}$. Since the polynomials (2)–(4) belong to the ideal generated by the polynomials (6), Theorem 5 implies the following corollary that has been proved in [3, 7].

Theorem 6 (see [3, 7]). *If $\frac{1}{3} \in K$ then $T^{(4)}$ is generated as a two-sided ideal of $K\langle Y\rangle$ by the polynomials (1), (5) and (6).*

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ON SOME ISSUES OF NUMBER-THEORETIC METHOD IN APPROXIMATE ANALYSIS [1]

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In the papers [5]–[7] some directions for the future research in Korobov's number-theoretic method in approximate analysis were given. In this paper, we will discuss unsolved problems of the theory of the lattice hyperbolic zeta function. This function is defined in the right semiplane $\alpha > 1$ by the zeta-series[2]

$$\zeta(\Lambda|\alpha) = {\sum_{\vec{x}\in\Lambda}}' (\overline{x_1}\ldots\overline{x_s})^{-\alpha}.$$

Problem of the right order. As is known ([4]), the lattice hyperbolic zeta function has the right order of decreasing on the class of algebraic lattices as a determinant of lattice grows. Moreover, there is an asymptotic formula for these lattices. From continuity of the hyperbolic zeta function on the lattice space follows that the right order of decreasing of the lattice hyperbolic zeta function is achievable on the class of rational lattices. Indeed, it is sufficient to consider rational lattices from small neighborhoods of algebraic lattices. A natural question arises: is the right order of decreasing achievable on integer lattices or not? If it is achievable then we must provide an algorithm for construction of optimal parallelepipedal nets such that the right order of approximate integration error on the classes E_s^α is achieved on them. In other words, in this case we must construct an algorithm for calculation of a module N and optimal coefficients modulo N such that $\zeta(\Lambda(1, a_1, \ldots, a_{s-1}; N)|\alpha) = O\left(\frac{\ln^{s-1} N}{N^\alpha}\right)$, $(\alpha > 1)$. If the right order of decreasing is not achievable on integer lattices, then for algebraic lattices we obtain some type of analog of the Liouville—Thue—Sigel—Roth theorem ([10]), because this means that we cannot approximate algebraic lattices by integer lattices.

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[2]$\sum'$ means that $\vec{x} = \vec{0}$ is excluded from summation

Analytic continuation existence problem. As is shown in the papers [6], [7] there is an analytic continuation of the hyperbolic zeta function of any Cartesian lattice. Moreover, for any Cartesian lattice a functional equation for the explicit form of such an analytic continuation was obtained ([6]). Natural questions arise about existence of an analytic continuation of the hyperbolic zeta function or an explicit form of such a continuation in the following cases:

for S. M. Voronin lattices $\Lambda(F,q)$, where F is an arbitrary algebraic field of order s over the field of rational numbers $\mathbb{Q}$, q is a prime number, and an integer lattice $\Lambda(F,q)$ corresponds to an ideal $\mathfrak{L} \subset \mathbb{Z}_F$ with norm $N(\mathfrak{L}) = q$, if the fundamental lattice $\mathbb{Z}^s$ corresponds to the ring $\mathbb{Z}_F$ of the integer algebraic numbers of the field F. S. M. Voronin and his disciple N. Temirgaliev considered the case of the ring of integer Gaussian numbers and the case of circular fields (see the papers [1], [2], [3], [9]). This comes from the fact that the quadratic Gaussian number field and circular fields are among the most extensively studied algebraic fields. In particular, for these fields there are theorems about corresponding ideals and about distribution of their norms in arithmetic progressions explicitly given by algebraic fields.

for joint approximation lattice $\Lambda(\theta_1,\dots,\theta_s) = \{(q, q\theta_1 - p_1, \dots, q\theta_s - p_s) \mid q, p_1, \dots, p_s \in \mathbb{Z}\}$, where $\theta_1,\dots,\theta_s$ are arbitrary irrational numbers. Such lattices are important because they are directly linked with the Littlewood problem. It is readily seen that the dual lattice $\Lambda^*(\theta_1,\dots,\theta_s)$ has the form $\Lambda^*(\theta_1,\dots,\theta_s) = \{(q - \theta_1 p_1 - \ldots - \theta_s p_s, p_1, \dots, p_s) \mid q, p_1, \dots, p_s \in \mathbb{Z}\}$. The natural presumption is that the hyperbolic zeta functions of these lattices are linked by some functional equation between values over the left and the right semiplanes.

for algebraic lattice $\Lambda(t,F) = t\Lambda(F)$.

for arbitrary lattice Λ. If the hyperbolic zeta function of any lattice cannot be analytically continued to the whole complex plane (our presumption is that it can be continued), then we must describe a class of all lattices such that their hyperbolic zeta function can be analytically continued to the whole complex plane except the point $\alpha = 1$, where it has a pole of order s.

It appears that the key to solving the analytic continuation problem is further studying of possibility of the passage to the limit for hyperbolic zeta functions of Cartesian lattices in the left semiplane over a convergent sequence of Cartesian lattices. If such a limit always exists, then passing to the limit in the functional equation, we obtain a functional equation for the limit lattice. It appears that obtaining a functional equation only in terms of dual lattices offers the greatest promise since convergence of a lattice sequence is equivalent to convergence of the sequence of corresponding dual lattices. It must be emphasized that we expect the greatest challenge in the case when a limit lattice is not Cartesian and has only one main component. For example, this is the case for all algebraic lattices.

Problem of behavior in the critical strip. N. M. Korobov pointed out the importance of this problem in personal communications. He made an assumption that the analytic continuation of the lattice hyperbolic zeta function from the right semiplane to the critical strip and the analytic continuation of the hyperbolic zeta function of dual lattice or adjoint lattice from the left semiplane to the critical strip will allow to obtain constants in the corresponding transfer theorems.

Problem of lattice exponential sums values. Normalized exponential sums of parallelepipedal nets can take two values: 0 and 1. Normalized exponential sums of two-dimensional Smolyak nets can take three values: 0, 1, and -1 (see the paper [8]). For non-regular nets there is either the good uniform estimate $O\left(\frac{1}{\sqrt{N}}\right)$ or the sum is equal to 1. It is very important to obtain estimates for normalized exponential sums of algebraic lattices. If the spectrum of values of these sums is not concentrated around 0 and 1, then algebraic nets can not be well approximated by parrallelepipedal nets and algebraic lattices can not be well approximated by integer lattices.

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ON NET HYPERBOLIC PARAMETER[1]

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Suppose

$$S_{M,\vec{\rho}}(m_1,\ldots,m_s) = \sum_{k=1}^{N} \rho_k e^{2\pi i[m_1\xi_1(k)+\ldots+m_s\xi_s(k)]}$$

is an exponential sum of weighted net, $S^*_{M,\vec{\rho}}(\vec{m}) = \frac{1}{N} S_{M,\vec{\rho}}(\vec{m})$ is a normed exponential sum of weighted net.

Definition 1. *[1] The zeta function of a net M with weights $\vec{\rho}$ and parameter $p \geq 1$ is the function $\zeta(\alpha, p|M, \vec{\rho})$ such that in the right semiplane $\alpha = \sigma + it$ $(\sigma > 1)$*[2]

$$\zeta(\alpha, p|M, \vec{\rho}) = \sideset{}{'}\sum_{m_1,\ldots,m_s=-\infty}^{\infty} \frac{|S^*_{M,\vec{\rho}}(\vec{m})|^p}{(\overline{m}_1 \ldots \overline{m}_s)^\alpha} = \sum_{n=1}^{\infty} \frac{S^*(p, M, \vec{\rho}, n)}{n^\alpha}, \tag{1}$$

where

$$S^*(p, M, \vec{\rho}, n) = \sum_{\vec{m} \in N(n)} |S^*_{M,\vec{\rho}}(\vec{m})|^p. \tag{2}$$

[1]This research was partially supported by the RFBR grant №11-01-00571a
[2]$\overline{m} = \max\{1, |m|\}$

Theorem 1. *If $f(x_1,\ldots,x_s)\in E_s^\alpha(C)$, then*

$$|R_N[f]|\leq C\left|\frac{1}{N}S_{M,\vec{\rho}}(\vec{0})-1\right|+\frac{C}{N}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' \frac{|S_{M,\vec{\rho}}(\vec{m})|}{(\overline{m}_1\ldots\overline{m}_s)^\alpha}=$$
$$=C\left|S^*_{M,\vec{\rho}}(\vec{0})-1\right|+C\cdot\zeta(\alpha,1|M,\vec{\rho}). \tag{3}$$

This estimate cannot be improved on the class $E_s^\alpha(C)$.

Let us consider the class $E_s^{\alpha,q}$ with the norm

$$\|f(\vec{x})\|_{E_s^{\alpha,q}}=\left(|C(\vec{0})|^q+\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' |C(\vec{m})|^q(\overline{m}_1\ldots\overline{m}_s)^{\frac{q\alpha}{p}}\right)^{\frac{1}{q}}<\infty.$$

Theorem 2. *If $f(\vec{x})\in E_s^{\alpha,q}$ and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$|R_N[f]|\leq$$
$$\leq\|f(\vec{x})\|_{E_s^{\alpha,q}}\left(\left|\frac{1}{N}S_{M,\vec{\rho}}(\vec{0})-1\right|^p+\frac{1}{N^p}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' \frac{|S_{M,\vec{\rho}}(\vec{m})|^p}{(\overline{m}_1\ldots\overline{m}_s)^\alpha}\right)^{\frac{1}{p}}=$$
$$=\|f(\vec{x})\|_{E_s^{\alpha,q}}\left(\left|S^*_{M,\vec{\rho}}(\vec{0})-1\right|^p+\zeta(\alpha,p|M,\vec{\rho})\right)^{\frac{1}{p}}. \tag{4}$$

This estimate cannot be improved on the class $E_s^{\alpha,q}$.

Definition 2. *[8] The hyperbolic parameter of a net M with weights $\rho(\vec{x})$ is*

$$q\left(M,\rho(\vec{x})\right)=\min_{\vec{m}\in\mathbb{Z}^s\setminus\{\vec{0}\},|S(\vec{m})|>0}\overline{m_1}\ldots\overline{m_s}\,.$$

In [1] the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net on the space of periodic functions E_s^α is considered for arbitrary net M with weights $\vec{\rho}$. $A_{M,\vec{\rho}}$ is defined by the equality

$$g(\vec{x})=A_{M,\vec{\rho}}f(\vec{x})=\frac{1}{N}\sum_{k=1}^{N}\rho_k f[x_1+\xi_1(k),\ldots,x_s+\xi_s(k)]. \tag{5}$$

As far as a normed exponential sum of a net with weights is concerned it is natural to define five subsets of the fundamental lattice $\mathbb{Z}^s$:

$$K_0=K_0(M,\vec{\rho}) = \{\vec{m}\in\mathbb{Z}^s\,|\,S^*_{M,\vec{\rho}}(\vec{m})=0\}, \tag{6}$$
$$K_1=K_1(M,\vec{\rho}) = \{\vec{m}\in\mathbb{Z}^s\,|\,S^*_{M,\vec{\rho}}(\vec{m})=1\}, \tag{7}$$
$$K_2=K_2(M,\vec{\rho}) = \{\vec{m}\in\mathbb{Z}^s\,|\,S^*_{M,\vec{\rho}}(\vec{m})\neq 1,\,|S_{M,\vec{\rho}}(\vec{m})|=1\}, \tag{8}$$
$$K_3=K_3(M,\vec{\rho}) = \{\vec{m}\in\mathbb{Z}^s\,|\,0<|S^*_{M,\vec{\rho}}(\vec{m})|<1\}, \tag{9}$$
$$K_4=K_4(M,\vec{\rho}) = \{\vec{m}\in\mathbb{Z}^s\,|\,|S^*_{M,\vec{\rho}}(\vec{m})|>1\}. \tag{10}$$

It is clear that $\mathbb{Z}^s = K_0 \bigcup K_1 \bigcup K_2 \bigcup K_3 \bigcup K_4$. This partitioning is called the Korobov partitioning.

In [1] the regular and unbiased linear operator $A_{M,\vec{\rho}}$ of weighted mean over net was defined (see the paper [1], pp. 195 and 199). A regular operator does not increase a norm of a function. Therefore $K_4 = \varnothing$. Further, $S^*_{M,\vec{\rho}}(\vec{0}) = 1$ because $A_{M,\vec{\rho}}$ is unbiased.

Definition 3. *The hyperbolic parameter $q(K)$ of a subset K of the fundamental lattice $\mathbb{Z}^s$ is*

$$q(K) = \min_{\vec{m}\in K} \overline{m_1} \ldots \overline{m_s}. \qquad (11)$$

If K is the empty set, then $q(K) = \infty$.

Definition 4. *Suppose M is a net with weights $\vec{\rho}$ such that the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net is regular and unbiased, then the first, the second and the third hyperbolic parameters of the net M with weights $\vec{\rho}$ are*

$$q_\nu \left(M, \rho(\vec{x})\right) = q(K_\nu \left(M, \rho(\vec{x})\right)) \quad (\nu = 1, 2, 3). \qquad (12)$$

Suppose M is a rational net with denominator p, then in $G_s = \{\vec{x}\,|\, 0 \leqslant x_i < 1\, (i = 1, \ldots, s)\}$ there are exist N rational points of the form

$$\left(\frac{x_1^{(k)}}{p}, \ldots, \frac{x_s^{(k)}}{p}\right) \qquad k = 1, \ldots, N, \qquad (13)$$

where $x_i^{(k)}$ are integer, $0 \le x_i^{(k)} \le p - 1$, p are natural.

Theorem 3. *Suppose M is a rational net with denominator p and with weights $\vec{\rho}$ such that the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net is regular and unbiased, then*

$$p \cdot \mathbb{Z}^s \subset K_1 \left(M, \rho(\vec{x})\right),$$

and the number of different values of exponential sums $S^*_{M,\vec{\rho}}(\vec{m})$ with weights $\vec{\rho}$ is finite and is not greater than p^s.

Theorem 4. *Suppose M is a rational net with denominator p and with positive weights $\vec{\rho}$ such that the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net is regular and unbiased, then the set $K_1(M, \vec{\rho})$ is an integer lattice.*

Theorem 5. *Suppose M is a rational net with denominator p and with positive weights $\vec{\rho}$ such that the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net is regular and unbiased, then the set $K_1(M, \vec{\rho}) \bigcup K_2(M, \vec{\rho})$ is an integer lattice.*

Definition 5. *A net M with weights $\vec{\rho}$ such that the linear operator $A_{M,\vec{\rho}}$ of weighted mean over net is regular and unbiased is of type $\Delta(N, s) < 1$ if for any $\vec{m} \in K_3(M, \vec{\rho})$*

$$|S^*_{M,\vec{\rho}}(\vec{m})| \leqslant \Delta(N, s).$$

Theorem 6. **(Generalized Bakhvalov — Korobov's theorem for the lattice hyperbolic zeta-function)** *Suppose M is a rational net of type $\Delta(N,s)<1$ with denominator p and with positive weights $\vec\rho$ such that the linear operator $A_{M,\vec\rho}$ of weighted mean over net is regular and unbiased, then*

$$\zeta(\alpha,p|M,\vec\rho)\le 2^{(\alpha+1)s+1}\alpha\left(\frac{\alpha}{\alpha-1}\right)^s\frac{(\ln q(\Lambda)+1)^{s-1}}{q^\alpha(\Lambda)}+$$
$$+\Delta^p(N,s)\frac{1}{t^{\alpha-1}}\left(\frac{\ln^{s-1}t}{(\alpha-1)(s-1)!}+\right.$$
$$\left.+\sum_{m=0}^{s-2}\frac{\ln^m t}{m!}\left(\sum_{k=m}^{s-2}\zeta(\alpha)^{s-2-k}C_k^m\frac{\alpha-1+\zeta(\alpha)}{\alpha-1}+\frac{C_{s-1}^m}{\alpha-1}\right)\right), \tag{14}$$

where $\Lambda=K_1(M,\vec\rho)\bigcup K_2(M,\vec\rho)$ and $t=q_3\left(M,\rho(\vec x)\right)$.

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ON FREE SUBGROUPS IN COXETER AND ARTIN GROUPS WITH TREE-STRUCTURE

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Let G be finitely generated Coxeter group with tree-structure defined by the presentation $G=<a_1,...,a_n;(a_ia_j)^{m_{ij}},i,j=\overline{1,n}>$, where m_{ij} is number that corresponds to symmetrical matrix of Coxeter, and $m_{ii}=1,m_{ij}\ge 2$ a group G matches the end coherent tree-graph Γ such that if the tops of some edge e of the graph Γ match the form a_i and a_j, then the edge e corresponds to the ratio of the species $(a_ia_j)^{m_{ij}}$ [1].

The group G can be represented as the tree product 2-generated of the groups, united by a finite cyclic subgroups. We will go from the graph Γ of a group G to the graph $\overline{\Gamma}$ as follows: the tops of the graph $\overline{\Gamma}$ put in conformity the Coxeter groups of two forming $G_{ij} =< a_i, a_j; a_i^2, a_j^2(a_ia_j)^{m_{ij}} = 1 >$, every edge $\overline{e}$, connecting tops corresponding to G_{ij} and G_{jk} put in conformity the cyclic subgroup $< a_j; a_j^2 >$. Let G be finitely generated Artin group by the copresentation $G =< a_1, ..., a_n; \langle a_ia_j\rangle^{m_{ij}} = \langle a_ja_i\rangle^{m_{ji}}, i, j = \overline{1, n}, i \neq j >$, where $\langle a_ia_j\rangle^{m_{ij}}$ is the word with the length m_{ij}, is consisting of m_{ij} alternating letters a_i and $a_j, i \neq j$, and m_{ij} is number that corresponds to symmetrical matrix of Coxeter, where $m_{ij} \geq 2, i \neq j$. If the group G corresponds the finite coherent tree-graph Γ such that if the tops of some edge e of the graph Γ correspond forming a_i and a_j, then for the edge e corresponds to the relation of the form $\langle a_ia_j\rangle^{m_{ij}} = \langle a_ja_i\rangle^{m_{ji}} i \neq j$. In this case, we have an Artin group with a tree-structure [2].

We represent the group G as the tree product 2-generated of the Artin groups united by an infinite cyclic subgroups. We will go from the graph Γ of a group G to the graph $\overline{\Gamma}$ as follows: the tops of the graph $\overline{\Gamma}$ put in conformity the Artin groups of two forming $G_{ij} =< a_i, a_j; \langle a_ia_j\rangle^{m_{ij}} = \langle a_ja_i\rangle^{m_{ji}}, i \neq j >$ and $G_{ij} =< a_i, a_j; \langle a_ia_j\rangle^{m_{ij}} = \langle a_ja_i\rangle^{m_{ji}}, i \neq j >$, every edge $\overline{e}$, connecting tops corresponding to G_{ij} and G_{jk} put in conformity the cyclic subgroup $< a_j >$.

Problem of freedom is to determine whether a given subgroup of the group of the free. In [3] this problem is considered for Coxeter groups of extra large type.

This paper considers the theorem on the freedom of the Artin and Coxeter groups with a tree-structure.

Theorem 1. *A finitely generated without torsion free subgroup of the Coxeter group with a tree-structure is free.*

Theorem 2. *Let H be finitely generated subgroup of the Artin group G with a tree-structure, while for any $g \in G$ and every subgroup $G_{ij}, i \neq j$, executed equality $H \cap gG_{ij}g^{-1} = E$ then H is free.*

Theorem 3. *Let H be finitely generated subgroup of the Artin group G with the tree-structure, it is possible to allocate effectively free part of subgroups H.*

In the proof of use of the ideas V.N. Bezverkhnii on bringing many forming of the subgroup to the special set [4,5].

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ON WEIGHTED NUMBER OF INTEGRAL POINTS ON SOME FOUR-DIMENSIONAL QUADRATIC SURFACES

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The question about the distribution of integral points on quadratic surfaces long since attracts many investigators' attention. In more general statement of a question a problem about weighted quantify of integral points is also considered on such surfaces (see, [1, 2]).

We will considered the unconic four-dimensional surface given by the equation

$$Q_1\left(x_1, x_2\right) - Q_2\left(x_3, x_4\right) = h, \tag{1}$$

where $h \neq 0$ is integer; $Q_1\left(x_1, x_2\right)$ and $Q_2\left(x_3, x_4\right)$ are the polyhedron positive binary quadratic forms (quadric quartic) with matrices A_1 and A_2, $\det A_1 = \det A_2 = -\delta_F$, where δ_F is a discriminant of imaginary quadratic field $F = Q(\sqrt{d})$ and d is negative square-free number.

We will connect the function with the equation (1)

$$I_{h,\lambda}(n) = \sum_{Q_1(x_1,x_2)-Q_2(x_3,x_4)=h} e^{\frac{-\lambda(Q_1(x_1,x_2)+Q_2(x_3,x_4))}{n}}, \tag{2}$$

which is called the weighted number of integral points on the surface (1) taken with the weight $e^{-\frac{1}{n}\omega(x_1,x_2,x_3,x_4)}$, where

$$\omega\left(x_1, x_2, x_3, x_4\right) = \lambda Q_1\left(x_1, x_2\right) + \lambda Q_2\left(x_3, x_4\right),$$

$\lambda > 1$.

The introduction of natural parameter λ due to the fact that further we will use the results from [1, 2], obtained by other approach, providing with this parameter $\lambda > 1$ the condition of positivity $Q_1\left(x_1, x_2, x_3, x_4\right) + \omega\left(x_1, x_2, x_3, x_4\right)$, where

$$Q_1\left(x_1, x_2, x_3, x_4\right) = Q_1\left(x_1, x_2\right) - Q_2\left(x_3, x_4\right).$$

For the function $I_{h,\lambda}(n)$ for $\lambda = 1$ and $h = 1$ in [3] asymptotic formula with remainder term $O\left(n^{\frac{3}{4}+\varepsilon}\right)$, where $\varepsilon > 0$ is arbitrarily small positive number is obtained. Using the approach from [3] we obtained the following asymptotic result about the value $I_{h,\lambda}(n)$ in which the dependence of remainder term δ_F, λ, h is established.

THEOREM 1. *For the weighted number of integral points $I_{h,\lambda}(n)$ on the four-dimensional quadratic unconic surface $Q_1(x_1, x_2) - Q_2(x_3, x_4) = h$ with weighting function $\lambda(Q_1(x_1, x_2) + Q_2(x_3, x_4))$ asymptotic formula is true*

$$I_{h,\lambda}(n) = \frac{2\pi^2 n e^{\frac{-\lambda h}{n}}}{\lambda|\delta_F|} \sum_{q=1}^{\infty} q^{-4} \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{-2\pi i \frac{\ell h}{q}} \cdot$$

$$\cdot G_1\left(q, \ell, \bar{\mathrm{O}}\right) \cdot G_2\left(q, -\ell, \bar{\mathrm{O}}\right) + O\left(\lambda^2 h n^{\frac{3}{4}+\varepsilon}\right),$$

where $G_1\left(q, \ell, \bar{\mathrm{O}}\right)$ and $G_2\left(q, -\ell, \bar{\mathrm{O}}\right)$ are homogeneous double Gauss sums; ε is arbitrarily small positive number.

We note that theorem makes precise the corresponding result from [2] in which in case of four-dimensional quadratic surfaces the remainder depending on δ_F in our designations has the form $O\left(|\delta_F|^3 n^{\frac{3}{4}+\varepsilon}\right)$.

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TORSION FREE ABELIAN GROUPS OF FINITE RANK

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One of the important problems is the problem of description for torsion free abelian groups of finite rank. This problem has been researched by many authors. We mention classical works by L.S. Pontryagin [1], A.I. Maltsev [2], A.G. Kurosh [3], F. Levi [4], R. Baer [5], D. Derry [6], R. Beaumont and R. Pierce [7,8]. The question how to find convenient terms for torsion-free finite-rank groups is actual up to now, because this class of groups is very complicated.

The notion of the quotient divisible mixed group has been introduced in [9]. It generalizes the classic notion of the quotient divisible torsion free group by R. Beaumont and R. Pierce [8].

An abelian group A is called quotient divisible, if it contains a free subgroup F of finite rank such that the quotient group A/F is torsion divisible, but the group A itself doesn't contain nonzero torsion divisible subgroups. A free basis of the group F is called the basis of the quotient divisible group A.

It is shown in [9] that the category $\mathcal{QD}$ of all quotient divisible groups with quasi-homomorphisms is dual to the category $\mathcal{QTF}$ of all torsion-free finite-rank groups with quasi-homomorphisms.

The new method of the description is proposed in [10,11,12]. We describe pairs consisting of two mutually dual groups: a quotient divisible group and a torsion-free finite-rank group. The terms of the description are finite sequences of finitely presented modules over the ring of polyadic numbers. The method is the following.

The product $\widehat{\mathbf{Z}} = \prod\limits_p \widehat{\mathbf{Z}}_p$ of the rings of p-adic integers over all prime numbers p is called the ring of polyadic numbers.

Let $\alpha = (\alpha_p) \in \prod\limits_p \widehat{\mathbf{Z}}_p$ be a polyadic number. We denote as m_p the maximal power of the prime number p such that p^{m_p} divides the p-adic integer α_p in the ring $\widehat{\mathbf{Z}}_p, m_p = \infty \Leftrightarrow \alpha_p = 0$. The characteristic $char(\alpha) = (m_p)$ is called the characteristic of the polyadic number α. A polyadic number α divides a polyadic number β if and only if $char(\alpha) \leq char(\beta)$. Every finitely generated ideal $\langle \alpha_1, \dots, \alpha_n \rangle_{\widehat{\mathbf{Z}}}$ of the ring $\widehat{\mathbf{Z}}$ is generated by one element $\langle \alpha_1, \dots, \alpha_n \rangle_{\widehat{\mathbf{Z}}} = \langle \alpha \rangle_{\widehat{\mathbf{Z}}}$, where α is the greatest common divisor of the numbers $\alpha_1, \dots, \alpha_n$. This ideal is of the form $I_\chi = \{\gamma \in \widehat{\mathbf{Z}} \,|\, char(\gamma) \geq \chi\}$, where $\chi = char(\alpha)$. The quotient ring $Z_\chi = \widehat{\mathbf{Z}}/I_\chi$ is of the form $Z_\chi = \prod\limits_p K_p$, where $K_p = Z_{p^{m_p}}$ if $m_p < \infty$ or $K_p = \widehat{\mathbf{Z}}_p$ if $m_p = \infty, \chi = (m_p)$. We consider the ring Z_χ also as a cyclic $\widehat{\mathbf{Z}}$-module.

Let R be a ring. An R-module M is called finitely presented if there exists an exact sequence of homomorphisms

$$R^n \to R^k \to M \to 0$$

for some positive integers n and k.

The following two theorems take place , see [13] or [14].

Theorem A. *A $\widehat{\mathbf{Z}}$-module M is finitely presented if and only if it is of the form $M \cong Z_{\chi_1} \bigoplus \ldots \bigoplus Z_{\chi_n}$ for some characteristics $\chi_1, \ldots, \chi_n$.*

The sequence of the characteristics $\chi_1, \ldots, \chi_n$ in **Theorem A** is determined not uniquely by the module M. Nevertheless, there exists a sequence $\chi_1, \ldots, \chi_n$ satisfying the condition $\chi_1 \geq \ldots \geq \chi_n$. The decreasing sequence $\chi_1 \geq \ldots \geq \chi_n$ is determined uniquely by the module M and it is called the generalized characteristic of the finitely presented $\widehat{\mathbf{Z}}$-module M.

Theorem B. *Let N be a finitely generated submodule of a finitely presented $\widehat{\mathbf{Z}}$-module M. Then the modules N and M/N are finitely presented.*

We define now a category $\mathcal{S}$ of sequences. This category has been introduced in [12] as well as the categories $\mathcal{D}$ and $\mathcal{TF}$. An object of the category $\mathcal{S}$ is a finite sequence of elements $a_1, \ldots, a_n$ of a finitely presented module M over the ring of polyadic numbers $\widehat{\mathbf{Z}}$. The order of elements is essential, repetitions in the sequence are possible.

Morphisms from an object $a_1, \ldots, a_n$ to an object $b_1, \ldots, b_k$ are pairs (φ, T), where $\varphi : \langle a_1, \ldots, a_n \rangle_{\widehat{\mathbf{Z}}} \to \langle b_1, \ldots, b_k \rangle_{\widehat{\mathbf{Z}}}$ is a homomorphism of $\widehat{\mathbf{Z}}$-modules generated by these sets of elements and T is a matrix of size $k \times n$ with integer entries, such that the following matrix equality takes place

$$(\varphi a_1, \ldots, \varphi a_n) = (b_1, \ldots, b_k)T.$$

That is $\varphi(a_i) = t_{1i}b_1 + \ldots + t_{ki}b_k, i = 1, \ldots n, T = \|t_{ij}\|$.

Let two morphisms be given

$$a_1, \ldots, a_n \xrightarrow{(\varphi,T)} b_1, \ldots, b_k \xrightarrow{(\psi,S)} c_1, \ldots, c_m$$

The composition of the morphisms is defined as $(\psi, S)(\varphi, T) = (\psi\varphi, ST)$. The identity morphism of an object consists of the identity homomorphism and the identity matrix.

The objects of the category $\mathcal{S}$ are the terms which describe objects of the following two categories of groups. Objects of the category $\mathcal{D}$ are all quotient divisible groups with marked bases. Objects of the category $\mathcal{TF}$ are all torsion-free finite-rank groups with marked bases (maximal linearly independent sets). Morphisms of these two categories $\mathcal{D}$ and $\mathcal{TF}$ are usual group homomorphisms such that the matrices of them with respect to the marked bases consist of integers.

The main result is the following theorem.

Theorem 1. *The commutative diagram of six functors takes place*

$$\begin{array}{ccccc} & & \mathcal{S} & & \\ & \nearrow\swarrow & & \searrow\nwarrow & \\ \mathcal{TF} & & \longleftrightarrow & & \mathcal{D} \end{array}$$

At that, the functors $\mathcal{TF} \leftrightarrow \mathcal{S}$ and $\mathcal{TF} \leftrightarrow \mathcal{D}$ are dualities and the functor $\mathcal{S} \leftrightarrow \mathcal{D}$ is an equivalence.

Some applications of this theorem are given in [11,12]. In particular, this approach is applied to the almost completely decomposable groups in [11], where the famous example of the group by A.L.S. Corner [15] is considered.

It is interesting to remark the following. Let $a_1, \ldots, a_n$ be a sequence of elements of a finitely presented $\widehat{\mathbf{Z}}$-module M, that is an object of the category $\mathcal{S}$. Choosing a direct decomposition of the module M according to the Theorem A, we can present the elements $a_1, \ldots, a_n$ as columns of "generalized" numbers, i.e. elements of the rings Z_χ. Then we obtain a matrix. It occurs that the p-components of this matrix are matrices which have been used by A.I. Maltsev in his paper [2], they are so called perfect matrices in his terminology. Thus, the duality $\mathcal{TF} \leftrightarrow \mathcal{S}$ actually coincides with the description by Maltsev [2]. The duality $\mathcal{TF} \leftrightarrow \mathcal{D}$ is a modification of the duality $\mathcal{QTF} \leftrightarrow \mathcal{QD}$ [9]. As for the p-primitive torsion free groups by A.G. Kurosh [3], they are also quotient divisible. It means that we can consider the equivalence $\mathcal{S} \leftrightarrow \mathcal{D}$ as a generalization of the theorem by Kurosh [3].

As a result of all this consideration, a new interest occurred to the matrices by Maltsev [2]. So Yu.V. Kostromina [16,17] has found recently the Maltsev's matrices for the dual in the sense of R. Warfield [18] groups in the class of locally free groups, she has found also the Maltsev's matrices for the dual in the sense of D. Arnold [19] groups in the class of quotient divisible torsion free groups.

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ON ALMOST NILPOTENT VARIETIES

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This work describes a new result on the varieties of Leibniz algebras. A Leibniz algebra is called a linear algebra with multiplication satisfying the identity $(xy)z \equiv (xz)y + x(yz)$. It is well-known that any Lie algebra is a Leibniz algebra.

Let F be a field of characteristic zero. The necessary concepts can be found in books [1] and [2]. Let us recall, that a variety $\mathbf{V}$ is almost nilpotent if $\mathbf{V}$ is not nilpotent and every his own subvariety is nilpotent. It is well-known that in case of associative algebras the variety of all associative and commutative algebras is the unique almost nilpotent variety, and in case of Lie algebras the variety of all metabelian Lie algebras $\mathbf{A}^2$ is the unique almost nilpotent variety of Lie algebras. The variety $\mathbf{A}^2$ is defined by identity $(x_1x_2)(x_3x_4) \equiv 0$.

The identity $x(yz) \equiv 0$ defines the variety ${}_2\mathbf{N}$ of all left nilpotent of the class not more than two algebras. A full description of this variety is given in the paper [3].

Theorem 1. *In the case of a field of zero characteristic there are exactly two almost nilpotent varieties of Leibniz algebras. There is the variety* $\mathbf{A}^2$ *of all metabelian Lie algebras and the variety* ${}_2\mathbf{N}$ *of all left nilpotent of the class not more than two algebras.*

In proving the theorem, we have used the results of papers [4] and [5]. We give a basic sketch of a proof. We will use the left normed notation and write $((ab)c) = abc$. We denote the multiplication operator on the right, such as the element z, by capital letter Z: we suppose that $xz = xZ$. In particularly the Engel identity looks like $\underbrace{xyy...y}_{m} = xY^m \equiv 0$ in our notation.

Let $\mathbf{U}$ be another almost nilpotent variety of Leibniz algebras, then $\mathbf{U} \bigcap {}_2\mathbf{N}$ is nilpotent and the identity $x_1X_1^m \equiv 0$ is satisfy in $\mathbf{U}$. The identity $x^2Y^m \equiv 0$ follows from the last identity. Consider a Leibniz algebra $L \in \mathbf{U}$. It is well-known, that there is an ideal $I_f = \{a|aX^m \equiv 0, \text{ for any } x \in L\}$. As $x^2 \in I_f$, so the identity $x^2 \equiv 0$ is satisfy in quotient algebra L/I_f and then L/I_f is a Lie algebra. There are two possibilities:

In the first case the quotient algebra L/I_f is not nilpotent in the variety $\mathbf{U}$, and so there is a non-nilpotent variety of Leibniz alebras $varL/I_f$, generated by Lie algebra L/I_f. So minimal non-nilpotent variety $\mathbf{A}^2 \subset varL/I_f \subset \mathbf{U}$ and $\mathbf{U} = \mathbf{A}^2$.

In the second case the quotient algebra L/I_f is nilpotent and identity $x_1x_2\dots x_c \equiv 0$ is satisfy in L/I_f. By definition of ideal I_f the identity $x_1x_2\dots x_cX^m \equiv 0$ is satisfy in Leibniz algebra L, and then Engel identity is satisfy in L. So by the theorem from paper [5] the variety $\mathbf{U}$ is nilpotent.

Thus, ${}_2\mathbf{N}$ и $\mathbf{A}^2$ — exactly two almost nilpotent varieties of Leibniz algebras.

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CRYSTALLINE COHOMOLOGY AND THEIR APPLICATIONS

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A review of the theory of crystalline cohomology and crystalline representations that includes some new results is given. Examples and applications are included.

1 Introduction

Purposes of the communication are: 1) a review of recent results that extend: Grothendieck‘s approaches and their implementation by P. Bertelot, expositions by L. Illusie and W. Messing; the theory of Dieudonne modules, Dieudonne crystals and F-crystals, cristalline cohomology; 2) applications of the results to $K3$ surfaces [1] and more general Calabi-Yau varieties. Presenting results are connected with:

(i) with theory of formal groups [4, 5] and p - divisible groups;

(ii) with families of varieties over basic schema S of characteristic p (and include the review of the case of two dimensional abelian varieties);

(iii) with loop groups of reductive groups by Faltings, by Hartl abd by Viehmann;

(iv) with Newton polygons and the supremum of Newton polygons of p-divisible groups by Ekedahl, by van der Geer and by Harashita.

2 Preliminaries and Definitions

Let F be a commutative formal group low of n variables over commutative ring R with unit. In the case $n = 1$, following to the known results by M. Lazard, there is only one $1-$ bud of the form $x + y + \alpha xy$.

Proposition 1. *Let $n = 2$, $A = Z_p[\alpha, \beta]$ be the ring of polynomials with integer p-adic coefficients from α, β. $1-$ buds are*

$$F(x,y) = \begin{cases} x_1 + y_1 + \alpha x_1 y_1 \\ x_2 + y_2 + \beta x_2 y_2, \end{cases}$$

$$F_a(x,y) = \begin{cases} x_1 + y_1 + \alpha x_1 y_1 \\ x_2 + y_2 + \beta x_1 y_1, \end{cases}$$

$$F_b(x,y) = \begin{cases} x_1 + y_1 + \alpha x_2 y_2 \\ x_2 + y_2 + \beta x_2 y_2, \end{cases}$$

$$F_c(x,y) = \begin{cases} x_1 + y_1 + \alpha(x_1 + x_2)(y_1 + y_2) \\ x_2 + y_2 + \beta(x_1 + x_2)(y_1 + y_2), \end{cases}$$

Note 1. *$1-$ buds given in Proposition 1 are also two-dimensional formal group lows, whose coefficients under terms of degrees $\geqslant 3$ are zeros.*

Note 2. *These group lows define classes of group lows. In particular, the class F_a contains under values of parameters $\alpha = 0, \beta = -1$, the Witt group, that corresponds to prime number $p = 2$.*

Let now the ring R is the field k. Recall, that formal $k-$ scheme is formal $k-$ functor, that is the limit of directed inductive system of finite $k-$ schemes, and a formal group is a group object in the category of formal $k-$ schemes. The notion of a stack, as one of category theory variants of moduli space is defined by P. Deligne and D. Mumford.

Proposition 2. *There exist formal stacks, that are categories that are bundled on formal groupoids and that satisfy axioms of decent theory.*

Let R be a complete discrete valuation ring with quotient field K and perfect residue field k of characteristic p. Under Calabi-Yau variety over K we understand smooth projective scheme $\mathcal{V}$ over K of dimention n with trivial canonical bundle $\omega_X = \Omega^n_{X \setminus K}$ [1, 2]. A weak N'eron model of the variety X is called smooth proper scheme $\mathcal{V}$ of finite type over R with the isomorphism $\mathcal{V} \otimes_R K \simeq X$, that satisfies next property: for every finite unramified extension $R' \supset R$ with quotient field K', the canonical mapping $\mathcal{V}(R') \to X(K')$ is bijection [3].

3 Conclusions

We apply above mentioned constructions and results to investigation of smooth Calabi-Yau varieties [1, 2], their N'eron model, weak N'eron model, and to reductions of the models over residue fields k. Examples of simple crystalline representations will be given.

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ON ALGEBRAIC PROBLEMS OF OPTIMIZATION OF BODIES SHAPES THAT WITHSTAND EXTREME STRESS

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An overview of algebraic problems and methods for optimizing the shape of flexible body in terms of buckling load of such a body is given. Examples and applications are provided.

1 Introduction

Shape optimization problems for flexible bodies, that can withstand extreme stress have been considered by Bernoulli and Euler[1]. Evolving the work [1], researches of J. L Lagrange, T. Clausen, E. Nicolai, and others, N. Olhoff and S. Rasmunssen have found [2], that problems under investigation can be reduced to problems of non-differentiable optimization. The author of review [3] believes, that the multiple eigenvalue may occur in many optimization problems in terms of stability criteria. Algebra allows us to resolve situations in which problems with the applicability of the classical differential methods arise. The present report gives an overview of algebraic approaches, techniques and their applications in nondifferentiable optimization problems for the shape of flexible bodies in terms of buckling load of such a body.

2 Preliminaries and definitions

The known results from sources [4, 5] are provided and used below. Suppose X is a matrix of size $n_1 \times n_2$, whose elements are real numbers. Let X^* be a matrix conjugate with respect to the matrix X. It is known that nonzero eigenvalues of matrices XX^* and X^*X are coincident and positive. Arithmetic values of the square roots of common eigenvalues of matrices XX^* and X^*X are called singular values of matrix X. In the following we suppose that σ_k is k-largest singular values of matrix X. Also we suppose that these singular values are arranged in the order of decreasing $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$, where σ_n is the smallest singular value. Suppose singular values $\sigma_{n+1} \cdots$ equal zero.

2.1 Norms

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For vectors $x, y \in \mathbb{R}^n$ with an inner product (x, y) we will use Euclidean norm l_2 , which would be denoted by $\|x\| = \sqrt{(x, x)}$. The inner product in $\mathbb{R}^{n_1 \times n_2}$ of matrix X, Y, X_0 from $\mathbb{R}^{n_1 \times n_2}$ would be denoted by $(X, Y) = tr(X^*Y)$. Euclidean norm of a matrix X would be denoted by $\|X\|_E = (X, X)^{\frac{1}{2}}$. Spectral norm of a matrix X equels the greatest singular values of matrix X. It would be denoted by $\|X\|$. Nuclear norm of X would be denoted by $\|X\|_*$.

Note 1. *For a matrix $A \in \mathbb{R}^{n_1 \times n_2}$*
$\|A\|_E = (trAA^*)^{\frac{1}{2}} = (trA^*A)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \sigma_i^2\right)^{\frac{1}{2}}$,
where $\sigma_1, \sigma_2, \cdots \sigma_n$ are nonzero singular values of the matrix A.

2.2 Subgradient.

Recall the definition of the subgradient of a convex function $f : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}$.

Definition 1. *A matrix $g_f(X_0)$, satisfying the condition*
$f(X) - f(X_0) \geqslant (g_f(X_0), X - X_0)$
for all $X \in \mathbb{R}^{n_1 \times n_2}$ is called a subgradient f in X_0.

The set
$\partial f(X_0) = \{X^* \in \mathbb{R}^{n_1 \times n_2} | f(X) - f(X_0) \geqslant (X^*, X - X_0)\}$
is called the subdifferential of f in X_0.

Let $A \otimes B$ be a tensor product (Kronecker's product) of two rectangular matrices whose elements are real numbers. In the following it is used mainly for vectors.

Note 2. *Let $A \in \mathbb{R}^{n_1 \times n_2}$ be a matrix whose rank is r. Matrix A has singular value decomposition which is represented as $A = \sum_{1 \leqslant i \leqslant r} \sigma_i u_i \otimes v_i{}^*$. Then subgradient of a nuclearl norm A is known and it be represented as*

$g_*(A) = \sum_{1 \leqslant i \leqslant r} u_i \otimes v_i^* + W$.

Matrix W meets the known properties (in particular, $\|W\| \leqslant 1$).

2.3 r- Algorithm

One of the most effective methods of nondifferentiated optimizing is subgradient method with space dilation in the direction of the difference of two successive subgradients [4]. Following by [4, 5] the application of matrix r-algorithms scheme will be presented. Since $\mathbb{R}^{n_1 \times n_2}$is an Euclidean space then we will concider elements of $\mathbb{R}^{n_1 \times n_2}$ as elements of an Euclidean space E^n, where an inner product would be denoted by $(,)$. Operator of space dilation in the direction ξ with coefficient α will be denoted by $R_\alpha(\xi)$.This operator under its application to an element x of the Euclidean space E^n dilates in α times $(x, \xi)\xi$ and does not alter the $x - (x, \xi)\xi$.

2.4 Projections.

Let V be a subspace of dimension r in E^n and P_V be the orthogonal projection onto V. During calculations we need to project with application of P_V, also we need to project points of space E^n onto a closed convex subspace S from E^n. The problem of projection of the point $a \in E^n$ on S has the representation

$d(x) = \|x - a\| \to min, x \in S$,

and its solution is a solution $d(x) = min\|x - a\|, x \in S$ of this minimization problem.

3 Conclusion

We apply the above-mentioned constructions and results, as well as other methods of linear algebra to problem of nondifferentiable optimization the shape of flexible body in terms of its buckling load.

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UDC 511.3

THEORETICAL MODELS OF FLOW CONTROL IN NEAR-WALL AREAS

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Flow control in near-wall areas attracts wide interest in both the fluid mechanics and engineering because of many potential applications. Drag reduction of bodies, lift enhancement, mixing advancement etc. are some of many problems in which near-wall flow control is urgent. The proposed control strategy is based on special changes of the near-wall vorticity field by means of generation of stationary vortex zones. Artificial surface irregularities such as cross grooves, interceptors, ribs and so on may be applied to create the desired vortex. To minimize energy costs for realization of this control way, one has to take into account dynamic properties of the vortices created. The most effective control algorithms use information about critical points and other topology features of a flow pattern. In some cases, the goal of control lies in generating the necessary flow topology.

In this work, we consider cross grooves as the instrument for generation of local separation zones on the wall. A simplified model when the zone is replaced by a point vortex locating in the vorticity center is applied to derive dynamic properties of the flow [I. M. Gorban and O. V. Homenko, Dynamics of Vortices in Near-wall Flows with Irregular Boundaries, Continuous and Distributed Systems: Theory and Applications, Series: Solid Mechanics and Its Applications, 2014, Vol. 211, 115-128, Zgurovsky, Mikhail Z., Sadovnichiy, Victor A. (Eds.)].

Motion of the vortex is governed by a set of non-linear differential equations with the vortex velocity in the right part. The critical points of such a flow are determined from the condition of vortex equilibrium and their types in the conservative system depend on the "sign" of Jacobean only. The analysis points out on three stationary points in this case and one point only that lies on the groove axis is stable. To determine the strength of the vortex corresponding to this point, the Kutta condition in the sharp groove edges is used. The vortex obtained is immovable in the global sense but it rotates periodically near the critical point along an infinite small trajectory.

The rotation frequency is the important characteristic of the stationary vortex; just it is responsible for its reaction to external perturbations. Investigations demonstrate resonance behavior of the stationary vortex in the periodically perturbed flow when the amplitude of deviation of the vortex from the critical point grows rapidly when the external frequency approaches the vortex eigenfrequency.

To reduce sensitivity of the vortex to external perturbations, we propose the scheme of active flow control with ejection of fluid from the region. The dynamic system under consideration includes now the external flow, the stationary vortex and a sink locating on the groove boundary. If one coordinate of the vortex is fixed, the equations of vortex equilibrium and the Kutta condition in the groove edges will be the sufficient system for determining another vortex coordinate, vortex circulation and sink parameters (strength and angular coordinate). As a result, the curve of stationary vortices in xy-plane is obtained and corresponding sink parameters are evaluated. The dynamical analysis of the system shows that ejection changes the flow topology. If without ejection the critical point is the conditionally stable center that now it will be either stable or unstable focus that depends on point position on the stationary curve. The stable vortices are obtained to be located the left of the central axis of groove. The sinks corresponding to the stable flow patterns lie on the groove wall that is opposite to the flow. The stable zone width will rise when the groove depth decreases that important for a practice because of shallow grooves are primary applied for near-flow control in technical applications. To demonstrate operating the proposed control strategy, the direct numerical simulation of the flow in the cross groove was carried out. The obtained vorticity patterns demonstrate stabilization of flow in the groove when fluid ejecting.

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WARING'S PROBLEM INVOLVING NATURAL NUMBERS OF A SPECIAL TYPE

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In 2008–2011, we solved several well-known additive problems such that Ternary Goldbach's Problem, Hua Loo Keng's Problem, Lagrange's Problem with restriction on the set of variables. Asymptotic formulas were obtained for these problems ([1]–[3]). The main terms of our formulas differ from ones of the corresponding classical problems.

In the main terms the series of the form

$$\sigma_k(N,a,b) = \sum_{|m|<\infty} e^{2\pi i m(\eta N - 0{,}5k(a+b))} \frac{\sin^k \pi m(b-a)}{\pi^k m^k}.$$

appear.

These series were investigated by the authors in [4].

Suppose that $k \geqslant 2$ and $n \geqslant 1$ are naturals. Consider the equation

$$x_1^n + x_2^n + \ldots + x_k^n = N \tag{1}$$

in natural numbers $x_1, x_2, \ldots, x_k$. The question on the number of solutions of the equation (1) is Waring's problem. Let η be the irrational algebraic number. In this report we represent the variant of Waring's Problem involving natural numbers such that $a \leqslant \{\eta x_i^n\} < b$, where a and b are arbitrary real numbers of the interval $(0,1)$.

Let $J(N)$ be the number of solutions of (1) in natural numbers of a special type, and $I(N)$ be the number of solutions of (1) in arbitrary natural numbers. Then the equality holds

$$J(N) \sim I(N)\sigma_k(N,a,b).$$

The series $\sigma_k(N,a,b)$ is presented in the main term of the asymptotic formula in this problem as well as in Goldbach's Problem, Hua Loo Keng's Problem.

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BOMBIERI-VINOGRADOV THEOREM FOR PRIMES OF SPECIAL FORM

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To solve the binary additive problems with primes from the intervals of the form

$$[(2m)^c, (2m+1)^c), \tag{1}$$

where $m \in \mathbb{N}$, и $c \in (1, 2]$ it is required analog known theorem of Bombieri - Vinogradov received in 1965 in the works [1] and [2].

D. Tolev proved a special variant of the Bombieri - Vinogradov theorem [3]:

Theorem 1. *Let the inequalities*

$$0 < \lambda < \frac{1}{4},\ 0 < \theta < \frac{1}{4} - \lambda,\ A > 0.$$

Then

$$\sum_{k \leqslant x^\theta} \max_{y \leqslant x} \max_{(a,k)=1} \mid \psi_\lambda(y; k, a) - \frac{y^{1-\lambda}}{\varphi(k)(1-\lambda)} \mid \ll x^{1-\lambda} \ln^{-A} x,$$

where

$$\psi_\lambda(y; k, a) = \sum_{\substack{n \leqslant y \\ n \equiv a \pmod k \\ \{\sqrt{n}\} < n^{-\lambda}}} \Lambda(n).$$

In theorem Bombieri - Vinogradov boundary of the parameter k is smaller than $\sqrt{x}$, and in Theorem Tolev $k < x^{\frac{1}{4}}$. This circumstance makes it impossible apply Theorem 1 to the solution of binary additive problems with primes of the intervals of the form (1).

In this work a refinement of Theorem 1 in the special case is proved.

Theorem 2. *Let $c \in (1, 2]$ — constant. Let also $\pi_c(x, q, l)$ — the number of such primes $p \leqslant x$ and $p \equiv l \pmod q$ that $\{0.5p^{1/c}\} \leqslant 0.5$.*

Then for any $A > 0$ it is exist $\varepsilon > 0$ such that

$$\sum_{q \leqslant x^{1/3-\varepsilon}} \max_{(l,q)=1} |\pi_c(x, q, l) - \frac{\operatorname{Li} x}{2\varphi(q)}| \leqslant c\, x(\ln x)^{-A},$$

where $c = c(A) > 0$.

In the proof of the theorem we use the formula obtained in the [4]:

$$\sum_{\substack{n \leqslant N \\ n \equiv r (\mathrm{mod} q)}} \Lambda(n) e^{2\pi i f(n)} = O(Nq^{-1}N^{-\varkappa}),$$

where $c \in (1, 2]$, $A > 1$ — arbitrary constant, $f(n) = 0.5mn^{1/c}$,
$1 \leqslant m \leqslant (\ln N)^{2A}$, $0 < \varepsilon \leqslant 0.001$, q — natural number, $q \leqslant N^{1/3-\varepsilon}$, $\varkappa = \varkappa(\varepsilon) > 0$.

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PÓLYA PROBLEM FOR PERMANENT[1]

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The talk is based on the papers [1, 2, 3].

Two important functions in matrix theory, determinant and permanent, look very similar:

$$\det A = \sum_{\sigma\in S_n} (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \quad \text{and} \quad \operatorname{per} A = \sum_{\sigma\in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

here $A = (a_{ij}) \in M_n(\mathbb{F})$ is an $n\times n$ matrix and S_n denotes the set of all permutations of the set $\{1,\ldots,n\}$.

While the computation of the determinant can be done in a polynomial time, it is still an open question, if there are such algorithms to compute the permanent. Due to this reason, starting from the work by Pólya [4], 1913, different approaches to change the matrix in such a way that the permanent of original matrix would be equal to the determinant of the new matrix (to convert the permanent into the determinant) were under the intensive investigation.

Among our results we prove the following theorem:

Theorem 1. *Suppose $n \geqslant 3$, and let $\mathbb{F}$ be a finite field with* $\operatorname{char}\mathbb{F} \neq 2$. *Then, no bijective map $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfies*

$$\operatorname{per} A = \det T(A).$$

[1]The work is partially financially supported by the grants RFBR № 12-01-00140 and MD-962.2014.1

Also we investigate Gibson barriers (the maximal and minimal numbers of non-zero elements) for sign-convertible $(0,1)$-matrices and solve several related problems. In particular, we find Gibson barriers for symmetric and weak symmetric conversion.

Our results are illustrated by the number of examples.

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REPRESENTATION OF SEVERAL NATURAL NUMBERS AS A SUM OF PAIRWISE DISTINCT SUMMANDS

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We study the problem of representing several of natural numbers as a sum of pairwise different terms. There is given the estimation of the minimal number of components sufficient for such representation, provided that representation exists. For convenience, the initial numbers are treated as a sets, and their constituent components are treated as elements of these sets.

Theorem 1. *Let there are m sets containing pairwise distinct natural numbers. For each set the amount of incoming numbers is counted. Then the numbers in the sets can be replaced so that the sums of numbers in the sets will stay the same, the numbers will remain different, and their quantity will be not more then $2m-1$.*

The proof is based on the transformation of sets, so that the sums of the elements in them remained unchanged, while the quantity of items in each set does not exceed two. At the same time there is the set with a single element. The algorithm of such transformation is given.

The quantity of elements in the sets depends on the number of different values that the sums of the elements in the sets are taken. If all amounts are the same then the minimum number of elements is equal to $2m-1$. It can be shown that if the number of the values is equal to two, it is sufficient $2m-2$ elements. If all m values are different, it is enough $m = 2m - m$ elements. Hypothesis appears that if the quantity of different values of amounts is equal to k then it is sufficient $2m-k$ items. This hypothesis is disproved by the following example: $1+5$, 6, 7, 8, $3+4+5$, 9. The quantity of sets is equal to 6, the quantity of numbers in them is equal to 9, the quantity of different values is equal to 4. It is impossible to reduce the quantity of numbers, as they can not be greater than 9, and all the numbers, not exceeding 9, are presented already. If you reduce the quantity of numbers, you decrease the total amount.

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UDC 511.3

ON A NEW MEASURE ON INFINITE DIMENSIONAL UNITE CUBE

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1 Introduction

Let Ω be an infinite dimensional unite cube:

$$\Omega = \{\omega = (\alpha_n) \,|0 \leqslant \alpha_n \leqslant 1, n = 1, 2, ...\}.$$

In this cube a product Lebesgue measure can be introduced (see [1, p. 219]). There is another construction of a measure in Ω also called the Haar measure. The Haar measure is defined in locally compact topological groups. It was proven uniqueness of this measure if it is invariant in regard to transitions (see [2, p.241], [3, p. 303]). Many of measures could be considered as a Haar measure. Particularly, the product of Lebesgue measures in $[0, 1]^n$ is a Haar measure for any natural number n and, hence, is unique in this cube. Really, to prove this, consider the topological group R^n. The group $Z \oplus \cdots \oplus Z$ is a subgroup and the factor group $T_n = R^n/(Z \oplus \cdots \oplus Z)$ is locally compact. Let $A \subset [0, 1]^n$ be a measurable set in the product Lebesgue meaning in $[0, 1]^n$. Consider the union of intersections $(\bar{a}+A)\bigcap(\bar{m}+[0, 1)^n), \bar{m} \in Z^n$ for any given vector $\bar{a} \in R^n$. Only no more than 2^n of these intersections have non-zero measure, and the sum of their measures is equal to the measure of the set A. Therefore, the product measure is invariant in regard to the transitions $\bar{x} \mapsto \bar{x} + \bar{a}(mod1)$, $\bar{x} \in [0, 1)^n, \bar{a} \in R^n$. So, this is a unique Haar measure.

Despite that the most of told above are true for Ω, the situation is currently different in infinite dimensional case. We can define the Haar measure in Ω, as in the factor group, invariant in regard to transitions (mod 1). We get, then, a some unique measure defined in R^∞.

Consider now the unite cube Ω. In this case the "number" of non-empty intersections $(\bar{a}+A)\bigcap(\bar{m}+\Omega'), \bar{m} \in \prod_1^\infty Z,\ A \subset \Omega'$ (here $\Omega' : 0 \leqslant \omega_n < 1$) is non-countable. So, the product measure in Ω is not invariant in regard to transitions $\bar{x} \mapsto \bar{x}+\bar{a}(mod 1), \bar{x} \in \Omega', \bar{a} \in \Omega$. Therefore, in Ω it would be introduced the measure different from the Haar measure. Some of sets being measurable in the Haar meaning can stand now nonmeasurable. Moreover, another measure, different from product Lebesgue measure in Ω can be introduced also. The theorem below shows justness of this statement. To formulate the main theorem we have to introduce some designations.

We begin with studying of distribution of special curves of a kind $(\{t\lambda_n\})_{n\geqslant 1}$ (the sign $\{\}$ means a fractional part, and $\lambda_n > 0, \lambda_n \to \infty$ as $n \to \infty$) in the subsets of infinite dimensional unite cube. In the works [4,6,7,11,12] the finite case has been studied. It is necessarily to note that this curve has a zero measure in the product Lebesgue meaning. We below show that in Ω it can be introduced a *new measure* (we denote it μ_0) connected with metric, and in which the curve considered above acquires a property of nonmeasurability.

2 The basic result

Our basic result is formulated below.

THEOREM 1. *Let the sequence (λ_n) be an unbounded sequence of positive real numbers every finite subfamily of elements of which is linearly independent over the field of rational numbers. Then the curve $(\{t\lambda_n\}), t \in [0,1]$ is not μ_0-measurable set in Ω.*

DEFINITION 1. *Let $\sigma : N \to N$be any one to one mapping of the set of natural numbers. If for any $n > m$ there is a natural number m such that $\sigma(n) = n$, then we call σ a finite permutation. A subset $A \subset \Omega$ is called to be finite-symmetrical if for any element $\theta = (\theta_n) \in A$ and any finite permutation σ one has $\sigma\theta = (\theta_{\sigma(n)}) \in A$.*

Let Σ denote the set of all finite permutations. We shall define on this set a product of two finite permutations as a composition of mappings. Then Σ becomes a group which contains each group of n degree permutations as a subgroup (we consider each n degree permutation σ as a finite permutation, in the sense of definition 1, for which $\sigma(m) = m$ when $m > n$). The set Σ is a countable set and we can arrange its elements in a sequence.

In Ω we define the Tychonoff metric by following expression

$$d(x.y) = \sum_{n=1}^{\infty} e^{1-n} |x_n - y_n| .$$

Then (Ω, μ) becomes a compact metric space. At first we define, using usual product Lebesgue measure, volume of the ball of a radius $r > 0$ in $\Omega_0 = \{x = (x_n) \mid |x_n| \leqslant 1\}$:

$$B(0, r) = \{E \in \Omega_0 | d(x, 0) < r\} .$$

On this bases it may be introduced the measure in the Ω by known way by using of open sets. The open ball we define as an intersection $\Omega \bigcap B(\theta, r)$. An elementary set we define as a set being gotten by using of finite number of operations of unionize, taking differences or complements. It is clear that every elementary set is μ_0-measurable. The set of elementary subsets of Ω is an algebra of subsets. The σ-algebra of subsets in Ω can be defined by known way and the set function defined in the subsets' algebra can be extended to the σ-algebra (see [1, p. 152]). The outer and inner measures can be introduced by known way. Given any set in Ω, we call it measurable, if and only if, when it's outer and inner measures are equal (see also [2,p.16]). Defined measure will be, as it seen from the reasoning above, a regular measure, and outer measure of a set in the meanings of product and μ_0-meaning are the same. Our basic auxiliary result is a following lemma.

Lemma 1. *Let $A \subset \Omega$ be a finite-symmetric subset of zero measure and $\Lambda = (\lambda_n)$ is an unbounded, monotonically increasing sequence of positive real numbers any finite subfamily of elements of which is linearly independent over the field of rational numbers. Let $B \supset A$ be any open, in the Tychonoff metric, subset with $\mu_0(B) < \varepsilon$,*

$$E_0 = \{0 \leqslant t \leqslant 1 | \{t\Lambda\} \in A \wedge \Sigma'\{t\Lambda\} \subset B\}.$$

Then, we have$m(E_0) \leqslant c_0\varepsilon$ where $c_0 > 0$ is an absolute constant, m designates the Lebesgue measure.

The lemma 1 delivers the first fundamental difference. The main tool in the proof of this lemma is that fact that if we have some covering of a closed set by a union of a family of balls with finite total measure and none of which containing other then there is a finite number of balls only having with this set a nonempty intersection. This is somewhat different property than compactness, and the same property is not satisfied by cylindrical sets. Another difference stands clear after the theorem proved above. But in applications it is very important that every measurable set in a new meaning is measurable in the meaning of product measure.

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SOME FUNCTIONS ON FREE GROUPS

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The notions of pseudocharacters and quasicharacters are connected with the such questions and properties of groups as stability of equations on groups, width of verbal subgroups and groups of cohomologies. This notions were introduced by A.I. Shtern in 1983 ([1]). Questions about the existence of non-trivial pseudocharacters on different types of groups were considered in the work of R.I. Grigorchuk ([2, 3]), V.G. Bardakov ([4]), V.A. Faiziev ([5]), author ([6, 7]).

A pseudocharacter on a group G is a function f from the group G to the space of real numbers R such that

1) $|f(ab) - f(a) - f(b)| < \varepsilon$ for some positive number ε and for any $a, b \in G$

2) $f(a^n) = nf(a)$ for any $a \in G$ and $n \in Z$.

A pseudocharacter is called non-trivial if $\varphi(ab) - \varphi(a) - \varphi(b) \neq 0$ for some elements $a, b \in G$

R. I. Grigorchuk has raised questions about the existence of non-trivial pseudocharacters on groups with one defining relation, and two generators and also about special pseudocharacters on free groups.

Question 1 (Grigorchuk). *Let G be a non-amenable group with one defining relation. Is it true that there are non-trivial pseudocharacters on G?*

Question 2 (Grigorchuk). *Let F be a free group of rank ≥ 2, and $\alpha : F \longrightarrow F_0$ is an isomorphism on its subgroup $F_0 \in F$. Is it true that there exists a nonzero α-invariant (i.e. $f(\alpha(x)) = f(x)$) pseudocharacter on F?*

It is necessary to determine functions on free group F_n, which can be used for building pseudocharacters invariant with respect to endomorphisms of free group.

Conditions on endomorphisms of free group, under which such non-trivial pseudocharacters exist, are found. Let $F_n =< a_0, \ldots, a_{n-1} >$ be a free group. Consider these endomorphisms F_n, at which the generators of this group transform as follows:

$$a_0 \to a_1, a_1 \to a_2, \ldots, a_{n-2} \to a_{n-1}, a_{n-1} \to U_0(a_0, \ldots, a_{n-1}),$$

where U_0 is an element of a free group $U_0(a_0, \ldots, a_{n-1})$.

Consider irreducible form of the element U_0 in generators a_i. Irreducible form of element U_0 is divided into 3 parts $U_0 \equiv U_{01}U_{00}U_{02}$, according to a special rule. U_{00} is the part of the word U_0, which contains all letters a_0, lying in the U_0 and is bounded by them.

THEOREM 1. *Let $F_n =< a_0, \ldots, a_{n-1} >$ be the free group of rank $n > 1$, and let α be an endomorphism, which is defined on F_n, and in which*

$$a_0 \to a_1, \ldots, a_{n-2} \to a_{n-1}, a_{n-1} \to U_0(a_0, a_1, \ldots, a_{n-1}).$$

Let U_{00} is cyclically reduced. Suppose that irreducible form of U_{00} contains the letter $a_{n-1}^{\pm 1}$. Then on the free group F_n there exists a non-trivial pseudocharacter invariant with respect to the endomorphism α.

THEOREM 2. *Let $F_n =< a_0, \ldots, a_{n-1} >$ be the free group of rank $n > 1$, and let α be an endomorphism, which is defined on F_n, and in which*

$$a_0 \to a_1, \ldots, a_{n-2} \to a_{n-1}, a_{n-1} \to U_0(a_0, a_1, \ldots, a_{n-1}).$$

Let U_{00} is cyclically reduced. Suppose also that U_0 contains $a_{n-1}^{\pm 2}$ with additional conditions $U_0 \neq a_0^{\pm 1}$, U_0 is cyclically reduced. Then on the free group F_n there exists a non-trivial pseudocharacter invariant with respect to the considered endomorphism α.

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ON THE ANTIVARIETY OF UNARY ALGEBRAS SATISFYING THE IDENTITY $fg(x) = x$.

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An algebra $\mathcal{A} = \langle A, \Omega \rangle$ is called *unary* if its signature Ω consists of unary symbols.

Unary algebras play an important role in algebra due to their applications and numerous associations with other parts of this area of mathematics. For example, unary algebras are identical to outputless automatons. They also can be interpreted as orgraphs. This attracts to them the attention of specialists in Russia and abroad.

A significant place in investigation of unary algebras have algebras and classes of algebraic systems which connected with given algebras in some special way. They are semigroups of endomorphisms, groups of automorphisms, subalgebra lattices, congruence lattices, topology lattises, and so on ([1], [2]). A range of important conclusive results has been obtained for unars (i. e. algebras with one unary operation) on problems from [1], [2].

Theory of unary algebras with more than one operation turned out to have a tangible specific character ([3], [4]). Therefore, results for unars not always can be used for effective construction of hypotheses in general theory of unary algebras. So the investigation of unary algebras with two unary operations is becoming essential.

Nowadays algebras of variety $\mathcal{A}_{1,1}$ defined by the identity $fg(x) = gf(x) = x$, where f, g are functional unary symbols, are the most deeply investigated among algebras with two unary operations (see, e.g., [5]).

In this paper we consider variety $\mathcal{B}_{1,1}$ of all algebras satisfying the identity $fg(x) = x$. This variety has been investigated in [6], [7].

The following results have been obtained by the author.

Theorem 1. *The variety $\mathcal{B}_{1,1}$ is a $\mathcal{A}_{1,1}$ cover in the lattice of all varieties of algebras with two unary operations.*

A unary algebra is called *strongly connected* if it is generated by its any element.

Theorem 2. *For any strongly connected algebra $\mathfrak{A}$ from variety $\mathcal{B}_{1,1}$ the following equation is valid*

$$End\mathfrak{A} = Aut\mathfrak{A}.$$

(where $End\mathfrak{A}$ and $Aut\mathfrak{A}$ are respectively the semigroup of endomorphisms and the group of automorphisms of $\mathfrak{A}$).

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ON CONGRUENCE LATTICES AND TOPOLOGY LATTICES OF COMMUTATIVE UNARY ALGEBRAS

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Let $\mathcal{A} = \langle A, \Omega\rangle$ be an algebra and σ a topology on the set A. A n-ary operation $F \in \Omega$ is called *continuous* with respect to σ if for any elements $a_1, a_2, \ldots, a_n \in A$ and a neighborhood U of the element $F(a_1, a_2, \ldots, a_n)$ there are neighborhoods $U_1, U_2, \ldots, U_n$ of the elements $a_1, a_2, \ldots, a_n$ respectively such that $F(U_1, U_2, \ldots \ldots, U_n) \subseteq U$. If any signature operation of $\mathcal{A}$ is continuous with respect to σ, then σ is said to be *a topology on the algebra* $\mathcal{A}$. It's not difficult to see that such topologies form a complete lattice under inclusion. This lattice will be called *the topology lattice of the algebra* $\mathcal{A}$ and denoted $\Im(\mathcal{A})$. As usual, the congruence lattice of $\mathcal{A}$ is denoted by $Con\mathcal{A}$.

Let $\widetilde{Con}\mathcal{A}$ be the dual lattice to $Con\mathcal{A}$. Then $\widetilde{Con}\mathcal{A}$ is isomorphic to a sublattice of $\Im(\mathcal{A})$ (see [1]).

A unary algebra $\langle A, \Omega\rangle$ is called *commutative* if $f(g(a)) = g(f(a))$ for all $f, g \in \Omega$, $a \in A$.

A commutative unary algebra is finite if and only if its congruence (topology) lattice is finite ([2]). There are infinite noncommutative unary algebras with finite congruence lattices and finite topology lattices (see [2]).

The class of all commutative unary algebras with linearly ordered congruence lattices is characterized in [3]. In [2] the class of all commutative unary algebras with linearly ordered topology lattices is described.

In this paper we describe commutative unary algebras whose nontrivial congruences (topologies) form an antichain.

A unary algebra is called *strongly connected* if it is generated by its any element.

We'll say that an algebra $\mathfrak{A}' = \langle A', \Omega'\rangle$ is obtained from $\mathfrak{A} = \langle A, \Omega\rangle$ *by the addition of loop* e if the following conditions are valid

1) $e \notin A$, $A' = A \cup \{e\}$, $\Omega \subseteq \Omega'$;

2) $\mathfrak{A}$ is a subalgebra of the reduct $\langle A', \Omega\rangle$ of $\mathfrak{A}'$;

3) $(\forall f \in \Omega')(f(e) = e)$;

4) $(\forall f \in \Omega' \setminus \Omega)(f(A) = \{e\})$.

Let us note, that if $\Omega = \Omega'$ then $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ by the addition of e as a new connected component.

A monogenic algebra $\langle A, f\rangle$ with a generator a and defining relation $f^n(a) = f^{n+m}(a)$, where $n \geqslant 0$, $m > 0$ is denoted by C^n_m.

THEOREM 1. *All nontrivial congruences of an arbitrary algebra $\mathfrak{A} = \langle A, \Omega\rangle$ form an antichain if and only if at least one of the following conditions is valid:*

1) $|A| \leqslant 3$;

2) $\mathfrak{A}$ is a strongly connected algebra whose order is equal to pp_1 or p^k, where p, p_1 are prime numbers, $0 \leqslant k \leqslant 2$;

3) $\mathfrak{A}$ can be obtained from a some strongly connected algebra of a prime number by the addition of a loop;

4) there is a operation $f \in \Omega$ such that the reduct $\langle A, f\rangle$ of $\mathfrak{A}$ is isomorphic to an algebra C^1_p, where p is a prime number.

Theorem 2. *All nontrivial topologies of an arbitrary algebra $\mathfrak{A} = \langle A, \Omega \rangle$ form an antichain if and only if at least one of the following conditions is valid:*
1) $|A| \leqslant 2$;
2) $\mathfrak{A}$ is a strongly connected algebra whose order is equal to pp_1 or p^k, where p, p_1 are prime numbers, $0 \leqslant k \leqslant 2$.

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ON ALGEBRAS OF SUBSETS OF UNARS

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Let $\mathfrak{A} = \langle A, f \rangle$ be an arbitrary unar, i. e. an algebra with one unary operation f. Suppose P(A) is the set of all subsets of A and

$$f(B) = \{f(b) | b \in B\}$$

for any $B \in P(A)$. Then $\langle P(A), f \rangle$ is called *the unar of subsets* of $\mathfrak{A}$.

Theorem 1. *A variety generated by a unar coincides with the variety generated by its unar of subsets.*

The union of two disjoint unars $\mathfrak{B}$ and $\mathfrak{D}$ is denoted by $\mathfrak{B} + \mathfrak{D}$. If $\mathfrak{A} = \mathfrak{B} + \mathfrak{D}$ and $B \neq \emptyset$, then the unar $\mathfrak{B}$ is called *a component* of $\mathfrak{A}$. A unar is called *connected* if it doesn't have proper components (see, e.g. [1]).

Let $\mathbb{N}$ be the set of all positive integers, $\langle A, f \rangle$ a unar, $a \in A$, and $n \in \mathbb{N}$. Then the result of n-tuple application of f to a is denoted by $f^n(a)$.

A unar $\langle A, f \rangle$ is said to be *a cycle of length n* ($n \in \mathbb{N}$) if $\langle A, f \rangle$ is a monogenic unar and $f^n(a) = a$, $f^k(a) \neq a$ for all $a \in A$, $0 < k < n$.

For any $k, n \in \mathbb{N}$ we'll write $k|n$ if n is divisible by k.

Theorem 2. *Let $\mathfrak{A}$ be a finite connected unar with a cycle of length n. Then the number of all connected components of the unar of subsets of $\mathfrak{A}$ is equal to $\sum\limits_{k|n, d|k} \frac{\mu(d) 2^{\frac{k}{d}}}{k}$, where $\mu : \mathbb{N} \to \mathbb{N}$ is Mobius function.*

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CONGRUENCES OF RIGHT ACTS OVER RIGHT AND LEFT ZERO SEMIGROUP

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A right act over a semigroup S (or a *right S-set*; see [1]) is a set X with a mapping $X\times S \to X$, $(x,s)\mapsto xs$ such that the axiom $x(st)=(xs)t$ is held for $x\in X$, $s,t\in S$. A left act is defined dually. *A congruence of the (right) act X* over the semigroup S is an equivalence relation ρ on X such that $(x,y)\in\rho \Rightarrow (xs,ys)\in\rho$ for all $x,y\in X$, $s\in S$. The least congruence is the relation of equality $\Delta_X=\{(x,x)\,|\,x\in X\}$.

A *coproduct* $\coprod_{i\in I} X_i$ of a family of acts $\{X_i\,|\,i\in I\}$ over the semigroup S is the disjoint union of these acts (if X_i have a non-empty intersections then we take their isomorphic disjoint copies). An act X is called *co-indecomposable* if it is not a non-trivial coproduct of the acts.

Theorem 1. *Every act is a coproduct of co-indecomposable acts.*

A *main congruence* $\rho_{a,b}$ of act X over a semigroup S is the congruence generated by a pair (a,b) where $a,b\in X$ and $a\neq b$.

We want to describe the main congruences of arbitrary right act over a right zero semigroup. For this we need the description of these acts obtained in [2], we give a formulation of that Theorem.

Theorem 2. *([2], Corollary 10). Let X and S be nonempty sets, σ be an equivalence relation on X. Further, the subsets $Y_s\subseteq X$ are given for each $s\in S$ such that $|Y_s\cap x\sigma|=1$ for all $x\in X$, $s\in S$ (here $x\sigma$ denotes σ-class containing the element x). Put $st=t$ for all $s,t\in S$ and $xs=y$ where y is the unique element of the set $Y_s\cap x\sigma$ for $x\in X$, $s\in S$. Then S is a right zero semigroup, and X is a right S-set. Moreover, any right act over the right zero semigroup is isomorphic to an act constructed by this way.*

Let S be a right zero semigroup, X be a right act over the S. Let σ and Y_s $(s\in S)$ have the same meaning as in the previous theorem. The unique element of the set $Y_s\cap xs$ we denote by x_s. Let K, K' be different classes of the equivalence relation σ and a_s be the unique element of the set $Y_s\cap K$, a'_s be the unique element of the set $Y_s\cap K'$. Denote by $\rho_{K,K'}$ the least equivalence relation on the set X containing all the pairs (a_s,a'_s) for $s\in S$. Differently $\rho_{K,K'}$ can be described in the language of graphs theory. Consider bipartite graph Γ whose vertex set is the set $K\cup K'$ and the set of edges is the set of pairs (a_s,a'_s), $s\in S$. The equivalence classes of the relation

$\rho_{K,K'}$ are exactly the connected components of the graph Γ. The following theorem describes the main congruences on a right act over a right zero semigroup.

THEOREM 3. *Let S be a right zero semigroup, X be a right act over S (its structure is described in Theorem 2). Let $a, b \in X$ and $a \neq b$. Then the common view of the main congruences is so:*

$$\rho_{a,b} = \begin{cases} \{(a,b),(b,a)\} \cup \Delta_X, \text{ if } (a,b) \in \sigma, \\ \{(a,b),(b,a)\} \cup \rho_{K,K'} \cup \Delta_X \text{ if } a \in K,\ b \in K' \text{ and } K,\ K' \text{ are} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{different } \sigma\text{-classes.} \end{cases}$$

Let us again consider an arbitrary act X over a right zero semigroup S. In view of the Theorem 1 we have $X = \coprod_{i \in I} X_i$ where X_i are the co-indecomposable acts. Note that X_i are exactly the classes of the relation σ appearing in Theorem 2. Let Y_s have the same meaning as in this Theorem. The unique element of the set $X_i \cap Y_s$ we will denote by a_{is}. For any subset $J \subseteq I$ let ρ_J be the least equivalence relation on the set $\bigcup_{j \in J} X_j$ containing all the pairs (a_{is}, a_{js}) where $i, j \in J$, $s \in S$. Note that the relation ρ_J generalizes the relation $\rho_{K,K'}$ which was used to construct the main congruences.

THEOREM 4. *([3]). Let X be an act over a right zero semigroup S, $X = \coprod_{i \in I} X_i$ be its decomposition into a coproduct of co-indecomposable acts. Let Y_s ($s \in S$) and σ have the same meaning as in Theorem 2. Consider an arbitrary equivalence relation τ on the set I. Let $I = \bigcup\{J_\alpha \mid \alpha \in \Omega\}$ be a decomposition of the set I into classes of the equivalence relation τ. For each $\alpha \in \Omega$ let us take any equivalence relation ρ'_α on the set $\bigcup\{X_i \mid i \in J_\alpha\}$ satisfying the condition $\rho'_\alpha \supseteq \rho_{J_\alpha}$. Then $\rho = \bigcup\{\rho'_\alpha \mid \alpha \in \Omega\}$ is a congruence of the act X. Conversely, each congruence of the act X has this form.*

The following theorem is a reformulation of the Corollary 10 in [2]. It is needed for us to describe the structure of any right act over a right zero semigroup.

THEOREM 5.*Let X be a set, $Y \subseteq X$ be a nonempty subset, $A = X \backslash Y$ (it is possible that $A = \emptyset$), $\{\varphi_s \mid s \in S\}$ be a family of mappings $\varphi_s : A \to Y$. Put $st = s$ for all $s,\ t \in S$, $as = \varphi_s(a)$ for $s \in S$, $a \in A$ and $ys = y$ for $y \in Y$, $s \in S$. Then S is a left zero semigroup, $X = Y \cup A$ is a right act over the S. Moreover, any right act over the left zero semigroup can be constructed by this way.*

The congruence of any right act over a left zero semigroup may be desribed as follows::

THEOREM 6.*Let X be a right act over a left zero semigroup S. Let the sets Y, A and the mapping $\sigma_s : A \to Y$ (for $s \in S$) have the same meaning as in the previous theorem. Take any equivalence relation σ on the set Y. For $s \in S$ let $\varphi_s^{-1}(\sigma) = \{(a,b) \mid (\varphi_s(a), \varphi_s(b)) \in \sigma\}$, $\widetilde{\sigma} = \bigcap_{s \in S} \varphi_s^{-1}(\sigma)$. For each class K of the relations σ let $A_K = \bigcap_{s \in S} \varphi_s^{-1}(K)$ (this set can be empty). Take the subsets $A'_K \subseteq_K A_K$ and put $Z_K = K \bigcup A'_K$. Let σ' be an any equivalence relation on the set $A \backslash \cup_K A'_K$, contained in $\widetilde{\sigma}$. Then $\rho = \bigcup_K (Z_K \times Z_K) \cup \sigma'$ be a congruence of act X. Moreover, any congruence ρ of act X, for which $\rho|_Y = \sigma$, built so.*

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UDC 511.3

NEAR-WALL FLOWS WITH IRREGULAR BOUNDARIES[1]

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Vortical structure of fluid flows is a determining factor when moving a body in water or in air as well as when operating hydraulic systems. A lot of important technical problems in fluid dynamics connect with optimal transformation of flow vortical pattern near a body. The control strategy may be directed as on creating regular recirculation zones in near-wall flow as on its destruction to intensify flow mixing in the region. In both cases, "intellectual"flow of fluid is created, in which the vortices have been formed according to the control scheme and either theoretical or semiempirical model predicting the vortex behavior [1]. Control technics will be more effective, if one takes into account topological properties of the flow under consideration. Here we use the standing vortex model to derive near-wall flow topology when the local recirculation zone is simulated by a vortex of finite circulation [2]. Motion of the vortex is described by a set of non-linear differential equations:

$$\frac{dx_v}{dt} = v_x(x_v, y_v, t), \quad \frac{dy_v}{dt} = v_y(x_v, y_v, t), \tag{1}$$

where x_v, y_v are the vortex coordinates and v_x, v_y are the components of the vortex velocity.

To build the velocity field in the region, we map the physical plane z into an upper half-plane of the auxiliary plane ζ. Then the complex flow potential is the superposition of that of uniform flow and Green's function of the vortex near a flat

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wall. If the function $\zeta = f(z)$ realizes the conformal mapping, the following condition of vortex equilibrium may be used to derive the critical points of flow:

$$\left(\left. \frac{d\Phi_0}{d\zeta} \right|_{\zeta=\zeta_v} + \frac{\Gamma_v}{4\pi\eta_v} \right) \left. \left[\left(\frac{df}{dz} \right)^2 \Big/ \left(\frac{d^2 f}{dz^2} \right) \right] \right|_{\zeta=\zeta_v} - \frac{i\Gamma_v}{4\pi} = 0, \tag{2}$$

where Γ_v is the vortex circulation, $\zeta_v = (\xi_v, \eta_v)$ and $\Phi_0(\zeta)$ are the vortex coordinate and uniform flow potential in ζ-plane respectively. From Eq. (2), two transcendental equations for determining the standing vortex coordinates are derived. To calculate the vortex circulation, this set has to be completed by an equation that follows from physical conditions of the problem under consideration. For example, if the flow boundary has a sharp edge, the unsteady Kutta condition can be involved.

The model was applied to studying the standing vortex dynamics in the angular region and in a cross groove. The dependencies of the standing vortex circulation Γ_v on the representative parameter of boundary irregularity were derived. It was found also that standing vortex is characterized by its eigenfrequency which governs the dynamic behavior of the vortex in the periodically perturbed flow. Periodic oscillations of the flow velocity cause multi periodic large amplitude motion of the standing vortex. The maximal amplitude of deviation of the vortex from its stationary point depends on the external perturbation frequency in resonance manner. Resonance flow perturbations in the regions bounded non-regular wall cause intensification of fluid mixing in recirculation zones. They stimulate generation of vorticity in sharp boundary edges, lead to chaotization of motion of both fluid particles and small vortices, cause non-regular fluctuations of the flow.

The report based on the results of joint work "Dynamics of Vortices in Near-wall Flows With Irregular Boundaries"with I.M. Gorban [3].

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ON RADICALS OF NILPOTENT ALGEBRAS

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Suppose F is a partially ordered field and $L = \langle L; +; \cdot \rangle$ is a linear algebra over a field F. An algebra L over a partially ordered field F is called an algebra with a lattice $\mathcal{K}$-*order* $\leqslant$ (see, for example, [1]) or a *lattice* $\mathcal{K}$-*ordered algebra* if the following conditions hold:

(1) $\langle L; +; \leqslant \rangle$ is a lattice ordered group [2];

(2) if $x \leqslant y$, then $\gamma x \leqslant \gamma y$ for all elements $x, y \in L$ and $\gamma > 0$, $\gamma \in F$;

(3) from $x \geqslant 0$ it follows that $x + xy \geqslant 0$ and $x + yx \geqslant 0$ for all elements $y \in L$.

The notion of this order was defined in 1972 by V.M. Kopytov for Lie algebras (see [3]). Moreover, in [3] V.M. Kopytov noted that this definition of an order can be stated for arbitrary algebras over ordered fields. This concept was introduced by the author and by E.E. Shirshova in [1], [4]. Namely, in these papers it was investigated a generalization of a notion of a Kopytov's order to linear algebras over ordered fields.

In particular, in [4] it was proved that a necessary and sufficient condition for a finite-dimentional associative algebra (a Lie algebra) over a linearly ordered field to be a lattice $\mathcal{K}$-orderable algebra is that this algebra be a nilpotent algebra.

Recall that an l-ideal I of a lattice $\mathcal{K}$-ordered algebra L over a partially ordered field F is called an *l-prime ideal* if for any nonzero l-ideals A and B in the factor–algebra L/I the product $AB = \{x = \sum\limits_{i=1}^{n=n(x)} a_i b_i \mid a_i \in A, b_i \in B\}$ is not equal to the set $\{I\}$ (see [1]). The intersection of all l-prime ideals in L is said to be the *l-prime radical* of L (see [4]).

In this paper, following [5], we say that algebra is a $\mathfrak{B}$-*radical algebra* if this algebra and its prime radical are equal. Using results of [4], we shall give the following definition. A $\mathcal{K}$-ordered algebra is called a $\mathfrak{B}_l$-*radical algebra* if its l-prime radical is equal to this algebra.

Theorem 1. *For any associative algebra (a Lie algebra) A over a linearly ordered field such that A is a finite–dimentional nilpotent algebra it follows that A is a $\mathfrak{B}$-radical algebra and A is a $\mathfrak{B}_l$-radical algebra.*

Moreover, any finite–dimentional nilpotent associative algebra A is a radical by Jacobson algebra.

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THE ABSOLUTE JACOBSON RADICAL OF ABELIAN MT-GROUPS

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A multiplication on an abelian group G is a homomorphism $\mu : G \otimes G \to G$. A group G with a multiplication determined on it is called a ring on G. A mixed abelian group G is called a MT-group, if every multiplication on its torsion part can be uniquely extended to a multiplication on G. MT-groups were introduced in [1] and were studied later by many algebraists.

In [2] L.Fuchs formulated the problem [2, problem 94] of describing absolute radicals of abelian groups. An absolute Jacobson radical $J^*(G)$ of an abelian group G is the intersection of the Jacobson radicals of all associative rings on G. In this work a description of the absolute Jacobson radical of an abelian MT-group will be given.

Let G be an abelian group, and let $\Lambda(G)$ be the set of all primes p such that the p-primary component of G is nonzero.

THEOREM 1. *Let G be an abelian MT-group. Then $J^*(G) = \bigcap\limits_{p\in\Lambda(G)} pG$. Moreover, there exists an associative and commutative ring on G, whose Jacobson radical is $\bigcap\limits_{p\in\Lambda(G)} pG$.*

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ON SOME NUMERIC EXPERIMENTS CONSERNING BEHAVIOR OF DIRICHLET SERIES WITH PERIODICAL COEFFICIENTS IN THE CRITICAL STRIP

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In this report results of some numeric experiments are given. In these experiments zeroes and behavior of a module of certain functions defined by Dirichlet series on a critical line are examined. These experiments are based on a numeric algorithm of construction of Dirichlet polynomials that approximate Dirichlet series with exponential speed. These algorothm was developed in paper [1].

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ONE PROPERTY OF COXETER GROUPS

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We consider a finitely generated Coxeter group

$$G = \langle a_1, \ldots, a_n;\ a_i^2, (a_i a_j)^{m_{ij}}, i, j \in \overline{1, n}, i \neq j \rangle,$$

where m_{ij} are elements of symmetric Coxeter matrix, $\forall i, j \in \overline{1,n}$ $m_{ii} = 1$, $m_{ij} \geqslant 2\,(i \neq j)$.

One may construct graph Γ such that each pare of its vertices corresponds to some generators a_i and a_j $(i \neq j)$ of G, and so, the edge corresponds to the relation $(a_i a_j)^{m_{ij}} = 1$. If we obtain finite connected tree Γ in that way, group G is *Coxeter group with tree structure.*

It's obvious that G is also a tree product

$$G = \left\langle \prod_s *G_s;\ \mathrm{rel}G_1, \ldots, \mathrm{rel}G_s, \ldots;\ a_i = a_i' \right\rangle,$$

where each factor G_s is two-generated Coxeter group like $\langle a_i, a_j;\ a_i^2, a_j^2, (a_i a_j)^{m_{ij}} \rangle$ and isomorphisms $a_i = a_i'$ amalgamate cyclic subgroups of order 2 like $\langle a_i;\ a_i^2 \rangle$.

Various algorithmic problems of Coxeter groups with tree structure are discussed in papers of V. N. Bezverkhnii and O. V. Inchenko. In papers [1, 2] a generalization of Nielsen method (look [3]) is used to study free products with amalgamation and HNN-extentions. In [1] they prove the following

Theorem 1. *The problem of cosets intersection of finitely generated subgroups H_1 and H_2 is solvable in the class of Coxeter groups with tree structure and there exists an algorithm enumerating generators of the intersection $w_1H_1 \cap w_2H_2$.*

Definition 1 ([4]). *We say that group G satisfies R-property if the normalizer of each finitely generated subgroup H of G is finitely generated.*

The author of [4] also uses Nielsen method to examine free products. In particular, she proves the following

Theorem 2. *Suppose that factors A_i of free product $G = A_1 * A_2$ satisfy R-property. Then G inherits this property.*

Using the same technique we assert the next fact:

Theorem 3. *A normalizer $N_G(H)$ of finitely generated subgroup H of finitely generated Coxeter group G with tree structure is finitely generated as well.*

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ON THE NUMBER OF VECTORS WITH REAL ALGEBRAIC COORDINATES NEAR SMOOTH MANIFOLDS OF THE FORM $U = F(X, Y, Z)$

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We use a variant of the *essential* and *inessential* domains method [1] for solution the problem indicated in the name of our paper. Note that the result on the uniform distribution of the corresponding vectors near smooth manifold follows from it.

We consider a polynomial $P = P(t) = a_n t^n + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$, $n \geqslant 4$, $a_n \neq 0$, at the point $t \in \mathbb{R}$ when a height $H(P) = \max(|a_n|, \ldots, |a_0|)$ is increased and n is fixed. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $P(t)$ and $\mu_i > 0$ $(i = 1, 2, 3, 4)$ are fixed numbers. Take a parallelepiped $\mathcal{T} = \prod_{i=1}^{4} I_i = \prod_{i=1}^{4}[a_i, b_i] \subset [-1/2, 1/2]^4$ where $|I_i| = b_i - a_i = Q^{-\mu_i}$ when $Q > Q_0 > 0$ and a set $\mathcal{M} = \{(x_1, x_2, x_3, x_4) \in \mathcal{T} : |x_i - x_j| < 0,01,\ i \neq j, \}$. Suppose that $\mathcal{T}_1 = \mathcal{T} \backslash \mathcal{M}$.

Introduce a class of polynomials $\mathcal{P}_n(Q) = \{P : |a_n| \gg H(P), H(P) \leqslant Q\}$. Let $\mathcal{A}_n(\mathcal{T}_1, Q)$ is a vector set of $\overline{\alpha} = (\alpha_i, \alpha_j, \alpha_k, \alpha_l)$, $1 \leqslant i < j < k < l \leqslant n$ containing the roots of P, $P \in \mathcal{P}_n(Q)$ such that: **1)** $\alpha_i, \alpha_j, \alpha_k, \alpha_l \in \mathbb{R}$, **2)** $\overline{\alpha} \in \mathcal{T}_1$. Hence, the taken roots are distinct. We prove

Theorem 1. *If* $0 < \mu_i < 1/4$ $(i = 1, \ldots, 4)$ *then*

$$\sharp\mathcal{A}_n(\mathcal{T}_1, Q) \gg Q^{n+1-\mu_1-\mu_2-\mu_3-\mu_4}.$$

The proof of Theorem 1 is based on the construction of special integral polynomials with the following conditions: **1)** the values $|P(t)|$ are small when $(t, t, t, t) = (x_1, x_2, x_3, x_4) \in B \subset \mathcal{T}_1$ and a measure $|B|$ is greater than $\frac{1}{2}$ of the measure $|\mathcal{T}_1|$, **2)** $|P^{'}(t)| \asymp H(P) = Q$ at the points of the set B.

From Theorem 1 we obtain

Theorem 2. *Let* $u = f(x, y, z)$ *is a continued function at a parallelepiped* $\mathcal{K} = \prod_{i-1}^{3} K_i \subset [-1/2, 1/2]^3$. *Suppose that* $\mathcal{J}(Q, \lambda) = \{(x, y, z, u) : x \in K_1,\ y \in K_2,\ z \in K_3,\ |u - f(x, y, z)| < Q^{-\lambda},\ 0 < \lambda < 1/4\}$. *Then there are* $\geqslant c(n)Q^{n+1-\lambda}$ *of the vectors* $\overline{\alpha}$ *in* $\mathcal{A}_n(\mathcal{T}_1, Q)$ *such that* $\overline{\alpha} \in \mathcal{J}(Q, \lambda)$ *where* $c(n) > 0$ *is a constant depending only on* n.

For proving of Theorem 1 we use the *essential* and *inessential* domains method by Sprindžuk which is developed and improved by the representatives of the Number Theory schools in Byelorussian Academy of Sciences (Minsk, Belarus) and University York (York, UK) (1980 — 2014).

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DIAGONAL ACTS

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A right diagonal act over the semigroup S is a set $S \times S$ where S acts as follows: $(a, b)s = (as, bs)$ for $a, b, s \in S$. A left diagonal act over S and a diagonal bi-act over S are defined analogously. A subset $A \subseteq S \times S$ is called *a generating set* if $AS^1 = S \times S$. A right diagonal rank of a semigroup S (denoted $\operatorname{rdr} S$) is the least cardinality of generating sets of $(S \times S)_S$. A left diagonal rank $\operatorname{ldr} S$ and a bi-diagonal rank $\operatorname{bdr} S$ are defined analogously. Along with a right diagonal act we define *a right diagonal act of order* n as $(S^n)_S$. The least cardinality of generating sets of this act we call *a right diagonal rank of order* n *of the semigroup* S and denote by $\operatorname{rdr}_n S$.

Attention to diagonal acts was caused by a question from [1]: are there infinite semigroups which have a finite right diagonal act? It has turned out that there are plenty of such semigroups. For example, in [2] it was shown that the transformations monoid $T(X)$ of an infinite set X, the monoid of partial transformations $P(X)$ and the monoid of binary relations $B(X)$ have cyclic right, left and bi-diagonal acts. Papers on diagonal acts were mostly devoted to conditions on semigroups to have cyclic or finitely generated diagonal acts. In the paper [3] flatness properties of diagonal act were studied.

Obviously, $\operatorname{bdr} S \leqslant \operatorname{rdr} S, \operatorname{ldr} S$. Besides this inequality, diagonal ranks are independent from one another. So, it was shown in [2] that the monoid $I(X)$ of partial bijections of an infinite set X has the following property: $\operatorname{rdr} I(X) = \operatorname{ldr} I(X) = \infty$, $\operatorname{bdr} I(X) = 1$ which shows a relative independence of one-sided diagonal ranks from a bi-diagonal rank. Let X be an infinite set, $F(X)$ be a monoid of full finite-to-one transformations (that is, no infinite subset is mapped to a single point), multiplication of maps is from left to right. It was proved in [2] that $\operatorname{rdr} F(X) = 1$ but $\operatorname{ldr} F(X) = \infty$. This shows a mutual independence of right and left diagonal ranks.

A lot of known at that time results on cyclic and finitely generated diagonal acts are presented in [4]. We've extended a table of results from [4] with new results, which are marked with stars.

However, right diagonal ranks of different orders are not independent. Consider the following theorem.

Semigroup S	Non-trivial S, cyclic right act?	Infinite S, f.g. right act?	Non-trivial S, cyclic bi-act?	Infinite S, f.g. bi-act?
Finite	No	–	No	–
Commutative	No	No	No	No
Idempotent	No	No	No	No
With an identity	No	No	???	???
Inverse	No	No	Yes	Yes
Completely regular	No	No	No	No
Completely simple	No	No	No	Yes
Completely 0-simple	No	No	No	Yes
Group	No	No	No	Yes
Cancellative	No	No	No	Yes
Right cancellative	No	No	Yes*	Yes
Left cancellative	No	No	Yes*	Yes
Bruck-Reilly extension	No	No	No	No
Epigroup	No*	No*	No*	Yes
Periodic	No*	No*	No*	Yes
Locally finitely	No	No	No*	No*
Right invariant with 1	No*	No*	No	Yes
Left invariant with 1	No*	???	No	Yes

Таблица 1: Summary of results

Theorem 1. *Let S be an infinite semigroup. Then* $\operatorname{rdr}_n S \leq (\operatorname{rdr} S)^n$ *for an odd* n *and* $\operatorname{rdr}_n S \leq (\operatorname{rdr} S)^{n-1}$ *for an even* n*.*

The following theorem generalizes a number of known before results.

Theorem 2 ([7], theorem 4.8). *Let S be an infinite semigroup satisfying some non-trivial semigroup identity. Then the right diagonal act over S is not finitely generated.*

Under some natural conditions the rank of the Cartesian product of two semigroups is equal to the product of their ranks.

Theorem 3 ([7], theorem 2.6). *Let S and T be semigroups each of which satisfies one of the following:*

(i) infinite with a finite right diagonal rank;

(ii) finite with a right identity.

Then $\operatorname{rdr}(S \times T) = \operatorname{rdr} S \cdot \operatorname{rdr} T$.

This theorem gives us a way to construct infinite semigroups of any diagonal rank. Indeed, consider the Cartesian product of an infinite semigroup S such that $\operatorname{rdr} S = 1$ (say, $T(X)$) and a group of needed order.

An example of a semigroup such that $\operatorname{bdr} S = 1$, $\operatorname{bdr}_3 S = \infty$ was presented in [8].

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ABELIAN GROUPS WITH FINITELY APPROXIMATED ACTS

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A *right act* over a semigroup S (or a right S-act) is a set X with an action of the semigroup S on X, i.e. there is a mapping $X \times S \to X$, $(x, s) \mapsto xs$ satisfying the condition $x(st) = (xs)t$ for all elements $x \in X$, $s, t \in S$ (see [5]). As we shall not the left acts consider then we shall call the right acts simply acts.

The category of acts over a semigroup contains a great information on the structure of the semigroup, it is similar to the fact that the category of modules over a ring has a lot to say about the ring. In the works [1]–[4] the semigroups S were investigated satisfying the following conditions on acts:

$(*)$ *all the right S-acts are finitely approximated*;

$(**)$ *all the right$S$-acts are approximated by the acts of n or less number of elements.*

It was proved in [1] that a semigroup S satisfies the condition $(**)$ with $n = 2$ if and only if S is a semilattice (a commutative idempotent semigroup). In [2] it was be established the periodicity of the semigroups satisfying the condition $(**)$, and in [4] this assertion was strengthened: it turned out that such semigroup is uniformly locally finite, i.e. for every natural number t the orders of the t-generated subsemigroups are bounded in collection. In [2] and [3] the commutative and nil semigroups with the conditions $(*)$ and $(**)$ were investigated.

Note that in view of the Birkhoff's Theorem (which states that *every algebra is a subdirect product of subdirectly indecomposable algebras*), for every semigroup S the condition $(*)$ is equivalent to the following condition:

$(*')$ *all the subdirectly indecomposable S-acts are finite*;

and the condition $(**)$ is equivalent to the condition

$(**')$ $|X| \leqslant n$ *for every subdirectly indecomposable S-act X.*

We will describe the abelian groups satisfying the conditions $(*)$ and $(**)$. We use the additive record for them. For an abelian group A, we denote by $t(A)$ its torsion part, by $t_p(A)$ the p-component of the torsion part, by $r_0(A)$ the torsion free rank (if A is a torsion free abelian group then the torsion free rank is called simply the rank), and by $\exp(A)$ we call the exponent of the group A, i.e. the least common multiple of the orders of its elements (the exponent exists if and only if A is a bounded group, i.e. the orders of elements of the group A are bounded in collection).

Theorem 1. *An abelian group A satisfies the condition $(**)$ for some n if and only if the orders of elements of the group A are bounded in collection. If* $\exp(A) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ *where p_i are distinct prime number then the orders of subdirectly indecomposable unitary (resp., non-unitary) A-acts are exactly the numbers of view p_i^{β} (resp., $p_i^{\beta} + 1$) where $i \leqslant k$ and $\beta \leqslant \alpha_i$.*

Proposition 1. *A torsion abelian group satisfies the condition $(*)$ if and only if all its p-components are bounded groups.*

An abelian group A is called *locally free* if all its p-localizations $\mathbb{Q}_p \otimes A$ are free $\mathbb{Q}_p$-modules (here $\mathbb{Q}_p$ is the ring of all rational numbers whose denominators are coprime with p). It is well known that torsion free abelian group A of finite rank is locally free if and only if $\dim_{\mathbb{Z}_p}(A/pA) = r(A)$ for all prime numbers p.

Proposition 2. *A torsion free abelian group satisfies the condition $(*)$ if and only if it is a torsion free locally free group of finite rank.*

Theorem 2. *An arbitrary abelian group A satisfies the condition $(*)$ if and only if the groups $t(A)$ and $A/t(A)$ satisfy the condition $(*)$.*

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GRADED QUASI-FROBENIUS RINGS

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Quasi-Frobenius rings are the right self-injective rings with the minimum condition. The class of quasi-Frobenius rings contains semisimple rings with minimum condition and all group algebras of finite groups (not necessarily semisimple) [1, 2]. Special role quasi-Frobenius rings and modules in the theory of linear codes over rings and modules is shown in [3].

Everywhere bellow G is a multiplicative group with an identity element e, all considered rings are the associative G-graded rings with unit 1, gr.mod $-$ R is a category of right G-graded R-modules, which objects are unitary right G-graded R-modules, and morphisms are grading-preserving homomorphisms of the R-modules. The graded analogues of standard definitions will be denoted by the prefix gr-. Thus, a graded ring is called *gr-Artinian* (right) if it satisfies the descending chain condition of the right graded ideals.

If $M = \bigoplus_{g\in G} M_g \in gr.mod - R$ and $\sigma \in G$, then *σ-suspension* $M(\sigma)$ of M is the module M with the grading $M(\sigma)_g = M_{\sigma g}$ $(g \in G)$.

For $M, N \in gr.mod - R$ denote by $HOM_R(M,N)_g$ the set of *homomorphisms of degree* g, i.e. R-linear mappings such that $f(M_h) \subseteq N_{gh}$ for all $h \in G$. By definition,

$$Hom_{gr.mod-R}(M,N) = HOM_R(M,N)_e,$$

$$HOM_R(M,N)_g = Hom_{gr.mod-R}(M,N(g)) = Hom_{gr.mod-R}(M(g^{-1}),N)$$

and $HOM_R(M,N) = \bigoplus_{g\in G} HOM_R(M,N)_g$ is G-graded Abelian group.

Definition 1. *Graded ring will be called gr-quasi-Frobenius, if it is left and right gr-Artinian and each its one-sided graded ideal is annihilator.*

For characterization of the gr-quasi-Frobenius rings we need some definitions.

The definition of a gr-generator was given in [4]. Dually , using [5, lemma 2], we define the gr-cogenerator.

Lemma 1. *For graded R-module Q_R the following conditions are equivalent:*

1) Q is a cogenerator for $mod - R$;

2) for any nonzero morphism $f \in HOM_R(L, N)$, there exists $h \in HOM_R(N, Q)$ such that $hf \neq 0$;

3) $\mathrm{Ker}\psi = \bigcap_{f \in HOM_A(M,U)} \mathrm{Ker} f = 0$ for every graded module M_R;

4) $\prod_{g \in G} Q(g)$ is cogenerator for $gr.mod - R$.

A graded module Q_R will be called *gr-cogenerator* for $gr.mod - R$, if the conditions of Lemma 1 are correct.

We have the following proposition.

Proposition 1. *Let Q be a gr-injective module. Then Q is gr-cogenerator if and only if any gr-simple module may be embedding in $Q(g)$ for some $g \in G$.*

The left and right gr-Artinian ring will be called *gr-Artinian*.

Theorem 1. *The following conditions for the graded ring R are equivalent:*

1) R is gr-quasi-Frobenius.

2) R is gr-Artinian, graded Jacobson radical $J^{gr}(R)$ is left and right annihilator ideal and each graded minimal one-sided ideal is annihilator ideal.

3) R is gr-Artinian and right (left) gr-self-injective .

4) R gr-Artinian and it is gr-injective gr-cogenerator as right (left) R-module.

Follow theorem characterizes gr-quasi-Frobenius rings in the language of the theory of representations.

Theorem 2. *The following properties of the graded ring R are equivalent:*

1) R is gr-quasi-Frobenius.

2) Every gr-injective right R-module is gr-projective.

3) Every gr-projective right R-module is gr-injective.

Since any gr-projective module is projective, and any injective module is gr-injective (see, for example, [6, Chapter 2]), then from theorem 2 we get that every quasi-Frobenius ring is gr-quasi-Frobenius.

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ON ISOMORPHISM OF SEMIFIELD PLANES

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Let π be a finite projective plane, that is coordinatized by semifield W (a semifield plane). The left nucleus of semifield W is a subset

$$W_l = \{x \in W \mid x(yz) = (xy)z \ \forall y, z \in W\}.$$

W_l is a subfield of W and the semifield W is a vector space over W_l. The affine points of a plane π corresponds to the vectors (x, y), where $x, y \in W$, the affine lines corresponds to cosets of follows subgroups:

$$V_\infty = \{(0, y) \mid y \in W\},$$
$$V_m = \{(x, x\theta_m) \mid x \in W\}, \qquad m \in W.$$

Here the set of matrices $R = \{\theta_m \mid m \in W\}$ is a regular set of semifield plane (spread set). The set R contains zero and identity matrices and also it satisfies to follows condition: $\det(\theta_i - \theta_j) \neq 0$ for all $i \neq j$. In particular, all non-zero matrices of R are nondegenerated. The properties of a regular set determines the properties of a semifield plane and its automorphisms group.

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Let $\pi(W, R)$ be a semifield plane that defined by linear space W and regular set R. If the semifield planes $\pi(W, R)$ and $\pi(W, R')$ corresponds to the same linear space W, that isomorphism of these planes is determines by nondegenerated semi-linear transformation of a space W. The following results are proved.

THEOREM 1. *Let W be a n-dimensional linear space over the field $GF(p)$ (p be prime), $\pi_1 = \pi(W, R)$ and $\pi_2 = \pi(W, R')$ be the semifield planes of order p^n. The plane π_1 is isomorphic to π_2 if and only if there exist such nondegenerated linear transformation α of the linear space $V = W \times W$,*

$$\alpha = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

that the product $A^{-1}\theta(m)B$ is in the regular set R' for all matrix $\theta(m) \in R$.

THEOREM 2. *Let π be the semifield plane of order p^n. If W is n-dimensional linear space over the field $GF(p)$ then there exist a semifield plane $\pi(W, R)$ that is isomorphic to π.*

Using these results we created the computer program complex to verify are two semifield planes of the same order isomorphic. For this verification we can use the linear space over the prime order field and construct the matrix representation of regular set for each plane. As a regular set is also a linear space over $GF(p)$, that the check of isomorphism condition from theorem 1 is necessary only for its base elements.

We construct the examples of semifield planes defined by linear space over the prime order field that are isomorphic to some known semifield planes of rank 2 over $GF(4)$, $GF(8)$, $GF(9)$.

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EMBEDDED GRAPHS ON RIEMANN SURFACES: THEORY AND APPLICATIONS[1]

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The talk is based on the results of our joint works with professors N. Ya. Amburg and G.B. Shabat.

Classically Grothendieck dessin d'enfant is an embedded graph Γ on a smooth compact oriented surface M such that the complement $M \setminus \Gamma$ is homeomorphic to a disjoint union of open discs, see [1, 2, 4]. Each Grothendieck dessin d'enfant is in a natural correspondence to a unique (up to a linear-fractional transformation) Belyi

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pair. Belyi pair is an algebraic curve together with a non-constant meromorphic function on this curve with at most 3 critical values. This correspondence provides new approaches to construct and retrieve the connections between different branches of mathematics and mathematical physics, see [3, 4]. Among them there are connections between combinatorial and topological objects, visualization of finite group action, study of moduli space of algebraic curves, etc.

In the talk we provide the introduction to the theory of dessins d'enfants and formulate some new results. In particular, we introduce and investigate embedded graphs on unions of surfaces, their properties and correspondence between graphs on unions of surfaces and meromorphic functions on reducible curves. Shabat polynomials for dessins d'enfants on unions of surfaces will be discussed. Series of concrete examples will be provided.

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TO THE PROBLEM OF THE INTEGRITY OF THE ARTIN'S L-FUNCTIONS

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Consider Artin's L-function

$$L(s,\chi) = L(s,\chi,K/k) = \prod_{\text{unbranched } \wp} \left| E - M\left(\left[\frac{K/k}{\wp}\right]\right) N(\wp)^{-s}\right|^{-1}, \qquad (1)$$

where $\left[\frac{K/k}{\wp}\right]$ – Frobenius automorphism of the extension K/k, related to $\wp$.

In 1930 Artin formulated conjecture which states that in case of non-primary character Artin's L-function (1) is integral function in the whole complex plane [1].

In 1947 Brauer proved that Artin's L-function is meromorphic[2].

This article discusses result which states that Artin's L-function is meromorphic function, all it's poles lay on the critical line and in some cases they coinside with roots of Dedekind zeta-functions of some algebraic number fields.

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UDC 512.54

ON ASPHERICITY OVER SUBPRESENTATIONS

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Let $\{\mathcal{P}_i \mid i \in I\}$ be a collection of subpresentations of a group presentation $\mathcal{P}$, and let $\mathbf{X}_i$ denote the collection of all based spherical pictures over $\mathcal{P}_i$, $i \in I$. We shall say that a set $\mathbf{Y}$ of based spherical pictures over $\mathcal{P}$ *generates* $\pi_2\mathcal{P}$ *over* $\{\mathcal{P}_i \mid i \in I\}$ if $\mathbf{Y} \cup \bigcup_{i\in I} \mathbf{X}_i$ generates $\pi_2\mathcal{P}$. We will say that $\mathcal{P}$ is (A) *over* $\{\mathcal{P}_i \mid i \in I\}$ if $\pi_2\mathcal{P}$ is generated by the empty set over $\{\mathcal{P}_i \mid i \in I\}$. (See [1], [2] for details about pictures and their connection with $\pi_2\mathcal{P}$.)

Let us consider the free group F with a basis $\mathbf{x}$ and a set $\mathbf{r} = \bigcup_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{r_i}$ of cyclically reduced words on $\mathbf{x} \cup \mathbf{x^{-1}}$, where $\mathbf{r_1}, \dots, \mathbf{r_n}$ are mutually disjoint sets. Suppose that no word of $\mathbf{r}$ is trivial, nor is a conjugate of any other word of $\mathbf{r}$ or its inverse. Let R be the normal closure of $\mathbf{r}$ in the free group F and $G = F/N$ be the group defined by the presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r}\rangle$. For $i = 1, \dots, n$, let R_i be the normal closure of $\mathbf{r_i}$ in F, $\mathcal{P}_i = \langle \mathbf{x} \mid \mathbf{r_i}\rangle$, $N_i = \prod_{j\neq i} R_j$.

Theorem 1. *If a set* $\mathbf{Y}$ *of based spherical pictures over* $\mathcal{P}$ *generates* $\pi_2\mathcal{P}$ *over* $\{\mathcal{P}_i \mid i = 1, \dots, n\}$, *then for* $i = 1, \dots, n$, *the group* $\frac{R_i \cap N_i}{[R_i, N_i]}$ *is generated by the set*

$$\{W V_Y W^{-1}[R_i, N_i] \mid Y \in \mathbf{Y}, W \in \mathbf{W}\},$$

where V_Y *is a label of a simple closed path in a based spherical picture* Y, *separating the disks with* $\mathbf{r_i}$*-labels and the disks with* $(\mathbf{r} - \mathbf{r_i})$*-labels,* $\mathbf{W} \subseteq F$ *is a set of representatives of all the cosets of* N *in* F.

Note (see [3]) that $V_Y[R_i, N_i]$ is the image of $< \sigma_Y > \in \pi_2(\mathcal{P})$, where σ_Y is a sequence, represented by a picture $Y \in \mathbf{Y}$, under a G-homomorphism

$$\eta_i : \pi_2(\mathcal{P}) \to \frac{R_i \cap N_i}{[R_i, N_i]},$$

defined as follows: $\eta_i(< \sigma >) = V[R_i, N_i]$, where $< \sigma > \in \pi_2(\mathcal{P})$ and V is the product (taken in order) of the elements of σ of the form $US^{\epsilon}U^{-1}$, where $S \in \mathbf{r_i}, \epsilon = \pm 1$, $U \in F$.

The family $\{R_1, \ldots, R_n\}$ is said to be *independent* if

$$R_i \cap N_i = [R_i, N_i]$$

for $i = 1, \ldots, n$. This and related notions have been studied in [4, 5, 6, 7, 8]. For example [4, 5, 6], if $\{R_1, \ldots, R_n\}$ is independent, then $\mathcal{P}$ is (A) over $\{\mathcal{P}_i \mid i = 1, \ldots, n\}$. The inverse statement for $n = 2$ is shown in [4]. From Theorem 1 we have the following generalization.

Corollary 1. *If $\mathcal{P}$ is (A) over $\{\mathcal{P}_i \mid i = 1, \ldots, n\}$, then $\{R_1, \ldots, R_n\}$ is independent.*

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UDC 511.3

ON OPTIMALITY CONDITIONS FOR OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR p-LAPLACIAN

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In this paper we study an optimal control problem for a nonlinear monotone Dirichlet problem where the control is taken as $L^\infty(\Omega)$-coefficient of Δ_p-Laplacian. Namely, we consider the following minimization problem:

$$\text{Minimize} \quad \left\{ I(u,y) = \int_\Omega |\nabla y(x) - \nabla y_d(x)|^p \, dx \right\} \tag{1}$$

subject to the constraints

$$u \in \mathfrak{A}_{ad} \subset L^\infty(\Omega) \cap BV(\Omega), \quad y \in W_0^{1,p}(\Omega), \tag{2}$$
$$-\mathrm{div}\,\big(u|\nabla y|^{p-2}\nabla y\big) = f \quad \text{in } \ \Omega, \tag{3}$$
$$y = 0 \quad \text{on } \ \partial\Omega, \tag{4}$$

where $\mathfrak{A}_{ad}$ is a class of admissible controls and $f \in W^{-1,q}(\Omega)$, $y_d \in W_0^{1,p}(\Omega)$, $q = p/(p-1)$.

To be more precise, we define the class of admissible controls $\mathfrak{A}_{ad}$ as follows

$$\mathfrak{A}_{ad} = \Big\{ u \in BV(\Omega) \cap L^\infty(\Omega) \ \Big| \ TV(u) \leqslant \gamma, \ \|u\|_{L^1(\Omega)} = m, \ \alpha \leqslant u(x) \leqslant \beta \text{ a.e. in } \ \Omega \Big\}, \tag{5}$$

where α, β, γ, and m are given positive constants such that $0 < \alpha \leqslant \beta < +\infty$ and $\alpha|\Omega| \leqslant m \leqslant \beta|\Omega|$ and

$$TV(f) := \int_\Omega |Df| = \sup\Big\{ \int_\Omega f(\nabla, \varphi)_{\mathbb{R}^N}\, dx : \ \varphi \in C_0^1(\Omega;\mathbb{R}^N), \ |\varphi(x)| \leqslant 1 \,\text{for}\, x \in \Omega \Big\}.$$

It is clear that $\mathfrak{A}_{ad}$ is a nonempty convex subset of $L^1(\Omega)$ with empty topological interior.

Our main goal is to derive first order optimality conditions and provide their substantiation. We propose some ideas and new results concerning the differentiability properties of the Lagrange functional associated to considered control problem. Also, the obtained adjoint boundary value problem is not coercive and, hence, it may admit infinitely many solutions. That is why we concentrate not only on deriving of the adjoint system, but also, following the well-known Hardy–Poincaré Inequality, on formulation of sufficient conditions which would guarantee the uniqueness of the adjoint state to the optimal pair.

We show, that if under mentioned conditions $(u_0, y_0) \in L^\infty(\Omega) \times W_0^{1,p}(\Omega)$ is an optimal pair to problem (2)–(5), then there exists an element $\psi \in W_0^{1,p}(\Omega)$ such that the following system holds true:

$$\int_\Omega (u - u_0) \left(|\nabla y_0|^{p-2} \nabla y_0, \nabla \psi \right)_{\mathbb{R}^N} \, dx \geqslant 0, \quad \forall u \in \mathfrak{A}_{ad}, \tag{6}$$

$$-\mathrm{div}\left(u_0(x) |\nabla y_0|^{p-2} \nabla y_0 \right) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{7}$$

$$\begin{aligned} -\mathrm{div}&\left(u_0 |\nabla y_0|^{p-2} \left[I + (p-2) \frac{\nabla y_0}{|\nabla y_0|} \otimes \frac{\nabla y_0}{|\nabla y_0|} \right] \nabla \psi \right) \\ &= p \, \mathrm{div}\left(|\nabla y_0 - \nabla y_d|^{p-2} \left(\nabla y_0 - \nabla y_d \right) \right) \quad \text{in } \mathcal{D}'(\Omega). \end{aligned} \tag{8}$$

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UDC 511.3

GENERALIZED CHARACTERS OVER NUMERICAL FIELDS AND ANALYTICAL PROPERTIES OF EULER PRODUCTS WITH THESE CHARACTERS

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Let $\mathbb{K}$ be a finite extension of rational numbers field and let χ be a finite-valued character defined on a semigroup of integer ideals of ring of integers of $\mathbb{K}$.

We call χ generalized if

1. $\chi(\mathfrak{p}) \neq 0$ for almost all prime ideals $\mathfrak{p}$;
2. $S(x) = \sum\limits_{\substack{\mathfrak{a}, \\ N(\mathfrak{a}) \leqslant x}} = \alpha x + O(1)$,

Note that all Dirichlet characters of $\mathbb{K}$ satisfy these conditions when this field is a circular extension of a field of rational numbers.

For generalized characters over numerical fields a conjecture similar to Chudakov hypothesis for generalized numerical characters [1] is proposed: a generalized character over numerical field $\mathbb{K}$ is a Dirichlet character of this field.

Authors hope to obtain analytical solution for this problem. To support that a new result on analytical properties of Euler products for generalized characters is provided in this report.

THEOREM 1. *Let χ be a generialized character over $\mathbb{K}$. Then the following Euler product:*

$$f(s) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}, \quad s = \sigma + it$$

defines a function holomorphic in almost all points of $\sigma > 0$ semiplane, possibly except of $s = 1$, where it can have a pole of order 1. On $\sigma = 0$ boundary of the semilplate $f(s)$ does not have pole-like points, that is, points $s = it$ such that $|f(\sigma + it)| \to \infty$ when $\sigma \to 0$.

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UDC 511.3

ON A CERTAIN PROOF OF CHUDAKOV HYPOTHESIS IN CASE OF PRINCIPAL GENERALIZED CHARACTERS

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In 1950 N. G. Chudakov suggested hypothesis that predicated that finitely valued numerical character $h(n)$ that satisfies conditions

1. $h(p) \neq 0$ almost for all primes p;
2. $\sum\limits_{n \leqslant x} h(n) = \alpha x + O(1)$,

is a Dirichlet character [1], [2].

Character that satisfies conditions 1 and 2 is now known as generalized character. In case of $\alpha \neq 0$ it is called principal generalized character, in other cases — nonprincipal. Chudakov hypothesis for principal characters was proved by V. V. Glazkov in 1964 [3], [4]. The proof was performed with elementary method based on examination of target domain of the generalized character.

A new proof of Chudakov hypothesis in case of principal generalized characters is given in this report. It is based on examination of Diriclet series with generalized characters.

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PARALLELOGRAMM SUBSTITUTIONS AND TORIC TILINGS INTO BOUNDED REMAINDER SETS

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Let $\alpha \in \mathbb{R}^d$ be a vector that coordinates are linearly independed with one over $\mathbb{Z}$. This vector generates rotation of d-dimensional torus $\mathbb{T}^d$:

$$S_\alpha : x \to x + \alpha \bmod \mathbb{Z}^d.$$

Thus according to the Weyl theorem the sequence $\{S_\alpha^n(0)\}$ is uniformly distributed on torus $\mathbb{T}^d$. The set $X \subset \mathbb{T}^d$ is called bounded remainder set if there exist a constant $C > 0$ such that

$$|r(\alpha, X, n)| \le C$$

for any n. Here

$$r(\alpha, X, n) = \sharp\{k : 0 \le k < n, S_\alpha^k(0) \in X\} - n(X)$$

isa remainder terms of the uniform distribution problem.

Theorem 1. *Let $\mathbb{T}^d$:*

$$\mathbb{T}^d = \coprod_{i=1}^{R} R_i \sqcup \coprod_{i=1}^{G} G_i \sqcup \coprod_{i=1}^{B} B_i$$

be a toric tiling that consists of sets of three types. Suppose that

1) $S_\alpha(R_i) = R_{i+1}$, $S_\alpha(G_i) = G_{i+1}$, $S_\alpha(B_i) = B_{i+1}$ for any acceptable i.

2) There exist rotations S_R, S_G u S_B such that $S_R(S_\alpha(R_R)) = R_1$, $S_G(S_\alpha(G_G)) = G_1$ and $S_B(S_\alpha(B_B)) = B_1$.

Then all sets R_i, G_i, B_i are bounded remainder sets with effectively computated C.

Further we consider a new approach to generating tilings satisfing the condition of the theorem. This approach is based on parallelogramm substitutions from [1]. Particularly we study parallelogramm substitution from the definition of well-known Rauzy fractal. It is proved that this substitution produce a ssequence of toric tilings into bounded remainder sets. Moreover the estimate of the remainder term does not depend on the number of iteration. Also the prove a similar result for continuous deformations of this construction.

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ON DETERMINATIVE ANALITIC PROPERTIES OF DIRICHLET SERIES CORRESPONDING TO POWER SERIES WITH POLES OF FINITE ORDER AT $z = 1$

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Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it \tag{1}$$

is corresponded to power series (with same coefficients)

$$g(z) = \sum_{n=1}^{\infty} a_n z^n. \tag{2}$$

Proof of the following theorem is given in this report.

Theorem 1. *In a class of Dichlet series with finite abscissa of convergence next conditions are equivalent:*

1. *power series* (2) *determines a function with pole of order k in $z = 1$;*

2. *Dirichlet series* (1) *determines a function meromorphic on a complex plane. This function can have poles of order 1 at $s = 1, 2, \dots, k$, and at $s = k$ it definitely has a pole. Module of the function meets following condition in the left half-plane:*
$$|f(s)| = O\left(e^{|s|\ln|s| + A|s|}\right),$$
where A — some positive constant.

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ON CONGRUENCE LATTICES OF ALGEBRAS OF ONE CLASS OF UNARS WITH MAL'TSEV OPERATION

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A lattice $\langle L, \vee, \wedge \rangle$ with a zero 0 and a unit 1 is a *complemented lattice* if for any element $x \in L$ there is an element $x' \in L$ (a *complement* of the element x) such that $x \wedge x' = 0$ and $x \vee x' = 1$. If for any $a, b \in L$ with $a \leq b$ the interval $[a, b]$ is a complemented lattice, then L is a *relatively complemented lattice*.

If each element of the lattice has exactly one complement, it is an *uniquely complemented lattice*.

A complemented distributive lattice is a *Boolean lattice*.

In [1] the *unar with Mal'tsev operation* was introduced as an algebra with one ternary operation p that satisfies the Mal'tsev identities $p(x, y, y) = p(y, y, x) = x$ and has one unary operation permutable with p. Thus the unar with Mal'tsev operation is an algebra with an operator. Ternary Mal'tsev operation can be defined on any unar so that it permutes with the unary operation.

The ternary operation p is defined in [1] as follows. Let $\langle A, f \rangle$ be an arbitrary unar and $x, y \in A$. For any element x of the unar $\langle A, f \rangle$ by $f^n(x)$ we denote the result of f applied n times to an element x. Also $f^0(x) = x$. Assume that $M_{x,y} = \{n \in \mathbf{N}_0 \mid f^n(x) = f^n(y)\}$ and also $k(x, y) = \min\ M_{x,y}$, if $M_{x,y} \neq \emptyset$ and $k(x, y) = \infty$, if $M_{x,y} = \emptyset$. Assume further

$$p(x, y, z) \stackrel{def}{=} \begin{cases} z, & \text{if } k(x, y) \leqslant k(y, z) \\ x, & \text{if } k(x, y) > k(y, z). \end{cases} \tag{1}$$

Congruence properties of unars with Mal'tsev operation $p(x, y, z)$ defined by a rule (1) were studied in [2] - [4].

THEOREM 1. *Let $\langle A, f, p \rangle$ be an unar with Mal'tsev operation $p(x, y, z)$ defined by the rule* (1)*. The lattice $Con\langle A, f, p \rangle$ is a complemented lattice if and only if either the operation f is injective, or the unar $\langle A, f \rangle$ has one-element subalgebra $\{a\}$ such that $f(x) = a$ for all $x \in A$.*

COROLLARY 1. *Lattice $Con\langle A, f, p \rangle$ is an uniquely complemented lattice (Boolean lattice, relatively complemented lattice) if and only if $Con\langle A, f, p \rangle$ is a complemented lattice.*

COROLLARY 2. *Lattice $Con\langle A, f, p \rangle$ is a complemented lattice if and only if the algebra $\langle A, f, p \rangle$ is simple.*

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MSC2010 16R10

STANDARD BASES OF $T-$IDEALS

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Let M be any $T-$ideal in free associative algebra over a field of the zero characteristic from a countable set of variables, numbered by natural numbers. Variables are compared on their indexes, and multilinear words (monoms) — lexicographically. It is known that the ideal M is generated by its multilinear elements. We say, that the multilinear word v "covers" the multilinear word u, if there is an isoton action of variables to themselves, in which the line of images of variables from u is a subsequence in the line of variables from v. In a case, when u is the higher monom in multilinear element from the ideal M, than word v can be "reduced" modulo M. Thus, in a set of the multilinear elements from M it is possible to construct the standard basis in clear sense, in particular, to construct the reduced standard basis [1]. There is a hypothesis that basis is finite. As "test of the pen" the hypothesis is confirmed for $T-$ideal of identities of upper triangular matrixes, $T-$ideal of a Lie nilpotency with index 4, metabelian $T-$ideal [2], [3]. Most likely the hypothesis can be confirmed for non-matrix $T-$ideals.

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A DISCRETE VERSION OF THE MISHOU THEOREM

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Let $s = \sigma + it$ be a complex variable and α, $0 < \alpha \leq 1$, be a fixed parameter. As usual, denote by $\zeta(s)$ and $\zeta(s, \alpha)$ the Riemann zeta- and Hurwitz zeta-functions, respectively.

It is well known that the functions $\zeta(s)$ and $\zeta(s, \alpha)$ with transcendental or rational parameter α are universal in the Voronin sense, i. e., their shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$, approximate any analytic function. H. Mishou in [3] proved a joint universality theorem for the functions $\zeta(s)$ and $\zeta(s, \alpha)$. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, and let $H_0(K)$ and $H(K)$, $K \in \mathcal{K}$, be the classes of continuous non-vanishing and continuous functions on K, respectively, which are analytic in the interior of K. Moreover, let meas$A$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the Mishou theorem is the following statement.

Theorem 1. *Let $K_1, K_2 \in \mathcal{K}$, and that $f_1(s) \in H_0(K_1)$ and $f_2(s) \in H(K_2)$. Let the number α be transcendental. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

The latter theorem is of the so-called continuous type, in that theorem, for approximation of analytic functions, the shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha)$ when τ varies continuously in the interval $[0, T]$ are used. However, for the functions $\zeta(s)$ and $\zeta(s, \alpha)$, also a discrete universality is known. In this case, for approximation of analytic functions, discrete shifts $\zeta(s + ikh)$ and $\zeta(s + ikh, \alpha)$, where $h > 0$ is a fixed number and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are used. More precisely, the following discrete universality theorems are known.

Theorem 2. *Let $h > 0$ be an arbitrary number. Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

Theorem 2 was obtained by A. Reich in [5], and by B. Bagchi [1] by a different method.

Theorem 3. *Suppose that the number α is transcendental or rational $\neq 1, \frac{1}{2}$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let the number $h > 0$ be such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le m \le N : \sup_{s\in K} |\zeta(s+imh, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Theorem 3 with rational parameter α, under slightly different hypothesis on the set K, was proved in [1], and by a different method, in [6]. The case of transcendental α follows from a discrete universality theorem for periodic Hurwitz zeta-function obtained in [2].

Our aim is a discrete analogue of Theorem 1. Define the set

$$L(\mathbb{P}, \alpha, h) = \left\{ (\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), \tfrac{2\pi}{h} \right\},$$

where $\mathbb{P}$ is the set of all prime numbers.

Theorem 4. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ikh) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+ikh, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

For example, by the Nesterenko theorem [4], it is known that the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Therefore, the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$ with $\alpha = \pi^{-1}$ and rational h.

The proof of Theorem 4 is based on a limit theorem on weakly convergent probability measures in the space of analytic on D functions $H(D)$ equipped with the topology of uniform convergence on compacta. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$,

$$\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p, \qquad \text{and} \qquad \Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for $p \in \mathbb{P}$, and $\gamma_m = \gamma$ for $m \in \mathbb{N}_0$ and $\Omega = \Omega_1 \times \Omega_2$. Then Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ stands for the class of Borel sets of the space S, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ the projection of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, by $\omega_2(m)$ the projection of an element $\omega_2 \in \Omega_2$, $m \in \mathbb{N}_0$, and denote the elements of Ω by $\omega = (\omega_1, \omega_2)$. Now on the probability space $(\Omega, \mathbb{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element $\zeta(s, \alpha, \omega_1, \omega_2)$ by the formula

$$\zeta(s, \alpha, \omega_1, \omega_2) = \left(\prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_1(p)}{p^s} \right)^{-1}, \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s} \right),$$

and denote by P_ζ its distribution. Then we have the following theorem.

Theorem 5. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then the probability measure*

$$\frac{1}{N+1} \# \left\{ 0 \le k \le N : (\zeta(s+ikh), \zeta(s+ikh, \alpha)) \in A \right\}, \quad A \in \mathcal{B}(H^2(D)),$$

converges weakly to P_ζ as $N \to \infty$.

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ON GROBNER BASIS AND PARALLEL COMPUTING

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Introduction. A modification (in the algebraic sense) of Buchberger algorithm is proposed in the note. Consideration is kept on the conceptual level. The modification allows parallel computing (run) and does not exclude simultaneous using with the other parallel modifications proposed earlier.

Computation of Grobner Basis (GB) is one of the central points of the present computing algebra and geometry. Bases are used in the connection to solutions of the algebraic equations, analysis of algebraic varieties and so on. Thus the computations of GB represent significant scientific and practical interest. But this computations request significant computational resources (very often) what exceeds resources of the modern technology (often). In this regard needs to develop new approaches.

In the present time for computing of GB the Buchberger Algorithm is used (and it's manifold modifications) mainly. There are numerous papers dedicate to the Buchberger Algorithm and it's optimization, though a volume of the computation continues to stay very vast.

Touching modern computers we note that the growth of computing power flows at the expense of the growth of the number computer cores (up to some decades and hundreds of thousands) mainly. Therefore for using of the computing power completely it is necessary to used parallel algorithms otherwise resources of the computers will not work in practice.

Touching the Buchberger Algorithm we make the next note. Usually the tasks of the enumeration nature are parallelized sufficient simply. It is not related to the Buchberger Algorithm. There is a point of view that Buchberger algorithm is very heavy for parallelize (the reduction of $S-pairs$ is connate reciprocal). Here it can be noticed the Jean-Charles Faugere work (algorithm F4) though this work have a program character sooner then algebraic.

Generally it is known very well that the acceleration obtaining on multiprocessor computer is not more then liner (with respect to the number of the processors). Therefore the solving of the problem of computation of Grobner Basis is not associated with the multiprocessor computers though it is clear that if there are big number of processors and a good parallel algorithm then the acceleration can be very significant.

Proposed algorithm. Further we describe the general arrangement of the proposed algorithm shortly. The proposed approach is settled on the next fact: it is known that a process of the computing and a result of the Buchberger Algorithm depends on the choice of a monomial ordering.

Let us choose two (for simplicity) distinct monomial orderings previously: α and β. Further the algorithm initiates four processes (threads in the program terminology): A, B, A', B'. The first process A calculates GB (with the Buchberger Algorithm) in line with monomial ordering α. The second process B calculates *GB* in line with monomial ordering β.

The process A' executes the next operations. Let us denote by L_n^α and L_n^β the sets of S-pairs which obtained the processes A and B respectively towards the n-th step. On the step $n_1 > 0$ (and then on steps $n_2, n_3, \dots$) the process A' copies the sets $L_{n_1}^\alpha$ and $L_{n_1}^\beta$ and then executes the reduction of the set $L_{n_1}^\alpha \cup L_{n_1}^\beta$ (in line with monomial ordering α). Then the process A' adds to the process A the result $R_{n_1}^\alpha$ of the reduction. The process B' executes analogous calculates for the process B (adds to the process B the result $R_{n_1}^\beta$ of it's reduction).

Thus the process A obtains periodical makeup from the process B. The analogous computing runs for B (from the process A). Loosely speaking we calculate *GB* from „distinct directions".

Note that successive (not parallel) algorithms changing a monomial ordering from time to time are considered. However the optimality of the choice of an ordering and times is not fully clear. Presumably, these algorithms are closest to the the proposed algorithm.

As compared with the algorithms considered previously the proposed algorithm considered in this note is more radical: it calculates *GB* in line with distinct monomial orderings simultaneously and makes an exchange of information (of S-pairs) between them intermittently.

This algorithm can be considered as (partial) parallelization of *GB*. As the every process will be try to reduce degree of it's major variable primarily then intuitively the general process of the reduction will be run more intensive. This can be considered as a measure to abridge a number of S-pairs. According to the author it is enough interestingly to calculate *GB* simultaneous with degree monomial orderings.

Note the philosophical aspect. There are tasks in the theory of Grobner Basis that monomial ordering (using in during calculation) is not important for a result (for example consistency of systems). In this case the asymmetry calculation manifests clearly particularly. Taking several monomial orderings we symmetrize calculations.

The proposed modification must be tested very meticulous of course. Also it is necessary to determine the rational scheme, line up optimal parameters and so on.

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THE STRUCTURE OF FINITE QUASIFIELDS AND THEIR PROJECTIVE TRANSLATION PLANES[1]

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A ring $S = (S, +, \circ)$ with identity $e \neq 0$ is said to be *a semifield* if $S^* = S \setminus \{0\}$ is a loop, i.e., each of the equations $a \circ x = b$ and $y \circ a = b$ $(a, b \in S,\ a \neq 0)$ is uniquely solvable. Weakening double-sided distributivity in S to left or right one we obtain *a quasifield.* A quasifield is called *proper*, if it is not a field. When S is finite, then the identity e additively generates a subfield $\simeq Z_p$ of prime order p and the order of S is primary. It is well-known that any projective translation plane π of any order p^n can be constructed by using n-dimensional space W over Z_p, as a coordinatizing set of π. Using also a spread set of π we may to equip W by structure of a quasifield. The plane is said to be a *semifield plane* if W is a semifield. The plane is Desargues if W is a field. See monographs of Hughes (1973) and Lüneburg (1980).

Closely related problems of the construction and classification of different classes of finite non-Desargues translation planes and quasifields are being studied since in first of the last century; researches use computer calculations from 1950-th. It is well-known that semifield planes are isomorphic if and only if their semifields are isotopic. In accordance with Knuth [1] it is true the theorem: *A proper semifield of order p^n exists iff $n \geq 3$, $p^n \geq 16$.*

Unlike the finite fields, the structure of finite semifields and quasifield, even of the small orders, has been little studied. (Handbook (2007, [2]); see also Wene's problem [3] and Podufalov's questions 9.43, 10.48, 11.76, 11.77, 12.66 in [4]). On the other hand, it is *unknown even possible orders of maximal subfields* and *the number of subfields with fixed order*. It is proved

Theorem 1. *For every prime $p > 2$ the minimal subfield F of any proper semifield S of the order p^3 is a maximal subfield. Also, if $x \in S \setminus F$, then $\{e, x, x^2\}$ is F-base in S, at that $xx^2 \neq x^2x$ or S is a commutative and $xx^3 \neq x^2x^2$.*

The integer $n \geq 1$ is said to be *an order of an element $s \in S^*$* if n is smallest integer such that n-th power of s equals e for at least one of the brackets setup; if such n does not exist, then the order is considered infinite. The set of the orders of all elements of a loop is said to be *a spectrum.*

Also, the right principal powers $s^{n)}$ $(n \in Z,\ n \geq 0)$ is determined in [5] by recursively: $s^{0)} = e,\ s^{n+1)} = s^{n)} \circ s$. A loop is called right cyclic or right primitive, if its exhaust right principal powers of some element; otherwise loop is called right acyclic. The left principal powers $s^{(n}$ is defined analogously.

For a finite semifield or quasifield S we study the following questions.

(A) Find maximal subfields of S and spectrum of loop S^*.
(B) When loop S^* is singly generated ?
(C) Wene's conjecture [3]: Every finite semifield is (right)primitive.

In according with Rau [6], one of the Knuth semifields of order 32 yields a counterexample to the Wene conjecture. We proved

Theorem 2. *(i) There exists a semifield P of order 32 which has a unique maximal subfield M of order 4. Every element $x \in P \setminus M$ generates loop P^*.*

(ii) There exists a quasifield Q of order 16 such that each element is in a subfield of order 4.

Theorem 3. *Loop S^* for a proper semifield of order 27 is right primitive. The spectrum of loop S^* coincides with $\{1, 2, 5, 7, 8\}$.*

Using the right powers of elements of any loop we may consider its right spectrum. Then the right spectrum of the loop of Theorem 3 coincides with $\{1, 2, 13, 26\}$.

Evidently that spectrum of loop Q^* for the quasifield Q from Theorem 2, (ii), coincides with $\{1, 3\}$. Also, we may choose a semifield P in Theorem 2, (i), such that spectrum of loop P^* coincides with $\{1, 3, 4, 5, 6, 7, 8\}$.

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A PROBLEM ON LARGE ABELIAN SUBGROUPS AND THE GENERALIZED MAL'CEV PROBLEM[1]

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At the beginning of the last century I. Shur found the largest dimension of abeian subgroups in the groups $SL(n, C)$. These groups form the series of simple complex Lie groups. I. Shur proved that for $n > 2$ abelian subgroups of largest dimension are transformed to each other by automorphisms. In 1945 A.I. Mal'cev [1] pointed out *the problem of description of abelian subgroups of largest dimension in the rest simple, and so, semisimlpe complex Lie groups.* He solved this problem by the transition to the Lie algebras $L_\Phi(C)$, corresponding to the root systems Φ. Then he used the reduction to the same problem for the maximal nilpotent subalgabra. In terms of Chevalley algebras this problem may be formulated as the following one:

to describe abelian subalgebras of largest dimension in the nilpotent subalgebra $N\Phi(C)$ with the basis $\{e_r \mid r \in \Phi^+\}$ of the Chevalley algebra of type Φ.

Problem **(A)** *of description of large abelian subgroups in a group G of Lie type over a finite field* goes back to the Mal'cev problem. (Given a group-theoretic property $\mathcal{P}$, we recall that every $\mathcal{P}$-subgroup of largest order in a finite group is a *large $\mathcal{P}$-subgroup*). We transfer the worked out methods for the investigation of the following generalized problems.

A. I. Mal'cev generalized problem: *to descripe abelian subalgebras of largest dimension in the Chevalley algebra $L_\Phi(K)$ over an arbitrary field K.*

The generalized reduction problem: *to describe abelian subalgebras of largest dimension in the subalgebra $N\Phi(K)$ of the Chevalley algebra $L_\Phi(K)$.*

Similarly to the scheme of A. I. Mal'cev, Problem **(A)** is reduced to the analogous problem for the unipotent radical U of a Borel subgroup in G.

More exactly, a large abelian subgroup of the group G of Lie type over a finite field is a large abelian unipotent subgroup or a one of maximal tori in G. R. Carter and D.I. Deriziotis found maximal tori in G in 1978 − 1984. Problem **(B)** *on the description of the set $A(U)$ of large abelian subgroups in U and of the subset $A_N(U)$ of normal subgroups in U* was solved in the middle 1980th in the works of M.J.J. Barry and W.J. Wang for the classical types. For the rest groups G this problem pointed out A.S. Kondrat'ev in the survey [2, Problem (1.6)].

Let $U = UG(K)$ be a unipotent subgroup of a Chevalley group $G(K)$ of normal type $G = \Phi$ or of twisted type $G = {}^m\Phi$. At the beginning of 1990th V.M. Levchuk suggested an approach to Problem **(B)** based on the solution of Problem **(C)** *of description of maximal normal abelian subgroups in U*. At 2000th G.S. Suleimanova developed this approach and completed the solution of Problems **(B)** and **(A)** in 2013 [5].

Problem **(C)** was solved by the authors in [3], [4] for an arbitrary field K. The description of large normal abelian subgroups in finite groups U is a corollary. The authors also investigated the hypothesis: *Is it true that every large normal abelian subgroup in U is a large abelian subgroup?* The following theorem is proved in [4]:

THEOREM 1. *A subgroup in U over a finite field K is large normal abelian if and only if it is normal large abelian.*

The description of the set $A_N(U)$ brings also a calculation of the orders of large abelian subgroups in U. The methods of A.I. Mal'cev and their modification in the works of E.P. Vdovin (1999 – 2001) were used in the proof.

The solution of Problem **(C)** may by applied to the generalized problem of A.I. Mal'cev. Every element $\gamma \in U$ is uniquely represented as the product of root elements $x_r(\gamma_r)$ $(r \in \Phi^+)$, arranged according to a fixed (arbitrary) order of roots (*the canonical decompositoin*). We define the isomorphism π of the group U to "the adjoint" group $\langle N\Phi(K), \circ\rangle$, putting

$$\pi(\gamma) = \sum_{r\in\Phi^+} \gamma_r e_r \ (\gamma \in U\Phi(K)), \quad \alpha \circ \beta = \pi(\pi^{-1}(\alpha)\pi^{-1}(\beta)) \ \ (\alpha, \beta \in N\Phi(K)).$$

For $p(\Phi) = \max\{(r,r)/(s,s) \ \mid r,s \in \Pi(\Phi)\}$ the authors and L.A. Martynova showed earlier that if $p(\Phi)!K = K$, and for the types D_n and E_n also $2K = K$, then the sets of normal subgroups of the adjoint group and the set of ideals of the Lie ring $N\Phi(K)$ coincide. We make more precise the exceptions for the types D_n and E_n in terms of [4], [6].

THEOREM 2. *A normal subgroup H of the adjoint group $ND_n(K)$ over a field K is not a Lie ideal if and only if it has three p_{32}-connected corners with projections of order 2 on these corners and $Ke_{p_{3,-2}} \not\subseteq H$.*

THEOREM 3. *A normal subgroup H of the adjoint group $NE_n(K)$ over a field K is not a Lie ideal if and only if it has three α_4-connected corners from the set $\{\alpha_2, k_1\alpha_1 + \alpha_3, \alpha_5 + k_2\alpha_6 + k_3\alpha_7 + k_4\alpha_8\}$ (for a suitable $k_1, k_2, k_3, k_4 \in \{0,1\}$) with projections of order 2 on these corners and $Ke_{r+s+\alpha_4} \not\subseteq H$ for every such corners r, s.*

We find the largest dimension of abelian subalgebras in $N\Phi(K)$ and investigate the hypothesis: *Is it true that every large abelian ideal of the algebra $N\Phi(K)$ is a large abelian subalgebra?* By the established correspondence, *the description of maximal abelian ideals of the Lie ring $N\Phi(K)$* follows for $p(\Phi)!K = K$ from the solution of Problem **(C)**. The worked out methods allow to get the description in the rest cases.

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ABELIAN AND NON-ABELIAN GROUP CODES OVER NON-COMMUTATIVE GROUPS[1]

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The results presented in this abstract were obtained by the team of authors including C. García Pillado, S. González, C. Martínez (Oviedo), A. Nechaev, V. Markov (Moscow).

Definition 1. *Let G be a finite group, $G = \{g_1, \dots, g_n\}$, F a finite field, $R = FG$ the group algebra. An ideal I of the ring R defines a code*

$$\mathcal{C}(I) = \{(a_1, \dots, a_n) \in F^n : \sum_{i=1}^{n} a_i g_i \in I\}. \tag{1}$$

Any code defined by (1) is called a group code.

The study of the group codes began as early as in the middle of the last century. (cf. for instance [1]). It is clear that in fact the Definition 1 of the group code involves some enumeration of elements of the group G. In [2] this definition is refined as follows.

Definition 2 ([2]). *Let G be a finite group of order n. Код $\mathcal{C}$ of length n over a field F is called a G-code, if it is permutation equivalent to a code of the form $\mathcal{C}(I)$ for some ideal I in the group algebra FG, i.e. there exists a permutation $\sigma \in S_n$ such that*

$$\mathcal{C} = \{(a_1, \dots, a_n) \in F^n : (a_{\sigma(1)}, \dots, a_{\sigma(n)}) \in \mathcal{C}(I)\}.$$

It is clear that Definition 2 does not depend on the enumeration of elements of the group. Moreover it allows to consider a code as a group code for different groups simultaneously. In particular, the following definition was suggested.

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DEFINITION 3 ([2]). *A code $\mathcal{C}$ of length n over a field F is called an abelian group code if it is an A-code over F for some abelian group A of order n.*

Some class if abelian group codes is given by

THEOREM 1 ([2]). *If G is a finite group and*

$$G = AB = \{ab : a \in A, b \in B\}$$

for some abelian subgroups A, B of the group G then every G-code is abelian.

However, the authors could not present any examples of non-abelian group codes. It is easy to check that if $G < 24$ then the group G satisfies the conditions of Theorem 1, so it was natural to begin with study of groups of order 24. The first examples of the non-abelian S_4-codes were constructed in [3] with essential computer usage.

PROPOSITION 1 ([3]). *The weight distribution of two ideals with dimension 9 in the group algebra $\mathrm{GF}(5)S_4$ differs from the weight distribution of any abelian group code (the weight distribution of an ideal I is the vector $(m_0, m_1, \ldots, m_n)$, where m_i is the number of elements in I such that their expression contains exactly i non-zero coefficients). It follows that these ideals define non-abelian group codes.*

Later, we again used computation and the weight distribution comparison to prove

PROPOSITION 2 ([4]). *There exists an ideal of dimension 9 in the group algebra $\mathrm{GF}(3)S_4$ that defines a non-abelian group code.*

Quite recently we again used a computer to prove

THEOREM 2. *There exists an ideal of dimension 9 in the group algebra $\mathrm{GF}(2)S_4$ that defines a non-abelian group code.*

Note that the weight distribution of two ideals with dimension 9 in the group algebra $\mathrm{GF}(2)S_4$ coincides with the weight distribution of some abelian group code so we had to use the new algorithms to prove 2.

Propositions 1, 2 and Theorem 2 imply

COROLLARY 1. *If $p \in \{2, 3, 5\}$ then for any field F with $\operatorname{char} F = p$ there exists a non-abelian S_4-code over F.*

The codes in these examples have distances which are far from maximal ones for the given length and dimension. The next example does not have this drawback.

THEOREM 3. *If $G = \mathrm{SL}_2(\mathrm{GF}(3))$ and $F = \mathrm{GF}(2)$ then there exists a non-abelian G-code with dimension 6 and distance 10 over the field F. At the same time:*

1. *The distance of any abelian group code with the same length and dimension over F does not exceed 8.*
2. *The distance of any linear code with the same length and dimension over F does not exceed 10* [5].

Note that Theorem 3 is proved without computer, but the next one is proved using a computer again.

Theorem 4. *Let* $G = \mathrm{SL}_2(\mathrm{GF}(3))$ *and* F *be any field with* $\operatorname{char} F = p$,*where* $3 \leqslant p < 100$. *Then there exists a non-abelian G-code with dimension 4 over the field* F.

Hypothesis. *Over any finite field* F *with* $\operatorname{char} F = p > 2$, *there exists a non-abelian G-code with dimension 4 over the field* F.

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ON THE LENGTHS OF MATRIX ALGEBRAS AND SETS OF MATRICES[1]

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Definition 1 ([1]). *The* length of a finite system of generators $\mathcal{S}$ *for a finite-dimensional associative algebra* $\mathcal{A}$ *over an arbitrary field is defined as the least nonnegative integer* $l(\mathcal{S})$ *such that the words of length not exceeding* $l(\mathcal{S})$ *span this algebra (as a vector space). The maximum length for the systems of generators of an algebra is referred to as the* length of the algebra, *we denote it by* $l(\mathcal{A})$.

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It was established by A. Paz in [2] that the length of any commutative subalgebra in the matrix algebra of order n over the field of complex numbers $\mathbb{C}$ is not greater than $n-1$.

It was later proved in [3], [4] that this bound also holds for commutative algebras over arbitrary fields and the following description of commutative subalgebras of maximal length was obtained:

Theorem 1. *Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{A}$ be a commutative subalgebra in the matrix algebra $M_n(\mathbb{F})$. Then*
1. $l(\mathcal{A}) \le n-1$;
2. $l(\mathcal{A}) = n-1$ if and only if the algebra $\mathcal{A}$ is generated by a nonderogatory matrix C, i.e. by such a matrix $C \in M_n(\mathbb{F})$, that

$$\dim_{\mathbb{F}}(\langle C^0 = E_n,\ C,\ C^2, \dots,\ C^{n-1}\rangle) = n.$$

In the present talk we describe commutative subalgebras of the length $n-2$ in the algebra $M_n(\mathbb{F})$, i.e. of the length closest to maximal, over algebraically closed fields.

The following theorem shows that this question can be reduced to the case of nilpotent commutative subalgebras of the length $n-2$:

Theorem 2. *Let $\mathbb{F}$ be an algebraically closed field and let $n \in \mathbb{N}$, $n \ge 2$. Consider a commutative subalgebra $\mathcal{A}$ in $M_n(\mathbb{F})$ of the length $l(\mathcal{A}) = n-2$. Then there exist a number $m \in \mathbb{N}$, $2 \le m \le n$, a commutative subalgebra $\mathcal{B} \subseteq M_m(\mathbb{F})$ of the length $m-2$ and of the form $\mathbb{F}E + \mathcal{N}$, where $\mathcal{N}$ is a nilpotent algebra, and if $m < n$, a commutative subalgebra $\mathcal{C} \subseteq M_{n-m}(\mathbb{F})$ generated by a nonderogatory matrix, such that the algebra $\mathcal{A}$ is conjugated with the algebra $\mathcal{B} \oplus \mathcal{C}$.*

Applying the description of nilpotent commutative subalgebras in $M_n(\mathbb{F})$ with nilpotency indices n and $n-1$ (see [5] and [6]), we obtain our main result:

Theorem 3. *Let $n \ge 2$ and let $\mathbb{F}$ be an algebraically closed field. Consider the matrix $A = E_{1,2} + \dots + E_{n-2,n-1}$, where $E_{i,j}$ denotes the (i,j)-th matrix unit. Let $\mathcal{A}$ be a commutative subalgebra in $M_n(\mathbb{F})$, which contains the identity matrix I_n. Then $l(\mathcal{A}) = n-2$ if and only if the algebra $\mathcal{A}$ is conjugated in $M_n(\mathbb{F})$ with one of the following algebras:*
1. $\mathbb{F}I_n$, if $n = 2$;
if $n \ge 3$
2. $\mathbb{F}I_2 \oplus \mathcal{C}_{n-2}$, where $\mathcal{C}_{n-2} \in M_{n-2}(\mathbb{F})$ is a subalgebra generated by a nonderogatory matrix;
3. $\mathcal{A}_{0;n} = \langle I_n, A, A^2, \dots, A^{n-2}\rangle$;
4. $\mathcal{A}_{1;n} = \langle E_{1,n},\ C|\ C \in \mathcal{A}_{0;n}\rangle$;
5. $\mathcal{A}_{2;n} = \langle E_{n,n-1},\ C|\ C \in \mathcal{A}_{0;n}\rangle$;
6. if $n = 4$, $\mathcal{A}_{3;4}(1) = \langle E_{1,n} + E_{n,n-1},\ C|\ C \in \mathcal{A}_{0;n}\rangle$;
7. if $n = 4$, $\operatorname{char}\mathbb{F} = 2$, $\mathcal{A}_{4;4} = \langle E_4, E_{1,2} + E_{3,4},\ E_{1,3} + E_{2,4},\ E_{1,4}\rangle$;
8. $\mathcal{A}_{j;m} \oplus \mathcal{C}_{n-m}$, where $j = 0, 1, 2$, $3 \le m < n$, $\mathcal{C}_{n-m} \in M_{n-m}(\mathbb{F})$ is a subalgebra

generated by a nonderogatory matrix.
Algebras of types 3–7 are pairwise non-conjugate.

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UDC 511.3

POLYADIC REPRESENTATIONS OF POSITIVE INTEGERS

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We compare various types of representation of positive integers: DBNS, chain, and polyadic representation.

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UDC 511.3

ON ANALYTICAL PROPERTIES OF A CERTAIN CLASS OF DIRICHLET SERIES

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Consider a Dirchlet series:

$$f(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \quad s = \sigma + it, \tag{1}$$

where $h(n)$ is a non-zero finite-valued multiplicative function from naturals with sum function satisfying a condition:

$$S(x) = \sum_{n \leqslant x} h(n) = \alpha x + O(1).$$

For such Dirichlet series the following statement has been proved.

THEOREM 1. *Dirichlet series* (1) *defines a function holomorphic in almost all points of $\sigma > 0$ semiplane, possibly excluding $s = 1$, where it can have a pole of order 1. It also does not have pole-like points on the left boundary ($\sigma = 0$).*

Here we call a point $s = \sigma + it$ pole-like if $|f(\sigma + it)|$ tends to infinity when σ vanishes.

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UDC 511.3

ALMOST PERIODICAL FUNCTIONS AND DENSITY THEOREMS FOR DIRICHLET SERIES WITH PERIODIC COEFFICIENTS

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In this report the following statement is proved:

TEOPEMA 1. *Let $N_\epsilon(T)$ be a number of zeros of a Dirichlet series with non-zero periodic coefficients inside $\frac{1}{2} - \epsilon < \sigma < \frac{1}{2} + \epsilon,\ 0 < t \leqslant T$ rectangle. For arbitrary $\epsilon > 0$ the following asymptotic formula is valid:*

$$N_\epsilon(T) \sim \frac{1}{2\pi} T \ln T.$$

The basis of its proof is a statement that for Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it$$

with periodic coefficients, whose sum functions are bounded, there exists a sequence of Dirichlet polynomials $Q_n(s)$ which approximates $f(s)$ and its derivatives in any rectangle $0 \leqslant \sigma_0 < \sigma \leqslant 1,\ 0 < t \leqslant T$ with exponential speed.

Also several known results on zeros of almost periodical functions of a finite class [1] are used. Examples of such functions are Dirichlet polynomials $Q_n(s)$.

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AN EXAMPLE OF 2-FROBENIUS GROUP ISOSPECTRAL WITH SIMPLE GROUP $U_3(3)$

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Only finite groups are considered. Let G be a group. Denote by $\omega(G)$ the *spectrum* of G, i.e. the set of all element orders of G. Groups with identical spectrum is said to be *isospectral*. Since $\omega(G)$ is closed by the divisibility condition, it is uniquely determined by the subset $\mu(G)$ consisting of maximal elements of spectrum under this condition.

2-*Frobenius group* is a group G which contains a normal Frobenius subgroup B with Frobenius kernel A such that G/A is a Frobenius group with Frobenius kernel B/A. The *prime graph* or *Gruenberg-Kegel graph* $GK(G)$ of a group G is undirected graph whose vertices are prime divisors of order of G and edges are pairs of non-equal prime divisors p and q, such that G contains an element of order pq.

It is proved in [1] that every soluble group G with non-connected graph $GK(G)$ is a Frobenius or 2-Frobenius group. M.R Aleeva (Zinov'eva) [2] shoved that the list of simple groups which are isospectral with Frobenius groups (respectively, with 2-Frobenius groups) is exhausted by $L_3(3)$ and $U_3(3)$ (respectively, by $U_3(3)$ и $S_4(3)$), but the question about existence of such 2-Frobenius groups remained open until recently. It was proved in [3] that a simple non-abelian group isospectral with soluble one is isomorphic to one of groups $L_3(3), U_3(3), S_4(3)$ or A_{10}. A.M. Staroletov [4] described groups which are isospectral with A_{10}. They all are insoluble. A.V. Zavarnitsine [5] build an example of 2-Frobenius group of order 5 648 590 729 620 $= 2^2 \cdot 3^{24} \cdot 5$ which is isospectral with $S_4(3)$.

We consider the remaining case.

Theorem 1. *There exists a* 2*-Frobenius group isospectral with the simple group* $U_3(3)$.

Our example is a semidirect product of a group P of order 2^{18} with a Frobenius group of order 21.

The group P is generated by elements $x_1, x_2, x_3, y_1, y_2, y_3$ with the following defining relations:

$$x_i^4 = y_i^4 = 1, i \in \{1, 2, 3\}; \tag{1}$$

$$[x_i, x_j] = [y_i, y_j] = 1, i, j \in \{1, 2, 3\}; \tag{2}$$

$$[x_i, y_i] = 1, i \in \{1, 2, 3\}; \tag{3}$$

$$[x_i, y_j] * [x_j, y_i] = 1, 1 \leqslant i < j \leqslant 3; \tag{4}$$

$$[[x_i, y_j], x_k] = [[x_i, y_j], y_k] = 1, i, j, k \in \{1, 2, 3\}. \tag{5}$$

These relations imply

LEMMA 1. 1. *The nilpotency class of P is equal to 2 and P is a 2-group.*

2. *Subgroups $X = \langle x_1, x_2, x_3 \rangle$ and $Y = \langle y_1, y_2, y_3 \rangle$ are isomorphic to the direct product of three cyclic groups of order 4, and $P = \langle X, Y \rangle$.*

3. *The derived subgroup Z of P is generated by elements $z_1 = [x_1, y_2], z_2 = [x_1, y_3], z_3 = [x_2, y_3]$ whose orders are equal to 4. It is isomorphic to the direct product of three cyclic subgroups of order 4.*

4. *The order of P is equal to 2^{18}.*

5. *The exponent of P is equal to 8.*

Let r_x and r_y be automorphisms of X and Y, respectively, whose matrices in their basises x_1, x_2, x_3 and y_1, y_2, y_3 are equal to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

These automorphisms are uniquely extendable to an order 7 automorphism r of P. The matrix of restriction of r on Z in the basis z_1, z_2, z_3 is equal to

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

The automorphism r acts on P fixed-point-freely.

Let, further, s_x and s_y be automorphisms of groups X and Y, respectively, whose matrices in their basises x_1, x_2, x_3 and y_1, y_2, y_3 are equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{bmatrix}.$$

These automorphisms are uniquely extendable to an order 3 automorphism s of P. The matrix of restriction of s on Z in the basis z_1, z_2, z_3 is equal to

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

The fixed points subgroup of s in X is equal to $\langle x_1\rangle$, the fixed points subgroup of s in Y is equal to $\langle y_1\rangle$, the fixed points subgroup of s in Z is equal to $\langle z_1^2z_2^2z_3\rangle$. This implies that the subgroup C of fixed points of s in P is generated by $x_1, y_1, z_1^2z_2^2z_3$. By (3) and Lemma 1, C is abelian of exponent 4.

Besides, $s^{-1}rs = r^2$, i.e $F = \langle r, s\rangle$ is a Frobenius group of order 21.

The natural semidirect product P with F is a required group.

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ON A PROBLEM FOR ARTINIAN LIE ALGEBRAS

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The concept of artinian plays an important role in the theory of rings.

Associative ring R is called right (left) artinian if every descending chain of its right (left) ideals of ideals stabilizes [1], [2].

Examples of the right artinian rings are known, but left artinian rings are not. [1].

Further, saying artinian algebra or artinian ring we will have in mind right artinian.

The concept of artinian is used as one of the conditions of finiteness (finite dimensionality).

Artinian for Lie algebras through the ideals defined Y.A. Bahturin [3], S.A. Pikhtilkov [4] and V.M. Polyakov[5]. They considered special i-artinian Lie algebras.

In 1963, V.N. Latyshev has introduced a new class of Lie algebras [6], which he called special by analogy with Jordan algebras.

Lie algebra L is called a special Lie algebra if there is an associative PI-algebra A such that L is embedded in $A^{(-)}$ as a Lie algebra, where $A^{(-)}$ is Lie algebra defined on A by the operation $[x, y] = xy - yx$.

Perhaps the best analogs of the one-sided ideal for Lie algebras are subalgebras or inner ideals.

The notion of inner ideal for Jordan algebras was introduced by Jacobson [7]. G. Benkart introduced inner ideal for Lie algebras by analogy[8].

Subspace of B of the Lie algebra L is an inner ideal if the inclusion $[B, [B, L]] \subset B$.

F. Lopez, E. Garcia, G. Lozano explored the concept of the inner ideal with respect to artinian via Jordan pairs [9], [10].

Consider the definition artinian in three senses.

Let L is Lie algebra.

a) If the descending chain of ideals stabilizes, then the algebra is called i-artinian;

b) if decreasing chain algebras stabilized, then the algebra is called a-artinian;

c) if the descending chain of inner ideals stabilizes, then the algebra called inn-artinian.

In the paper [11] is an example of an infinite-dimensional inn-artinian Lie algebra.

We determined that of the inn-artinian may not follow the a-artinian.

It is easy to check that from inn-artinian follows i-artinian and from the a-artinian follows the i-artinian.

In the paper [11] also examples showing that of i-artinian can not follow a-artinian and inn-artinian.

Are interesting examples of infinite a-artinian algebras.

The next question arises.

Question 1. Are there infinite-dimensional a-artinian Lie algebras?

The answer to this question is unknown to the author.

O.Y. Schmidt formulated the problem on the existence of an infinite non-abelian groups all, subgroups of which are finite. This problem was solved, A.Y. Olshansky [12]. The example of [12] built by using geometric methods in group theory. Its postponement on Lie algebras are still unable to perform.

The main result of the paper is the consideration of question 1 to the next question.

Question 2. Are there primary infinite a-artinian Lie algebra?

Theorem 1. *Questions 1 and 2 for the Lie algebras are equivalent.*

Concepts a-artinian and inn-artinian are useful. For example, the following theorem gives a solution to the problem of A.V. Mikhaleva on the solubility of the primary radical Artinian Lie algebra.

THEOREM 2. *([13]). Let L be a-artinian or inn-artinian algebra. Then prime radical $P(L)$ Lie algebra L is solvable.*

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HOMOTOPY CLASSIFICATION OF TRANSITIVE LIE ALGEBROIDS[1]

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We call by the Lie algebroid a finite dimensional vector bundle $E \to M$ over a smooth manifold M with a homomorphism $a : E \to TM$ to the tangent bundle TM, called anchor, and the space $\Gamma^\infty(E)$ of smooth sections is provided with an additional structure, the commutator bracket $\{\bullet, \bullet\}$, which satisfies the natural properties of the structure infinite-dimensional Lie algebra, as well as the Newton-Leibniz identity with respect to the operation of multiplication of section by a smooth function. The anchor thus induces a homomorphism of the Lie algebra $\Gamma^\infty(E)$ into the Lie algebra $\Gamma^\infty(TM)$ of vector fields on the manifold M.

Examples of Lie algebroids are the tangent bundle TM, the bundle $\mathcal{D}(L)$ of covariant differentiations of all smooth sections $\Gamma^\infty(L)$ of any finite-dimensional vector bundle L over a smooth manifold M, as well as the tangent bundle of an arbitrary smooth foliation $\mathcal{F}$ on the manifold M without singular points. In the case where a anchor is surjective, the Lie algebroid is called transitive.

Transitive Lie algebroids were detaily studied in the book by K.Mackenzie ([1]) . In particular, it was shown that the smooth map of manifolds generate the inverse image (pullback) of transitive Lie algebroids, which depends only on the homotopy class of the map. From this observation it follows that the classification of transitive Lie algebroids can be reduced to the construction of the final objects for each fixed finite-dimensional Lie algebra g, associated to the transitive Lie algebroid and the classification is invariant up to homotopy. In spite of the evidence of the observation the construction of the final object still was not conducted.

We prove ([2], [3]), that the homotopy classification is reduced to the construction of the final space in the form of the classifying space BG, where G is the group $Aut(g)$ of automorphisms of the adjoint Lie algebra g with topology thinner than the classical topology. This construction, in particular, allows to calculate the cohomology Mackenzie obstacle for existence of a transitive Lie algebroid, which turns trivial in many cases. For example, Mackenzie obstacle is trivial for any simply connected manifold.

This work was done jointly with Li Xiaoyu (China) and Gasimov V.A-M. (Azerbaijan).

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NUMERICAL CHARACTERISTICS OF VARIETIES OF LINEAR ALGEBRAS

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The report will present the results from the beginning of the new millennium, associated with unusual behavior of the numerical characteristics of the varieties. It will focus on the main numerical characteristic of variety: its growth. Characteristic of the main field is equal to zero. All backgrounds was stated in [1].

When we talk about own variety $\mathbf{V}$ of associative algebras, it is well known that sequence of codimensions $c_n(\mathbf{V})$, $n = 1, 2, \dots$, behaves asymptotically as either polynomial or exponential function like $a^n, n = 1, 2, \ dots$, where a is a non-negative integer. In class of Lie algebras the situation is various. The growth of codimension can be over-exponential, and in the case of exponential growth of $a^n, n = 1, 2, \dots$, number a may be non-integer. This effect exist even for classical objects.

For example, as shown in paper [2], for the variety generated by the Lie algebra of vector fields on the plane, asymptotically, i.e. starting with some n, the following inequality holds $(13,1)^n \leqslant c_n(\mathbf{V}) \leqslant (13,5)^n$.

In general, there are more unusual examples of behavior numerical characteristics. We consider the situation when variety consists of left-nilpotent algebras order two. Hence, the identity takes place

$$x(yz) \equiv 0. \tag{1}$$

Denote by ${}_2\mathbf{N}$ the variety of all such algebras. From identity (1) follows that in all non-zero products parentheses can be placed only left-normed way, that is so: $(((x_1x_2)x_3)\dots x_n)$.

Let $\mathbf{V}$ be a variety of algebras with identity (1) and $F(\mathbf{V})$ denotes relatively free algebra of countable rank, generated by free variables $x_1, x_2, \dots$. Let $P_n(\mathbf{V})$ be vector space of multilinear elements on the variables $x_1, \dots, x_n$ in $F(\mathbf{V})$, and let $c_n(\mathbf{V}) = \dim P_n(\mathbf{V})$ be its dimension. The growth of sequence $c_n(\mathbf{V}), n = 1, 2, \dots,$

determines the growth of variety. If inequality $c_n(\mathbf{V}) < a^n$ holds for any n and appropriate number a, then there exist the lower and upper limits called the lower and upper exponents of the variety $\mathbf{V}$.

$$\underline{\mathrm{EXP}(\mathbf{V})} = \underline{\lim}_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{\mathrm{EXP}(\mathbf{V})} = \overline{\lim}_{n\to\infty} \sqrt[n]{c_n(A)},$$

If the limit exists, i.e. $\underline{\mathrm{EXP}(\mathbf{V})} = \overline{\mathrm{EXP}(\mathbf{V})}$, then it called the exponent of $\mathbf{V}$ and denoted by $\mathrm{EXP}(\mathbf{V})$.

In paper [3] was constructed examples of varieties with any exponents and in article [4] was constructed series of varieties, so-called intermediate growth. Recall that variety $\mathbf{V}$ have intermediate growth if for any $k > 0$, $a > 1$ exists such constant C_1, C_2, that for any n takes place inequality

$$C_1 n^k < c_n(\mathbf{V}) < C_2 a^n.$$

Note that in any class of associative algebras or class of Lie algebras such varieties doesn't exist.

Theorem 1. *(Giambruno A., Zaicev M. V., Mishchenko S. P.). For any real* $\alpha, \alpha > 1$, *exists such variety* $\mathbf{V}_\alpha \subset {}_2\mathbf{N}$, *that* $\mathrm{EXP}(\mathbf{V}_\alpha) = \alpha$.

Theorem 2. *(Giambruno A., Zaicev M. V., Mishchenko S. P.). For any real number* β, $0 < \beta < 1$, *exists such variety* $\mathbf{V}_\beta \subset {}_2\mathbf{N}$, *that*

$$\lim_{n\to\infty} \log_n \log_n c_n(\mathbf{V}_\beta) = \beta,$$

i.e. sequence $c_n(\mathbf{V}_\beta)$ *behaves as* n^{n^β}, $n = 1, 2, \ldots$.

Will be called a variety of almost nilpotent if it is not itself is nilpotent, but every own subvariety is nilpotent. For example, the variety of all commutative associative algebras and the variety of all metabelian Lie algebras is almost nilpotent. The growth of these varieties is small. More precisely, in the first case for any n codimension is equal to 1 and in the case of a metabelian variety of Lie algebras codimension is $n - 1$. In the class of algebras satisfying (1) the situation is different. Two years ago, an example of almost nilpotent variety with a significant growth of sequence of codimension with exponent is equal to two was found. A year later it was proved that there is almost nilpotent varieties with any integer exponent (see [5], [6]).

Theorem 3. *(Valenti A., Mishchenko S.P., Shulezhko A.V.). For any integer* m, $m \geqslant 2$, *exists almost nilpotent variety* $\mathbf{V}_m \subset {}_2\mathbf{N}$ *with exponent is equal to* m, *that is* $\mathrm{EXP}(\mathbf{V}_m) = m$.

In the last year (see [7]) was constructed an example of variety with various upper and lower exponents.

Theorem 4. *(Zaicev M.V.). For any real $\alpha > 1$ there exists such variety $\mathbf{V}_\alpha \subset {}_2\mathbf{N}$ that*

$$1 = \underline{\mathrm{EXP}(\mathbf{V}_\alpha)} \neq \overline{\mathrm{EXP}(\mathbf{V})} = \alpha.$$

Unusual examples was found among the varieties of polynomial growth. Recall that in the case of associative algebras, Lie or Jordan algebras the sequence of codimension of variety with polynomial growth asymptotically behaves as Cn^k, where k is integer. In the paper [8] first was built examples of variety with fractional polynomial growth.

Theorem 5. *(Zaicev M.V., Mishchenko S.P.). Over field of zero characteristic for any real $3 < \alpha < 4$ there exists variety of linear algebras $\mathbf{V}_\alpha \subset {}_2\mathbf{N}$, that for sufficiently large n the following condition holds*

$$C_1 n^\alpha < c_n(\mathbf{V}_\alpha) < C_2 n^\alpha,$$

where C_1, C_2 is some positive constants.

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LEIBNIZ ALGEBRAS VARIETY $\widetilde{\mathbf{V}}_1$

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The characteristic of the basic field is equal to zero and all the undefined notions can be found, for example, in the monograph [1]. Let's remind that Leibniz algebra is determined by the identity $(xy)z \equiv (xz)y + x(yz)$. This means that the operator of right multiplication is derivation of the algebra, that allows us to do conversions in products. For example, any product of elements we can express as a linear combination of left-normed products. We shall agree to omit brackets in these products, i.e. $abc = (ab)c$. In this work the variety $\widetilde{\mathbf{V}}_1$ of all Leibniz algebras is discussed, in which the next identity takes place

$$x_1(x_2x_3)(x_4x_5) \equiv 0.$$

Leibniz algebras variety $\widetilde{\mathbf{V}}_1$ was investigated rather extensively in the paper [2]. Almost polynomial growth of the variety is proved and also the construction of space of multilinear elements of relatively free algebra variety as the module of the symmetric group is described. In particular, formulas for finding of codimensions, multiplicities and colength are found. So, the codimension is expressed on the formula $c_n(\widetilde{\mathbf{V}}_1) = 2^{n-2}n(n-3) + 2n$, and for colength when $n > 2$ we have:

$$l_n(\widetilde{\mathbf{V}}_1) = \begin{cases} n^2 - \frac{7}{2}n + 6, & \text{if } n = 2k \text{ even}, \\ n^2 - \frac{7}{2}n + \frac{11}{2}, & \text{if } n = 2k+1 \text{ odd}. \end{cases}$$

Recently the authors have received a new result.

Theorem 1. . *The bases of multilinear part of variety* $\widetilde{\mathbf{V}}_1$ *consists of the elements of the form:* $x_{i_1}x_{i_2}\dots x_{i_n}$, *where* $i_2 > \dots > i_n$; $(x_{i_1}x_{i_2}\dots x_{i_{n-k}})(x_{j_1}x_{j_2}\dots x_{j_k})$, *where* $k = 2, \dots, (n-1)$, $i_2 > \dots > i_{n-k}$, $j_1 < j_2$ *u* $j_2 > j_3 > \dots > j_k$.

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ALMOST NILPOTENT VARIETIES OF ANY INTEGER EXPONENT

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The basic field has the zero characteristic. All undefined notions can be found in book [1]. We will say that a variety is almost nilpotent if it is non-nilpotent, but every its own subvariety is nilpotent. For example, variety of all associative and commutative algebras and variety of all metabelian algebras Lie are almost nilpotent. Recently the complete description of all almost nilpotent varieties was received in a class of Leibniz algebras [2]. In all these cases the growth of almost nilpotent variety was polynomial.

The main idea of this work is to prove the existence of almost nilpotent varieties, growth of sequence of codimensions of which is exponential. Ideologically we based on article [3] in which the almost nilpotent variety of exponent 2 is constructed and using Zorn's lemma they proved the next theorem

THEOREM 1. *Let* $\mathbf{U}$ *— any non-nilpotent variety of algebras. Then there is such subvariety* $\mathbf{V}$ *of variety* $\mathbf{U}$ *that variety* $\mathbf{V}$ *is almost nilpotent.*

We agree to omit brackets in case of their left-notmed arrangement, i.e. $abc = (ab)c$. Moreover, we denote R_a as the operator of multiplication on the right on an element a and write down the result of action of the operator R_a on an element b as bR_a, i.e. $bR_a = ba$. The last designation is convenient. For example, left-normed product $ba\dots a$ of degree $k+1$ can be written in a short form bR_a^k.

For any natural $m \geqslant 2$ we define not associative algebra A_m, which is generated by generators $\{z, a_1, a_2, \dots, a_m\}$ and with the following relations:

$$a_i a_j = a_i z = 0, \ 1 \leqslant i, j \leqslant m;$$

$$(zw(R_{a_1}, \dots, R_{a_m}))(zw'(R_{a_1}, \dots, R_{a_m})) = 0,$$

for some, probably empty, associative words w and w' from operators R_{a_i};

$$z(R_{a_1}\dots R_{a_m})^k a_{i_1}\dots a_{i_s} a_{i_{s+1}}\dots a_{i_t} + z(R_{a_1}\dots R_{a_m})^k a_{i_1}\dots a_{i_{s+1}} a_{i_s}\dots a_{i_t} = 0$$

for all $k \geqslant 0$ and $1 \leqslant s < t \leqslant m$, $1 \leqslant i_1, \dots, i_t \leqslant m$.

The variety generated by algebra A_m, we denote $\mathbf{U}_m$. The main result of the work is the next theorem.

THEOREM 2. *Let* $\mathbf{V}$ *be non-nilpotent subvariety of the variety* $\mathbf{U}_m$*. Then exponent of the variety* $\mathbf{V}$ *is equal to* m*.*

From this result and the theorem 1 follows that for any natural m, $m \geqslant 2$, there is an almost nilpotent variety of exponent m.

Note that in the work [3] was proved also that the variety $\mathbf{U}_2$ is almost nilpotent.

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TWO-DIMENSIONAL CONTOU-CARRÈRE SYMBOL AND RECIPROCITY LAWS AN ALGEBRAIC SURFACES[1]

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Let X be a smooth algebraic surface over a perfect field k. By any point $x \in X$ and any irreducible curve $C \subset X$ such that $x \in C$, one canonically constructs an Artinian ring $K_{x,C}$ which is a finite product of two-dimensional local fields. If x is a smooth point on a curve C, then the ring $K_{x,C}$ is isomorphic to the two-dimensional local field $k(x)((u))((t))$.

Let $\omega \in \Omega^2_{k(X)/k}$. For any pair $x \in C$ as above, the two-dimensional Parshin residue $\mathrm{res}_{x,C} : \Omega^2_{k(X)/k} \to k(x)$ is defined. The Parshin resiprocity laws are satisfied, [1]:

1. Fix a point $x \in X$. Then the following sum contains only finitely many terms distinct from 0 and
$$\sum_{C \ni x} \mathrm{Tr}_{k(x)/k} \circ \mathrm{res}_{x,C}(\omega) = 0.$$

2. Fix an irreducible curve $C \subset X$. Then the following sum contains only finitely many terms distinct from 0 and
$$\sum_{x \in C} \mathrm{Tr}_{k(x)/k} \circ \mathrm{res}_{x,C}(\omega) = 0.$$

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Analogous, multiplicative reciprocity laws are satisfied for two-dimensional tame symbols (in composition with norm maps) applied for any pair $x \in C$ to fixed three elements from the group $k(X)^*$. Here the two-dimensional tame symbol is a map from the group $K_3^M(k(x)((u))((t)))$ to the group $k(x)^*$. This map is a composition of boundary maps in Milnor K-theory.

A two-dimensional Contou-Carrère symbol was defined in [2] for any commutative ring R as some map (which is functorial with respect to R):

$$(\cdot,\cdot,\cdot) \; : \; R((u))((t))^* \times R((u))((t))^* \times R((u))((t))^* \longrightarrow R^*.$$

If the ring R is a field, then the two-dimensional Contou-Carrère symbol coincides with the two-dimensional tame symbol. If $R = L[\epsilon]/\epsilon^4$, where L is a field, then for any elements f, g, h from $L((u))((t))$ we have:

$$(1+\epsilon f, 1+\epsilon g, 1+\epsilon g) = 1 + \epsilon^3 \operatorname{res}(f dg \wedge dh),$$

where res is the Parshin residue, which is a map from the space $\Omega^2_{L((u))((t))/L}$ to the field L.

The following reciprocity laws for the two-dimensional Contou-Carrère symbol were proved in [2].

Theorem 1. *Let R be a local finite k-algebra. Let elements f, g, h be from the group $(k(X) \otimes_k R)^*$.*

1. *Fix a point $x \in X$. Then the following product contains only finitely many terms distinct from 1 and*

$$\prod_{C \ni x} \mathrm{Nm}_{k(x)/k}(f,g,h)_{x,C} = 1.$$

2. *Fix an irreducible curve $C \subset X$. Then the following product contains only finitely many terms distinct from 1 and*

$$\prod_{x \in C} \mathrm{Nm}_{k(x)/k}(f,g,h)_{x,C} = 1.$$

Here $(\cdot,\cdot,\cdot)_{x,C}$ is the two-dimensional Contou-Carrère symbol which is well-defined on the group $((K_{x,C} \otimes_k R)^*)^3$, since the ring $K_{x,C} \otimes_k R$ is a finite product of rings isomorphic to $(k' \otimes_k R)((u))((t))$ (for various $k' \supset k$).

This theorem is proved by means of (another) definition the two-dimensional Contou-Carrère symbol as generalized commutator in some categorical central extension of the group $R((u))((t))^*$. Such categorical central extensions and commutators in these central extensions were studied in [3].

We note that if the ground field k is a finite field, then for various n we can consider local k-algebras $R = k[s]/s^n$. Using these algebras R, from the two-dimensional Contou-Carrère symbol we derive multidimensional analogues of Witt symbols, which were used by A. N. Parshin and K. Kato for the construction of the two-dimensional class field theory, see [2].

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MATHEMATICAL MODEL OF INFORMATION SECURITY SYSTEMS BASED ON DIOPHANTINE SETS

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In this paper, we propose a mathematical model of polyalphabetic cryptosystem in which the inverse transform algorithm of the text is an algorithmically unsolvable problem for analyst. It permeates the idea of Shannon who believed that the greatest variability in the selection of keys is in cryptosystems containing Diophantine problems.

Based on the theoretical positions [1, 2, 3] for constructing persistent and effective models of information security systems (ISS), we note particularly, that all mathematical problems are models of concealment and protection information and solution of these problems corresponds to the correct key. Consequently, the choice of suitable problems, in particular NP-complete problem, helps to create the information security system at the appropriate level. Especially this problem, as noted Shannon [1], is associated with a task that contains Diophantine STI. Note that all non-standard knapsack problems KG (generalized task), KU (super-generalized task), KF (functional task) first introduced by the author [4], belong to the class of NP-complete problems.

In this paper, on basis of the multistage systems of Diophantine equations [4, 5, 6],

$$x_1, x_2, \ldots, x_m \overset{n}{=} y_1, y_2, \ldots, y_m \tag{1}$$

we propose mathematical model polyalphabetic cryptosystems, inverse transform algorithm (decryption) closed text which is to algorithmically unsolvable problem for policy analysis.

Let us associate with each numerical solution of (1)

$$a_1, a_2, \ldots, a_m \stackrel{n}{=} b_1, b_2, \ldots, b_m$$

two knapsack vectors $A = (a_1, a_2, \ldots, a_m)$ and $B = (b_1, b_2, \ldots, b_m)$, describing them as well - strong backpacks [4] of dimension m degree n. This ratio is recorded as followed:

$$A \stackrel{n}{=} B \quad \text{or} \quad (a_1, a_2, \ldots, a_m) \stackrel{n}{=} (b_1, b_2, \ldots, b_m).$$

and it has the following properties of equivalence.

For example, the following normal [4] two-parameter knapsack dimension $m = 5$, $n = 4$ degree 4.

$$(19a{+}b,15a{+}5b,11a{+}9b,3a{+}17b,2a{+}18b)\stackrel{4}{=}(a{+}19b,5a{+}15b,9a{+}11b,17a{+}3b,18a{+}2b)$$

are equivalent.

In particular, when $a = 1$, $b = 2$, then the followings are equivalent to the normal power numeric backpacks $n = 4$:

$$(21, 25, 29, 37, 38) \stackrel{4}{=} (39, 35, 31, 23, 22).$$

We consider Diophantine representation of the family of multistage systems

$$x_1, x_2, \ldots, x_m \stackrel{n}{=} p_1, p_2, \ldots, p_m \tag{2}$$

and define the set W (Diophantine set) [7]

$$W = \{p_1, p_2, \ldots, p_m | x_1, x_2, \ldots, x_m \stackrel{n}{=} p_1, p_2, \ldots, p_m\}$$

— non-negative integer values of ordered sets $p_1, \ldots, p_m$, for which the equation (2) is solvable with respect to unknown $x_1, \ldots, x_m$.

The paper presents the Diophantine representation of multistage systems of Diophantine equations for different parameters m and n, and form most common parametric solutions are given in the form of a system of linear equations . To build effective information security systems based on Diophantine representation of multistage systems of Diophantine equations , we can apply the following main theorem [3].

Theorem 1. *Suppose there are two pairs of equivalent numeric knapsacks, the first of which is an arbitrary parametric analytical solution multistage system of Diophantine equations of n-th degree*

$$x_1, x_2, \ldots, x_m \stackrel{n}{=} y_1, y_2, \ldots, y_m$$

and the second is any extension of the first :

$$(a_1, a_2, \ldots, a_m) \stackrel{n}{=} (b_1, b_2, \ldots, b_m), \ (\text{или } A \stackrel{n}{=} B), \ 1 \le n < m;$$

$$(c_1, c_2, \ldots, c_k) \stackrel{n+t}{=} (d_1, d_2, \ldots, d_k), \ (\text{или } C \stackrel{n+t}{=} D), \ t \ge 1, \ 1 \le n + t < k.$$

Then the problem of equivalent numeric backpacks (A, v) (or (B, v)) is solvable, and its solution coincides with the decision to enter (C, v) (or (D, v)).

Mathematical model to describe the alphabet cryptosystem used tuple different from the previously considered [8, 9, 10, 11]

$$\sum\nolimits_0 = \langle M^*, E(m), D(s), S^* | V(E(m), D(s)) \rangle \tag{3}$$

where M^* – the set of all messages $m = m_1, m_2, \ldots, m_k$ (open texts) over the alphabetic or numeric alphabet M; here m_i, $i = 1 \ldots k$, – elementary messages (in particular, letters or characters concatenation alphabet of M); S^* – the set of all ciphertext (cryptograms) $s = s_1, s_2, \ldots, s_k$ of cryptosystem (3); $E(m)$ – the algorithm of the direct transformation (encryption) of the message m to s; $D(s)$ – inverse algorithm of transition (decrypt) the ciphertext (cryptogram) s to $m \in M^*$.

We emphasize that the algorithm E and D alphabetic cryptosystem (3) are interconnected in such a way – $V(E, D)$, that any arbitrary message $m = m_1, m_2, \ldots$ $\ldots, m_k \in M^*$ can be uniquely transformed in corresponding cryptogram $s = s_1, s_2 \ldots$ $\ldots s_k \in S^*$ and vice versa.

For example, the model knapsack cryptosystem based on constructive knapsack is represented as:

$$\sum\nolimits_{D1} = \langle M^*, K_E(A, n), K_D(B, n), S^* | A \stackrel{5}{=} B \rangle.$$

We present alphabetical model of knapsack cryptosystem, main idea of which is that a legitimate user of the system due only one cipher determines Diophantine representation of the family of multistage system Diophantine equations. The difficulty for the non-legal user is to find the solution of multistage system Diophantine equations (1). This problem is easily solved legitimate user, because he knows how to find the solutions parametric Diophantine equations for higher degrees.

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UDC 511.512

ON SINGULAR SERIES THE PROBLEM ABOUT WEIGHTED NUMBER OF INTEGRAL POINTS ON FOUR-DIMENSIONAL QUADRATIC SURFACES

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Asymptotic formulas with remainder terms for the weighted number of integral points on s-dimensional quadratic surfaces were obtained in [1,2]; a diagonal case was considered in the first of them, but (and) the general case of four-dimensional quadratic surfaces the result of [2] has the form

$$I(n) = \pi^2 nW(n)H(p) + O\left(\|A\|\,|\delta_F|^3 n^{\frac{3}{4}+\varepsilon}\right),$$

where

$$I(n) = \sum_{p(x)=0} e^{-\frac{\omega(x)}{n}}$$

is the weighted number of integral points on the four-dimensional surface of second order

$$p(x) = Q_1(x_1, x_2) - Q_2(x_3, x_4) - h = 0,$$
$$\omega(x) = \lambda(Q_1(x_1, x_2) + Q_2(x_3, x_4)), \lambda > 1.$$

$W(n)$ is complex integral which in our case has the form

$$W(n) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{e^{\frac{h}{n}\cdot z}}{\lambda^2 - z^2}dz, \quad \lambda > 1;$$

$$H(p(x)) = \sum_{q=1}^{\infty} q^{-4} \sum_{\substack{l=1\\(l,q)=1}}^{q} S(lp(x), q)e^{-2\pi i \frac{lh}{q}}$$

is the singular series of Hardy-Littlewood, where

$$S(p(x), q) = \sum_{x_1,\dots,x_4=1}^{q} e^{2\pi i \frac{p(x_1,\dots,x_4)}{q}} \quad \text{is} \quad \text{Gaussian} \quad \text{sum},$$

$\|A\|$ is norm of a matrix $A = (a_{ij})$, defined by the equality

$$\|A\| = \max_{1\leqslant i\leqslant 4} \sum_{j=1}^{4} |a_{ij}|,$$

δ_F is the discriminant of imaginary quadratic field $F = Q\left(\sqrt{d}\right)$, d is free from quad rates corresponding to binary quadratic forms $Q_1(x_1, x_2)$ and $Q_2(x_3, x_4)$.

Estimations form $W(n)$ and $H(p(x))$ were given in [1, 2] as well. In our case of four-dimensional quadratic surface not being cone it succeeds in finding wore precise estimation over for $H(p(x))$ if use the following statement about the sum of Ramanujan

$$c_q(h) = \sum_{\substack{l=1\\(l,q)=1}}^{q-1} e^{-2\pi i \frac{lh}{q}}.$$

Lemma 1. *For the sum of Ramanujan the following inequality*

$$c_q(h) \leqslant \text{НОД}\,(q, h).$$

is valid.

Using this lemma from [3, theorems 1-3] we receive the following result.

Theorem 1. *For singular series $H(p(x))$, where*

$$p(x) = Q_1(x_2, x_2) - Q_2(x_3, x_4) - h$$

and $\det(Q_1) = \det(Q_2) = -\delta_F$ *following upper estimates are valid* $0 < H(p(x)) \ll |\delta_F| n^{\varepsilon}$ *for* $h \ll n^{\varepsilon}$ *and* $0 < H(p(x)) \ll |\delta_F| h$ *for* $h = O(1)$.

This theorem makes more precise corresponding estimations from [1,2] in four-dimensional case which in our designations have the form $0 < H(p(x)) \ll |\delta_F|^2 n^{\varepsilon}$, where ε is arbitrarily small positive number.

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UDC 512.5

CONTINUED AND NOT-CONTINUED PARTIAL SEMIGROUPS

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Partial semigroup is a set with a partial associative operation. Full operation is an operation whose values are defined for all sets of elements, partial is an operation whose values are determined, maybe not for all sets. An interesting question is the possibility of continue the partial operation to complete with the preservation of certain properties (eg associativity) VV Rosen [1] gave a definition of associative partial operation with two nonequivalent ways. Weak associativity means that for the elements a, b, c the equality $(ab)c = a(bc)$ is right, if both products $(ab)c$, $a(bc)$ are exist. Strong associativity means that for any elements a, b, c either the equality $(ab)c = a(bc)$, or both products $(ab)c$ и $a(bc)$ does not exist. Obviously, we can continue the operation, adding to the set of elements and all not defined products equate to this one (see ES Lyapin and AE Evseev [2]). Author compiled computer program that checks whether to continue the operation of partial semigroup defined on n-element set. With the help of the program found that when $n \leqslant 4$ any partial semigroup can be extended to the full without adding a new element . Obviously, if a partial semigroup S has a passive element (ie an element z, that $z \cdot a$ and $a \cdot z$ is not defined in any $a \in S$),than the operation can be extended to full without adding item.

Theorem 1. *Partial semigroup of nonzero elements of a completely 0 - simple semigroup can be continued.*

Example 1. Consider a partial semigroup S of all nonempty partial transformations of two-element set:

	1	2	3	4	5	6	7	8
1	1	1	1	5	5	5	–	–
2	1	2	3	4	5	6	7	8
3	3	3	3	6	6	6	–	–
4	1	4	7	2	5	8	3	6
5	1	5	–	1	5	–	1	5
6	3	6	–	3	6	–	3	6
7	7	7	7	8	8	8	–	–
8	7	8	–	7	8	–	7	8

Which can be extended in a following way:

	1	2	3	4	5	6	7	8
1	1	1	1	5	5	5	1	5
2	1	2	3	4	5	6	7	8
3	3	3	3	6	6	6	3	6
4	1	4	7	2	5	8	3	6
5	1	5	1	1	5	5	1	5
6	3	6	3	3	6	6	3	6
7	7	7	7	8	8	8	7	8
8	7	8	7	7	8	8	7	8

Example 2 Consider the subsemigroup of the partial semigroup S of all nonempty partial transformations of three - element set spanned by the elements:
$1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, $2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$, $5 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$,
$6 = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$, $7 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $8 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $9 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

The Keli matrix of this semigroup has this form:

	3	4	8	23	24	44	61	62	63
3	3	4	4	23	24	44	61	62	63
4	4	4	4	24	24	44	–	–	–
8	4	4	8	24	24	44	–	–	–
23	3	4	24	23	24	44	61	62	63
24	4	4	24	24	24	44	–	–	–
44	44	–	–	44	–	–	4	24	44
61	61	61	61	62	62	63	–	–	–
62	61	61	62	62	62	63	–	–	–
63	63	–	–	63	–	–	61	62	63

This semigroup can not be extended.

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ON SPECIAL LIE ALGEBRAS HAVING THE FAITHFUL MODULE WITH KRULL DIMENSION

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We will denote prime radical of associative or Lie algebra D through $P(D)$.
We will remind definition of Krull dimension of module [1].

Definition 1. *We will define Krull dimension $Kdim(M)$ of the left R-module M by means of transfinite induction:*
1) If $M = 0$, $\mathrm{Kdim}\,(M)$ *=-1;*
2) If $\mathrm{Kdim}\, M \nless \alpha$, $Kdim(M) = \alpha$ *if and only if there is no infinite decreasing chain of submodules*

$$M = M_0 \supset M_1 \supset \dots,$$

such that $\mathrm{Kdim}\ (M_{i-1}/M_i) \nless \alpha$ *for each* $i \in \mathbb{N}$*;*
3) If there is no such oder number alpha that

$$\mathrm{Kdim}\,(M) = \alpha,$$

then the module M has no Krull dimension.

Definition 2. *Krull dimension of a ring of R as left R - ${}_RR$ module is called Krull dimension of R.*

V. T. Markov proved the following statement [2].

Theorem 1. *Let R be PI-ring, M – faithful left R-module with Krull dimension. Then:*

(a) $P(R)$ is a nilpotent ideal;

(b) If moreover - the module M and the left ideal $P(R)$ - are finitely generated, then R has Krull left dimension and

$$\mathrm{Kdim}\,({}_RR) = \mathrm{Kdim}\,(M).$$

In 1963 V. N. Latyshev introduced a new class of Lie algebras [3] which he called special by the analogy with Jordan algebras.

We say that Lie algebra L the special or SPI- Lie algebra if there is associative PI - algebra A such that L is included in $A^{(-)}$ as Lie algebra, where $A^{(-)}$ – Lie algebra which has been obtained from A by means of operation $[x, y] = xy - yx$.

The analog of result of V. T. Markov for special Lie algebras take place.

We will define Krull dimension for modules over Lie algebras and Lie algebras the same as for associative algebras. As all ideals of Lie algebra are double-sided, there is not necessity to speak about Krull right and left dimension of Lie algebra, and we call it Krull dimension.

Definition 3. *Let $\bar{L}$ be a homomorphic image of Lie algebra L in algebra of endomorphisms* $\mathrm{End}(M)$ *of the module M. The set $\bar{L}$ is Lie algebra with operation $[x, y] = xy - yx$.*

We will denote through $A(L)$ associative subalgebra generated by $\bar{L}$ in algebra of endomorphisms $\mathrm{End}(M)$ of the module M and we will call it the associated algebra of representation M.

We will call PI - representation of Lie algebra L [4] the representation of algebra L in algebra of endomorphisms of $\mathrm{End}(M)^{(-)}$ of module M over algebra of L, for which associated algebra of representation $A(L)$ is PI-algebra.

The concept of PI-representations is required as associative algebra $A(L)$ of representation of special Lie algebra may not to be a PI - algebra.

As an example it is possible to take known irreducible representation of three dimensional nilpotent Lie algebra in algebra of endomorphisms of a ring of polynoms from one variable over a field of characteristic zero (see, for example, [5]).

For Lie special algebras the analog of the theorem of V. T. of Markov [2] takes place.

Theorem 2. *Let L - Lie be the special Lie algebra over field F, M – faithful PI-representation of Lie algebra L with Krull dimension. Then $P(L)$ - a solvable ideal.*

Lie algebra of L is called semiprime if it doesn't contain nonzero Abelian ideals. This definition is equivalent to equality to zero prime radical $P(L)$ of Lie algebra L [6].

The following analog of a proposition of [2] take place.

PROPOSITION 1. *If semiprime special Lie algebra L has the faithful module with Krull dimensionality, then L is a finite subdirect product of prime Lie algebras.*

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PARTIAL ORDERS ON A SET OF IDEMPOTENTS OF THE SEMIGROUP OF BOOLEAN MATRICES

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Let $\langle \mathbf{B}_{m\times n}, \cup, \cap, ', O, I\rangle$ be Boolean algebra of $m \times n-$matrices with entries belonging to Boolean algebra $\langle \mathbf{B}, \cup, \cap, ', 0, 1\rangle$. The operations $\cup, \cap, '$ and consequently relation of partial order $\subseteq$ are defined in an elementwise way. The matrices O and I whose entries are only zeroes 0 and units 1 accordingly, give zero and unit of such Boolean algebra.

DEFINITION 1. *Call matrix $C = A \sqcap B \in \mathbf{B}_{m\times k}$ with entries $C^i_j = \bigcup_{t=1}^{n}(A^i_t \cap B^t_j)$* **conjunctive composition** *of matrices $A = (A^i_j) \in \mathbf{B}_{m\times n}$ and $B = (B^i_j) \in \mathbf{B}_{n\times k}$. The* **disjunctive composition** *$A \sqcup B$ is defined in a dual way:$A \sqcup B = (A' \sqcap B')'$.*

Let symbol $\mathbf{M}(\mathbf{B})$ denote a set of all matrices of finite sizes, that is, $\mathbf{M}(\mathbf{B}) = \bigcup_{m,n\in\mathbf{N}} \mathbf{B}_{m\times n}$.

The pairs $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$ and $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$ form partial semigroups with partial binary operations. Moreover, inequality $A \subseteq B$ implicates $A \sqcap C \subseteq B \sqcap C$, $C \sqcap A \subseteq C \sqcap B$ and $A \sqcup C \subseteq B \sqcup C$, $C \sqcup A \subseteq C \sqcup B$. The complement of Boolean matrices is isomorphism of partial semigroups $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$ and $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$ because the equalities $(A \sqcap B)' = A' \sqcup B'$ and $(A \sqcup B)' = A' \sqcap B'$ are true. It is known [1], that conjunctive and disjunctive compositions do not form an associative couple.

Let symbol A^T designate transpose of matrix A and $A'^T = (A^T)' = (A')^T$. Note that $(A \sqcap B)^T = B^T \sqcap A^T$, $(A \sqcup B)^T = B^T \sqcup A^T$ and $(A \sqcap B)' = A' \sqcup B'$, $(A \sqcup B)' = A' \sqcap B'$.

A symbol E denotes below a square unit matrices $E = (\delta^i_j)$ with entries δ^i_j which are 1 if $i = j$, and it is 0 if $i \neq j$. And corresponding to the context size of matrix E will not be pointed out.

Definition 2. *Call an idempotent matrix $A = A \sqcap A$* **a primary idempotent** *if $E \nsubseteq A$, and* **a secondary idempotent** *of partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$ if $E \subseteq A$.*

Primary and secondary idempotents of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$ are defined in a dual way, that is, matrix $A = A \sqcup A$ is *a primary idempotent* if $A \nsubseteq E'$, and *a secondary idempotent* if $A \subseteq E'$

It was shown in [1] and [2] that using operations of dual compositions, complementation and transposition, any Boolean matrix of arbitrary size generates *a secondary idempotent of the right type:* $A^{\mathcal{R}} = A \sqcup A'^T$, $A_{\mathcal{R}} = (A^{\mathcal{R}})'^T = A \sqcap A'^T$ and *a secondary idempotent of the left type:* $A^{\mathcal{L}} = A'^T \sqcup A$, $A_{\mathcal{L}} = (A^{\mathcal{L}})'^T = A'^T \sqcap A$. Moreover, matrices $A^{\mathcal{R}}$ and $A^{\mathcal{L}}$ are idempotents of $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$, and $A_{\mathcal{R}}$, $A_{\mathcal{L}}$ are idempotents of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$. It was shown also that Boolean matrix A is a secondary idempotent of $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$ iff $A = A^{\mathcal{R}} = A^{\mathcal{L}}$. And a matrix A is a secondary idempotent of $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$ iff $A = A_{\mathcal{R}} = A_{\mathcal{L}}$.

It is known [3], [4] that a set of idempotents of the semigroup has a natural partial order. It is defined by $(B \leq^{\sqcap} A) \leftrightarrow (B = B \sqcap A = A \sqcap B)$ for the idempotent matrices A and B in the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$. And the natural partial order is defined by $(B \leq_{\sqcup} A) \leftrightarrow (B = B \sqcup A = A \sqcup B)$ for the idempotent matrices A and B in the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$.

Note that the sets of secondary idempotents of the partial semigroups $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$ and $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$ do not intersect.

Moreover, the partial order of Boolean algebra of matrices $\subseteq$ and the natural partial order $\leq_{\sqcup}$ coincide for a set of the secondary idempotent matrices of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcup\rangle$, and the partial order $\subseteq$ and the natural partial order $\leq^{\sqcap}$ are contrary for a set of the secondary idempotent matrices of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcap\rangle$. Precisely, we get

Theorem 1. *The equivalence*

$$A \leq_{\sqcup} B \leftrightarrow A \subseteq B$$

holds for any secondary idempotents A and B of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcup \rangle$.

If A and B are the secondary idempotents of the partial semigroup $\langle \mathbf{M}(\mathbf{B}), \sqcap \rangle$ *then*

$$A \leq^{\sqcap} B \leftrightarrow B \subseteq A.$$

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UDC 501.1

ON VARIETIES OF SEMIGROUPS OF RELATIONS WITH DESCRIPTOR OF FIXED POINT AND OPERATION OF REFLEXIVE DOUBLE CYLINDRIFICATION

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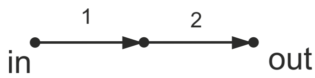
A set of binary relations Φ closed with respect to some collection Ω of operations on relations forms an algebra (Φ, Ω) which is called an *algebra of relations*. One of the most important operations on relations is the operation of product $\circ$. Algebra of relations $(\Phi, \circ)$ forms semigroup of relation and any semigroup isomorphic to some semigroup of relation.

Operations on relations can be defined by a logical formulas. Such operations are called *logical*. An important class of logical operations on relations is the class of diophantine operations. Operation is called *diophantine* [1, 2] (in other terminology – primitive positive [3]) if it can be defined by a formula, which contains in the prenex normal form only operations of conjunction and existence quantifiers.

Diophantine operations are described using graphs [1, 2, 3]. Denote by N the set of all natural numbers. A labeled graph is a pair $(V,\ E)$, where V is a finite set, called vertex set, and $E \subseteq V \times N \times V$ is a ternary relation.

A triple $(u, k, v) \in E$ is called an edge from u to v labeled by k, and graphically represented by $u\cdot \overset{k}{\to} \cdot v$. By an input-output-pointed labeled graph we mean a structure $G = (V, E, in, out)$ with two distinguished vertices, where (V, E) is a labeled graph; in and out are two distinguished vertices called input and output vertex respectively.

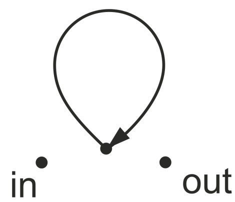
Note that the input-output-pointed labeled graph corresponding to the operation of relation product $\circ$ has the following form:



We shall concentrate our attention on the operations of relation product $\circ$, descriptor of fixed point ∇_1, and operation of reflexive double cylindrification ∇_2. These operations are defined as follows:

$$\nabla_1(\rho) = \{(x, x) : (\exists y)(y, y) \in \rho\}, \qquad \nabla_2(\rho) = \{(x, y) : (\exists z)(z, z) \in \rho\}.$$

The input-output-pointed labeled graphs corresponding to these operations have the following forms:



For the set Ω of operations on binary relations, denote by $R\{\Omega\}$ the class of algebras isomorphic to algebras of relations with operations from Ω. Let $Var\{\Omega\}$ be the variety generated by the class $R\{\Omega\}$.

The basis of identities of the variety generated by a class of semigroups of binary relations with descriptor of fixed point ∇_1 is obtained in [4].

The following theorem gives the basis of identities for the variety $Var\{\circ, \nabla_1, \nabla_2\}$.

Theorem 1. *An algebra $(A, \cdot, ^\star, ^*)$ of the type (2,1,1) belongs to the variety $Var\{\circ, \nabla_1, \nabla_2\}$ if and only if it satisfies the identities:*

$$(xy)z = x(yz), \quad (x^\star)^2 = x^\star, \quad (x^*)^2 = x^*, \quad xy^\star = y^\star x,$$

$$x^*xx^* = x^*, \quad (x^*y)^2 = x^*y, \quad (xy^*)^2 = xy^*,$$

$$(xy)^{\star} = (yx)^{\star}, \quad (xy)^{*} = (yx)^{*}, \quad x^{*}yz^{*} = z^{*}yx^{*}, \quad (xy^{\star})^{\star} = y^{\star}x^{\star},$$
$$(xy^{*}z)^{*} = y^{*}zxy^{*}, \quad x^{*}yx^{*}zx^{*} = x^{*}zx^{*}yx^{*}, \quad x^{\star\star} = x^{\star}, \quad x^{**} = x^{*},$$
$$(xy^{\star})^{*} = x^{*}y^{\star}, \quad (x^{*}y^{*})^{\star} = x^{\star}y^{\star}, \quad (x^{*}yz^{*})^{\star} = x^{\star}(yz^{*})^{\star},$$
$$(xy^{*}z)^{\star} = (xy^{*})^{\star}(y^{*}z)^{\star}, \quad x^{*}(x^{p})^{*} = x^{*} \text{ for any prime number } p.$$

The basis of identities found in the Theorem 1 is infinite. Is there a finite basis for this variety? The answer to this question is given by Theorem 2.

THEOREM 2. *The variety* $Var\{\circ, \nabla_1, \nabla_2\}$ *is not finitely based.*

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AN ASYMPTOTIC FORMULA FOR THE WARING – GOLDBACH PROBLEM WITH SHIFTED PRIMES

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In 1938 г. Hua Loo-Keng [1] proved that any sufficiently large natural number $N \equiv 5(mod\ 24)$ is a sum of five squares of primes. This paper deals with the analogue of this problem when the prime p is replaced by the shifted prime $p+1$, namely, we obtain an asymptotic formula for the number of such representations and determine arithmetic conditions under which the singular series of the problem is larger than absolute positive constant that depends solely on N. Statement of the problem belongs to professor V. N. Chubarikov.

Theorem 1. [2] *Let $I_2(N,1)$ denote the number of representations of a natural number N as a sum of five squares of shifted primes p_i+1, $i=\overline{1,5}$. Then for $I_2(N,1)$ the following asymptotic formula is valid:*

$$I_2(N,1)=\frac{4\pi^2\mathfrak{S}(N)N^{\frac{3}{2}}}{3\mathscr{L}^5}+O\left(\frac{N^{\frac{3}{2}}}{\mathscr{L}^6}\right),\quad \mathscr{L}=\ln N,$$

where singular series $\mathfrak{S}(N)$ converges absolutely and the following relation holds true

$$\mathfrak{S}(N)=\begin{cases} c(N), & \text{если } N\equiv 0 \pmod 4;\\ 0, & \text{если } N\not\equiv 0 \pmod 4,\end{cases}\qquad c(N)>2\prod_{p\geqslant 7}\left(1-\frac{10}{\varphi^2(p)}\right).$$

The proof of the theorem 1 is bases on author's papers [3, 4, 5]. These results are formulated as lemmas 1, 2, 3 and deal with

- the study of singular series

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}\frac{1}{\varphi^5(q)}\sum_{\substack{a=0\\(a,q)=1}}^{q}\left(\sum_{\substack{n=1\\(n,q)=1}}^{q}e\left(\frac{a(n+1)^2}{q}\right)\right)^5 e\left(-\frac{aN}{q}\right)$$

  and determining arithmetic conditions under which the singular series is larger than absolute positive constant that depends solely on N;

- the study of behavior of exponential sums over primes

$$V_m(\alpha;x,k)=\sum_{p\leqslant x}e(\alpha(p+k)^m),\ \alpha=\frac{a}{q}+\lambda,\quad (a,q)=1,\ |\lambda|\leqslant\frac{1}{q\tau},\ 1\leqslant q\leqslant\tau,$$

  when α is approximated by a rational number with small denominator. There we establish the relationship between this exponential sum and the density theorems for the zeroes of Dirichlet L – series in a short rectangle of a critical strip;

- estimating $V_2(\alpha;x,1)$ when α is approximated by a rational number with large denominator.

Lemma 1. *The following relationship is valid*

$$\mathfrak{S}(N)=\begin{cases} c(N), & \text{если } N\equiv 0 \pmod 4;\\ 0, & \text{если } N\not\equiv 0 \pmod 4,\end{cases}$$

where $c(N)$ – is an absolute positive constant that depends solely on N and

$$c(N)>2\prod_{p\geqslant 7}\left(1-\frac{10}{\varphi^2(p)}\right).$$

Lemma 2. *Suppose that $x \geqslant x_0$, $\tau \geqslant x^{m-\frac{3}{8}}\exp(\ln^{0,76} x)$, $q \leqslant x^{\frac{1}{4}} \exp\left(-\ln^{0,76} x\right)$, $b \geqslant 222(m+1)$ is an arbitrary fixed positive number, k is a fixed positive natural,*

$$T_m(a,q) = \sum_{\substack{n=1\\(n,q)=1}}^{q} e\left(\frac{a(n+k)^m}{q}\right), \quad F(q,x) = \begin{cases} \exp(-\ln^4 \ln x) & \text{если } q \leqslant (\ln x)^b, \\ (\ln x)^{B+4} & \text{если } q > (\ln x)^b. \end{cases}$$

Then the following estimate holds:

$$V_m(\alpha; x, k) = \frac{T_m(a,q)}{\varphi(q)}\gamma(\lambda; x, k) + \left(\frac{xq^{\frac{m}{m+1}}}{\varphi(q)}F(q,x)\right),$$

$$\gamma(\lambda; x, k) = \int_2^x \frac{e(\lambda(u+k)^m)}{\ln u} du.$$

Lemma 3. *Suppose that $x \geqslant x_0$, then the following estimate holds*

$$V_2(\alpha; x, 1) = \sum_{p \leqslant x} e(\alpha(p+1)^2) \ll \left(xq^{-\frac{1}{8}} + x^{\frac{15}{16}} + x^{\frac{3}{4}}q^{\frac{1}{8}}\right)\mathscr{L}^8.$$

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GENERALIZED ESTERMANN'S TERNARY PROBLEM FOR NON-INTEGER POWERS WITH ALMOST EQUAL SUMMANDS

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Estermann [1] proved an asymptotic formula for the number of solutions of the equation

$$p_1 + p_2 + n^2 = N, \tag{1}$$

where p_1, p_2 are primes and n is a natural number.

Professor V. N. Chubarikov stated the following problem: *To study the equation (1), where the summand n^2 is replaced by $[n^c]$, c is a fixed noninteger and impose more rigid conditions, namely, when the summands are almost equal.* We call it *the generalized Estermann's ternary problem with almost equal summands for non-integer powers of natural numbers.*

Theorem 1. [2] *Suppose that N is a sufficiently large positive integer, $\mathscr{L} = \ln N$, c is a fixed noninteger satisfying the following conditions*

$$\|c\| \geqslant 3c\left(2^{[c]+1} - 1\right)\frac{\ln \mathscr{L}}{\mathscr{L}}, \quad c \ > \ \frac{4}{3} + \mathscr{L}^{-0,3}. \tag{2}$$

Let us denote by $I(N, H)$ the number of solutions of the equation

$$p_1 + p_2 + [n^c] = N, \quad \left|p_i - \frac{N}{3}\right| \leqslant H, \quad i = 1, 2, \quad \left|[n^c] - \frac{N}{3}\right| \leqslant H$$

in primes p_1, p_2 and natural numbers n. Then for $H \geqslant N^{1-\frac{1}{2c}}\mathscr{L}^2$ the following asymptotic formula is valid:

$$I(N, H) = \frac{18}{3^{\frac{1}{c}}c} \cdot \frac{H^2}{N^{1-\frac{1}{c}}\mathscr{L}^2} + O\left(\frac{H^2}{N^{1-\frac{1}{c}}\mathscr{L}^3}\right).$$

Corollary 1. *Every positive integer $N > N_0$ is represented as a sum of two primes p_1, p_2 and non-integer power of a natural number m satisfying the following conditions*

$$\left|p_i - \frac{N}{3}\right| \leqslant N^{1-\frac{1}{2c}}\mathscr{L}^2, \quad \left|m - \left(\frac{N}{3}\right)^{\frac{1}{c}}\right| \leqslant \frac{3}{3^{\frac{1}{c}}c}N^{\frac{1}{2c}}\mathscr{L}^2 - \frac{9(c-1)}{3^{\frac{1}{c}}2c^2}\mathscr{L}^4 + 1,$$

where c is is a fixed noninteger that satisfies (2).

The proof of the theorem 1 is based Hardy-Littlewood-Ramanujan circle method in a form of Vinogradov's exponential sums. It consists of

- a lemma 1 on the estimate of short exponential sum $S_c(\alpha; x, y)$ with non-integer power a of natural number for $\alpha \in [-0,5,\ 0,5]$, including the neighbourhoods of points with small denominators except for a small neighbourhood of zero;
- a lemma 2 on the asymptotic formula for the sum $S_c(\alpha; x, y)$ with the reminder term for points α, lying in a small neighbourhood of zero;

- a lemma 3 on the behavior of short exponential sums over primes $S(\alpha;x,y)$ for points α, lying in a small neighbourhood of zero

LEMMA 1. [3, 4] *Suppose that $x\geqslant x_0>0$, $\mathscr{L}_x=\ln x$, A is a fixed positive integer larger than one, c is a noninteger satisfying*

$$1<c\leqslant \log_2\mathscr{L}_x-\log_2\ln\mathscr{L}_x^{6A}, \qquad \|c\|\geqslant \left(2^{[c]+1}-1\right)(A+1)\,\mathscr{L}_x^{-1}\ln\mathscr{L}_x.$$

Then for $y\geqslant\sqrt{2cx}\,\mathscr{L}_x^{A+\theta}$ and $x^{1-c}y^{-1}\mathscr{L}_x^{A}\leqslant|\alpha|\leqslant 0,5$ the following estimate is valid

$$S_c(\alpha;x,y)=\sum_{x-y<n\leqslant x}e(\alpha[n^c])\ll y\mathscr{L}_x^{-A},$$

where $\theta=0$ if $c\geqslant 1,1$ and $\theta=0,5$ if $c<1,1$.

LEMMA 2. [3, 4] *Suppose that $x\geqslant x_0>0$, A is a fixed positive integer larger than one, c is a noninteger satisfying*

$$1<c\leqslant \log_2\mathscr{L}-\log_2\ln\mathscr{L}^{6A}, \quad \|c\|\geqslant \left(2^{[c]+1}-1\right)(A+1)\,\frac{\ln\mathscr{L}}{\mathscr{L}}.$$

Then for $y\geqslant\sqrt{2c}x^{\frac{1}{2}}\mathscr{L}^{A}$ and $|\alpha|\leqslant x^{1-c}y^{-1}\mathscr{L}^{A}$ the following asymptotic formula is valid

$$S_c(\alpha;x,y)=\frac{\sin\pi\alpha}{\pi\alpha}\int_{x-y}^{x}e(\alpha(t^c-0,5))dt+O\left(\frac{y|\sin\pi\alpha|}{\mathscr{L}^{A}}\right).$$

LEMMA 3. [5] *Suppose that $x\geqslant x_0$, $y\geqslant x^{\frac{5}{8}}\exp(\ln x)^{0,67}$ and $|\alpha|\leqslant\dfrac{x}{y^2}$. Then the following equality holds:*

$$S(\alpha;x,y)=\sum_{x-y<n\leq x}\Lambda(n)e(\alpha n)=\frac{\sin\pi\alpha y}{\pi\alpha}e\left(\alpha\left(x-\frac{y}{2}\right)\right)+O\left(y\exp(-\ln^4\ln x)\right).$$

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SUM OF SHORT EXPONENTIAL SUMS OVER PRIME NUMBERS

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Vinogradov's method to estimate exponential sums over prime numbers allowed him to solve number of problems on prime numbers. One of the problems is a distribution of fractional parts of $\{\alpha p\}$, where he obtained more accurate results than in general case of distribution of fractional parts of $\{\alpha_n p^n + \ldots + \alpha_1 p\}$:

Theorem 1. ([1, 2]). *Let K be an integer, $K \leqslant N$, and α be a real number,*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad 1 \leqslant q \leqslant N,$$

then we have

$$V_K(N) = \sum_{k=1}^{K} \left| \sum_{p \leqslant N} e(\alpha k p) \right| \ll K N^{1+\varepsilon} \left(\sqrt{\frac{1}{q} + \frac{q}{N}} + N^{-0,2} \right).$$

The basis of this estimate are "Vinogradov's sieve" and nontrivial estimates of exponential sums of the form

$$W = \sum_{k \leqslant K} \left| \sum_{F_1 < m \leqslant F_2} a(m) \sum_{\substack{G_1 < n \leqslant G_2 \\ mn \leqslant x}} b(n) e(\alpha k m n) \right|,$$

where $a(m)$ and $b(n)$ are arbitrary complex valued-functions, K, F, G are natural numbers, $F \leqslant F_1 < F_2 \leqslant 2F$, $G \leqslant G_1 < G_2 \leqslant 2G$.

In this paper we will formulate theorems 2 and 3 on the estimate of exponential sums of the form

$$W(x, y) = \sum_{k \leqslant K} \left| \sum_{F_1 < m \leqslant F_2} a(m) \sum_{\substack{G_1 < n \leqslant G_2 \\ x - y < mn \leqslant x}} b(n) e(\alpha k m n) \right|.$$

These sums are obtained from $W(x)$ by replacing the condition $mn \leqslant x$ with the $x-y < mn \leqslant x$ where $\sqrt{x} < y \leqslant x\mathscr{L}_x^{-1}$, $\mathscr{L}_x = \ln xq$. We also formulate their application, namely, theorem 4 to estimate the sums of short exponential sums over prime numbers of the form

$$V_K(N,H) = \sum_{k=1}^{K} \left| \sum_{N-H<n\leqslant N} \Lambda(n)e(\alpha kn) \right|, \quad H \leqslant \frac{N}{\ln N}.$$

These sums arise in the study of distribution law of fractional parts of $\{\alpha p\}$, when prime number p lies in a short interval $N-H < p \leqslant N$.

Let us formulate the results. Theorem 1 deals with estimating the sum $W(x,y)$ that contains a "long" flat sum:

Theorem 2. *Suppose that the sum $W(x,y)$ satisfies the following conditions: $F \leqslant y$, $K \leqslant y$, $1 < q \leqslant Ky$,*

$$\sum_{t\leqslant 2KF} \left(\sum_{\substack{t=mk,\, 1\leqslant k\leqslant K \\ F<m\leqslant 2F}} |a(m)| \right)^2 \ll KF\mathscr{L}_x^{c_a}, \tag{1}$$

c_a is an absolute constant. If $b(n)=1$, then the following estimate is valid

$$W(x;y) \ll \begin{cases} Ky\left(\dfrac{1}{q}+\dfrac{F}{y}\right)^{\frac{1}{2}} \mathscr{L}_x^{\frac{c_a+1}{2}}, & \text{если } q < 4KF; \\ Ky \cdot \sqrt{\dfrac{q}{Ky}}\,\mathscr{L}_x^{\frac{c_a+1}{2}}, & \text{если } q \geqslant 4KF. \end{cases}$$

Corollary 1. *Suppose that A is an absolute constant. If $y \geqslant F\mathscr{L}_x^{2A+c_a+1}$ and $\mathscr{L}_x^{2A+c_a+1} \leqslant q \leqslant Ky\mathscr{L}_x^{-2A-c_a-1}$ then the following estimate is valid:*

$$W(x,y) = \sum_{k\leqslant K} \left| \sum_{F_1<m\leqslant F_2} a_m \sum_{\substack{G_1<n\leqslant G_2 \\ x-y<mn\leqslant x}} e(\alpha kmn) \right| \ll \frac{Ky}{\mathscr{L}_x^A}.$$

Theorem 2 deals with the sum $W(x,y)$, where sums constituting the double summation are "close" by order.

Theorem 3. *Suppose that the following conditions are satisfied in a sum $W(x,y)$: $F \leqslant y$, $G \leqslant y$, $K \leqslant y$, $1 < q \leqslant Ky^2x^{-1}$,*

$$\sum_{F_1<m\leqslant F_2} |a(m+m_1^*)|^2 \ll F\mathscr{L}_x^{c_a}, \qquad \sum_{G_1<n\leqslant G_2} |b(n)|^2 \ll G\mathscr{L}_x^{c_b}, \tag{2}$$

c_a and c_b are absolute constants, $m_1^* = 0$ or $F < m_1^* \leqslant 2F$. Then the following estimate is valid

$$W(x,y) \ll \begin{cases} Ky\left(\dfrac{1}{q} + \dfrac{F}{y} + \dfrac{x^2F^{-2}\mathscr{L}_x^{-4}}{y^2}\right)^{\frac14} \mathscr{L}_x^{\frac{c_a+c_b}{2}+1}, & \text{если } q < \dfrac{2Ky}{G}; \\ \\ Ky\left(\dfrac{qx}{Ky^2} + \dfrac{x^2F^{-2}\mathscr{L}_x^{-4}}{y^2}\right)^{\frac14} \mathscr{L}_x^{\frac{c_a+c_b}{2}+1}, & \text{если } q \geqslant \dfrac{2Ky}{G}. \end{cases}$$

Corollary 2. *Suppose that A is an absolute constant. If either of the two conditions are true:*

i. $y \geqslant \max(F\mathscr{L}_x^{4A+2c_a+2c_b+4}, xF^{-1}\mathscr{L}_x^{2A+c_a+c_b})$ *и* $\mathscr{L}_x^{2c_a+2c_b+4A+4} \leqslant q < 2KyG^{-1}$*;*

ii. $y \geqslant xF^{-1}\mathscr{L}_x^{2A+c_a+c_b}$ *и* $2KyG^{-1} \leqslant q \leqslant Ky^2x^{-1}\mathscr{L}_x^{-2c_a-2c_b-4A-4}$,

then the following estimate holds

$$W(x,y) \ll \frac{Ky}{\mathscr{L}_x^A}.$$

Theorem 4. *Suppose that K, H, N and q are natural numbers, $K \leqslant H$, A is an absolute constant, $\mathscr{L} = \ln Nq$, α is a real number and*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a,q) = 1, \quad \mathscr{L}^{4A+20} \leqslant q \leqslant \frac{KH^2}{N}\mathscr{L}^{-4A-20}.$$

If $H \gg N^{\frac23}\mathscr{L}^{4A+16}$ then the following estimate is valid

$$V_K(N,H) \ll \frac{KH}{\mathscr{L}^A}.$$

The proof of the theorem is based on Vinogradov's method of estimating exponential sums with prime numbers in a combination with the methods described in papers [3, 4, 5], using corollaries 1 and 2.

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TRIGONOMETRIC SUMS OF ALGEBRAIC NETS[1]

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The authors [6] formulate the problem of values of trigonometric sums nets.

Algebraic nets are a particular case of generalized parallelepipedal nets. General theory of generalized parallelepipedal nets was laid in the works [8] — [10].

The purpose of the work is to obtain a formula for trigonometric sums with grid scale, expressing the value of the funds through a number of points of a lattice.

In 1976, published K. K. Frolov [16], which first appeared algebraic nets. Most fully in the author's exposition of the method Frolov presented in his PhD thesis [17]. Later in the works of N. M. Dobrovol'skii [8] — [10] offered a modification of the method Frolov, using special weight functions. Modern full details of the method Frolov and its modifications by N. M. Dobrovolsky is given in the works [1] — [4].

We will use the notation and definitions from work [15].

Definition 1. *For an arbitrary Λ generalized lattice paralelipipedului net $M(\Lambda)$ is the set of $M(\Lambda) = \Lambda^* \cap G_s$.*

Net $M_1(\Lambda) = \Lambda^* \cap [-1; 1)^s$.

Generalized paralelipipedului mesh type II $M'(\Lambda)$ is the set $M'(\Lambda) = \{\vec{x} \,|\, \vec{x} = \{\vec{y}\}, \vec{y} \in M_1(\Lambda)\}$.

Definition 2. *Weight function of order r with B constant named smooth function $\rho(\vec{x})$ satisfying the conditions*

$$\sum_{\varepsilon_1,\ldots,\varepsilon_s=-1}^{0} \rho(\vec{x} + (\varepsilon_1, \ldots, \varepsilon_s)) = 1 \text{ if } \vec{x} \in G_s, \tag{1}$$

$$\rho(\vec{x}) = 0 \quad \text{if} \quad \vec{x} \notin (-1; 1)^s, \tag{2}$$

$$\left| \int_{-1}^{1} \ldots \int_{-1}^{1} \rho(\vec{x}) e^{2\pi i(\vec{\sigma}, \vec{x})} d\vec{x} \right| \leqslant B(\overline{\sigma}_1 \ldots \overline{\sigma}_s)^{-r} \quad \text{for any} \quad \vec{\sigma} \in \mathbb{R}^s. \tag{3}$$

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If conditions (1) and (2), we talk about just the weight function $\rho(\vec{x})$.

Definition 3. *A quadrature formula with a generalized paralelipipedului mesh typ II and the weight function $\rho(\vec{x})$ is called the formula of the form*

$$\int\limits_0^1 \cdots \int\limits_0^1 f(\vec{x})d\vec{x} = (\det\Lambda)^{-1} \sum_{\vec{x}\in M'(\Lambda)} \rho_{\vec{x}} f(\vec{x}) - R_{N'(\Lambda)}[f],$$

$$\textit{where} \quad \rho_{\vec{x}} = \sum_{\vec{y}\in M_1(\Lambda), \{\vec{y}\}=\vec{x}} \rho(\vec{y}), \quad N'(\Lambda) = |M'(\Lambda)|,$$

$R_{N'(\Lambda)}[f]$ — *the error of a quadrature formula.*

For the errors of quadrature formulas with a generalized paralelipipedului mesh type II on the class E_s^α a fair assessment (see [10], [15])

$$R_{N'(\Lambda)}[E_s^\alpha(C)] = \sup_{f\in E_s^\alpha(C)} |R_{N'(\Lambda)}[f]| \leqslant CB\cdot c_1(\alpha)^s \zeta_H(\Lambda|\alpha),$$

$$\text{where} \quad c_1(\alpha) = 2^{\alpha+1}\left(3 + \frac{2}{\alpha-1}\right), \quad \zeta_H(\Lambda|\alpha) = {\sum_{\vec{x}\in\Lambda}}' (\overline{x}_1 \ldots \overline{x}_s)^{-\alpha}.$$

Let $\vec{a} = (a_0, a_1, \ldots, a_{s-1})$ — integer vector such that the polynomial

$$P_{\vec{a}}(x) = \sum_{\nu=0}^{s-1} a_\nu x^\nu + x^s \tag{4}$$

is irreducible over the rationals and all the roots Θ_ν $(\nu = 1, \ldots, s)$ of equation (4) is valid.

Denote by $T(\vec{a})$ matrix degrees algebraically paired integer algebraic numbers $\Theta_1, \ldots, \Theta_s$ — roots of a polynomial $P_{\vec{a}}(x)$:

$$T(\vec{a}) = \begin{pmatrix} 1 & \ldots & 1 \\ \Theta_1 & \ldots & \Theta_s \\ \vdots & \vdots & \vdots \\ \Theta_1^{s-1} & \ldots & \Theta_s^{s-1} \end{pmatrix}, \tag{5}$$

and through $\vec{\Theta} = (\Theta_1, \ldots, \Theta_s)$ — is vector of the full set of algebraically paired numbers — roots of the polynomial $P_{\vec{a}}(x)$.

For any $t > 0$ lattice $\Lambda(t\cdot T(\vec{a}))$ is called algebraic. It has the form

$$\Lambda(t\cdot T(\vec{a})) = \left\{ \vec{x} = \left(t\sum_{\nu=1}^s \Theta_1^{\nu-1} m_\nu, \ldots, t\sum_{\nu=1}^s \Theta_s^{\nu-1} m_\nu \right) = t\cdot\vec{m}\cdot T(\vec{a}) \,\middle|\, \vec{m}\in\mathbb{Z}^s \right\}.$$

The set $M \subset G_s$ points $M_k = (\xi_1(k), \ldots, \xi_s(k)) \qquad (k = 1 \ldots N)$ is called it the grid M of N nodes, and themselves point - it the nodes of the quadrature formulas. Size $\rho_k = \rho(M_k)$ are called weights of the quadrature formula.

For an arbitrary integer $m_1, \ldots, m_s$ amount $S_{M,\vec{\rho}}(m_1, \ldots, m_s)$, defined by equality

$$S_{M,\vec{\rho}}(m_1, \ldots, m_s) = \sum_{k=1}^{N} \rho_k e^{2\pi i[m_1\xi_1(k)+\ldots+m_s\xi_s(k)]}, \tag{6}$$

called trigonometric sums grid with weights.

Let the matrix $T = T(\vec{a})$ and $t > 0$. Consider the algebraic net $M(t) = M'(t \cdot \Lambda(T))$ from $N'(t \cdot \Lambda(T))$ sites $\vec{x}_k$ $(k = 1, \ldots, N'(t \cdot \Lambda(T))$) with weights

$$\rho_k = \rho_{\vec{x}_k} = (\det(t \cdot \Lambda(T)))^{-1} \sum_{\{\vec{y}\}=\vec{x}_k,\, \vec{y}\in M_1(t\cdot\Lambda(T))} \rho(\vec{y})$$

and trigonometric sum with weights

$$S_{M(t),\vec{\rho}}(\vec{m}) = (\det(t \cdot \Lambda(T)))^{-1} \sum_{\vec{x}\in M(t)} \left(\sum_{\{\vec{y}\}=\vec{x},\, \vec{y}\in M_1(t\cdot\Lambda(T))} \rho(\vec{y}) \right) e^{2\pi i(\vec{m},\vec{x})}.$$

THEOREM 1. *For an arbitrary lattice Λ and an arbitrary weight function $\rho(\vec{x})$ true equality*[2]

$$S_{M,\vec{\rho}}(\vec{m}) = \delta(\vec{m}) + \sum_{\vec{x}\in\Lambda}{}' \int_{-1}^{1} \ldots \int_{-1}^{1} \rho(\vec{y}) e^{2\pi i(\vec{y},\vec{m}-\vec{x})} d\vec{y}, \tag{7}$$

where

$$\delta(\vec{m}) = \begin{cases} 1, & \text{if } \vec{m} = \vec{0}; \\ 0, & \text{if } \vec{m} \neq \vec{0},\ \vec{m} \in \mathbb{Z}^s. \end{cases}$$

Theorem 1 and the definition of a weight function $\rho(\vec{x})$ of order r with the constant B allows to obtain an estimate for the trigonometric sums generalized paralelipipedului grid with weight function

$$|S_{M,\vec{\rho}}(\vec{m}) - \delta(\vec{m})| \leqslant B \sum_{\vec{x}\in\Lambda}{}' (\overline{m_1 - x_1} \ldots \overline{m_s - x_s})^{-r}.$$

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ON SATURATED FORMATIONS OF FINITE MONOUNARY ALGEBRAS

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A class of algebraic systems which is closed under homomorphic images and finite subdirect products is called a *formation*.

Formations was widely used in group theory ([1]). Particularly, the saturated formations of groups is one of the most studied formations. A formation of groups is said to be a *saturated formation* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$ for an arbitrary group G and it's Frattini subgroup $\Phi(G)$.

Generalizations of these definitions had made by several authors ([2, 1] for example). We recall such definitions from [1]. A congruence θ on the algebraic system A is called a *Frattini congruence* if the union of all θ-classes generated by the elements of B differs from A for each proper subsystem B of the algebraic system A. A class $\mathfrak{X}$ is *saturated in the class* $\mathfrak{Y}$, if $A \in \mathfrak{Y}$ and $A/\theta \in \mathfrak{X}$ for some Frattini congruence θ on A implies $A \in \mathfrak{X}$.

We consider finite formations of monounary algebras ([3]) in this paper.

An element a of a monounary algebra $\langle A, f\rangle$ is *cyclic* if $f^n(a) = a$ for some positive integer n. A monounary algebra is *cyclic* if all of it's elements are cyclic.

Theorem 1. *The empty formation, the formation of all finite cyclic monounary algebras and the formation of all finite monounary algebras are the only saturated formations in the class of all finite monounary algebras.*

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HISTORY OF CREATION OF THE NUMBER-THEORETIC METHOD IN THE APPROXIMATE ANALYSIS[1]

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All events on formation of the new direction of mathematical researches - "A number-theoretic method in the approximate analysis occurred in Mathematical institute of V. A. Steklov of Academy of Sciences of the USSR within work of a seminar on number-theoretic methods in the approximate analysis, organized in 1956. As N. M. Korobov specifies: "Meeting of a seminar was held under the chairmanship of one of three of his heads — N. S. Bakhvalova (Moscow State University), N. N. Chentsov (IPM of Academy of Sciences of the USSR) and N. M. Korobov (MI of Academy of Sciences of the USSR)" [13].

According to associate professor E. A. Morozova, N. N. Chentsov's widow, the initiative of the organization of such seminar proceeded from Nikolay Nikolaevich Chentsov who worked at this time in I. M. Gelfand's dealing with computing problems of the domestic nuclear project the group.

In five years participants of a seminar did considerable work on which heads reported at the Fourth All-Union mathematical congress in 1961 in Leningrad. On behalf of heads the report was done by N. N. Chentsov. The list of publications was provided in the report from 27 names, from which 12 in Reports of Academy of Sciences of the USSR.

Results of work of a seminar of three K for the first six years of work were reflected in N. M. Korobov's monograph in 1963 [11] (the second edition left [14]in 2004). Abroad some monographs (see, for example,[21])]) were devoted to this problem.

Thus, we see that the solution of the vital problems of the computing practice which has arisen during implementation of the domestic nuclear project was motive

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of the organization of scientific activity on development of new multidimensional quadrature formulas.

Therefore the history of development of a number-theoretic method in the approximate analysis shares on two parts. The first part — is open theoretical part in which the first results and which continued to develop successfully all last 57 years were received. And the second part — is the applied closed part which it is possible to guess only.

Analyzing national history of development of a number-theoretic method in the approximate analysis, it is possible to allocate some stages of this development.

First, it is the initial stage 1956 — 1967. Rather complete actual idea of this stage can be received on works [11], [20] and [12].

The following stage of development of a number-theoretic method can be carried to 1976 - 1980гг. This stage first of all is connected with K. K. Frolov's works [18], [19]. At this stage there was a few operation, but K. K. Frolov's works made a basic contribution to the theory as optimum quadrature formulas on the class E_s^α were constructed.

About the Tula stage developments of a number-theoretic method in the approximate analysis it is possible to gain rather complete idea on works [1] — [10], [15] — [17].

Undoubtedly, studying of history of development of a number-theoretic method in the approximate analysis is now in a stage of formation and demands further systematic researches.

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NUMBER THEORETIC METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS[1]

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A partial differential equation for the function $u(t, \vec{x})$ is an equation of the form

$$\frac{\partial u}{\partial t} = Q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}\right) u(t, \vec{x}), \tag{1}$$

$$0 \leqslant t \leqslant T, \quad -\infty < x_\nu < \infty \quad (\nu = 1, \ldots, s),$$

$$u(0, \vec{x}) = \varphi(\vec{x}), \quad \vec{x} = (x_1, \ldots, x_s), \tag{2}$$

where

$$Q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}\right) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s} a_{j_1,\ldots,j_s} \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_s}}{\partial x_s^{j_s}} \tag{3}$$

is Differential operator of order $n(Q) = n_1 + \ldots + n_s$, and $\varphi(\vec{x}) = \varphi(x_1, \ldots, x_s)$ — periodic with unit period in each of its arguments a function of class E_s^α $(\alpha > m(Q) + 1)$.

Then,

$$\varphi(x_1, \ldots, x_s) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_s=-\infty}^{\infty} c_{m_1,\ldots,m_s} e^{2\pi i(m_1 x_1 + \ldots + m_s x_s)} \tag{4}$$

and the Fourier coefficients of the estimate

$$|c_{m_1,\ldots,m_s}| \leqslant \frac{\|\varphi\|_{E_s^\alpha}}{(\overline{m_1} \ldots \overline{m_s})^\alpha}. \tag{5}$$

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Quantity

$$\|\varphi\|_{E_s^\alpha} = \sup_{m_1,\dots,m_s} |c_{m_1,\dots,m_s}(\overline{m_1}\dots\overline{m_s})^\alpha| < \infty \tag{6}$$

is the norm in space E_s^α, with respect to which it is non-separable Banach space.

DEFINITION 1. *Discrete Cauchy problem with lattice Λ is an equation of the form*

$$\frac{\partial u}{\partial t} = Q\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_s}\right)u(t,\vec{x}) \tag{7}$$

$$0 \leqslant t \leqslant T, \quad -\infty < x_\nu < \infty \quad (\nu = 1,\dots,s). \tag{8}$$

Discrete initial displacement:

$$u(0,\vec{x}) = \varphi(\vec{x}), \quad \vec{x} \in M(\Lambda), \tag{9}$$

where $\varphi(\vec{x})$ is periodic function of class E_s^α $(\alpha > m(Q)+1)$.

Solution of the discrete Cauchy problem with the lattice Λ called trigonometric polynomial with variable coefficients $u(t,\vec{x}) \in \mathbb{T}(M^*(\Lambda))$, which satisfies (7) in (8) discrete initial conditions (9).

THEOREM 1. *Solution of the discrete Cauchy problem with the lattice Λ is trigonometric polynomial*

$$u(t,\vec{x}) = \sum_{\vec{m}\in M^*(\Lambda)} c(\vec{m})e^{Q(\vec{m})t}e^{2\pi i(\vec{m},\vec{x})}, \tag{10}$$

where

$$c(\vec{m}) = c_{M(\Lambda),M^*(\Lambda)}(\vec{m}) = \frac{1}{N}\sum_{\vec{y}\in M(\Lambda)} \varphi(\vec{y})e^{-2\pi i(\vec{m},\vec{y})}. \tag{11}$$

Discrete Cauchy problem with lattice Λ for partial differential equations можно is approximation to the solution of the Cauchy problem (1) — (3).

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ABOUT QUANTITATIVE MEASURE OF QUALITY OF OPTIMUM COEFFICIENTS[1]

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In 1959 year professor N. M. Korobov offered the new class of teoretiko-of numerical nets — parallelepiped nets:

$$M_k = \left(\left\{ \frac{a_1 k}{N} \right\}, \ldots, \left\{ \frac{a_s k}{N} \right\} \right) \quad (k = 1, 2, \ldots, N)$$

and the proper squaring formulas with equal scales

$$\int_0^1 \cdots \int_0^1 f(\vec{x})\, d\vec{x} = \frac{1}{N} \sum_{k=0}^{N-1} f\left(\left\{ \frac{a_1 k}{N} \right\}, \ldots, \left\{ \frac{a_s k}{N} \right\} \right) - R_N[f],$$

where $R_N[f]$ – is an error of squaring formula.

On the class of E_s^α of periodic functions with the quickly converging rows of Fourier's the best results were got

$$|R_N[f]| << \frac{\ln^{\alpha(s-1)} N}{N^\alpha} \quad \text{(N. S. Bakhvalov [2], N. M. Korobov [10])}.$$

The special place of parallelepiped nets with optimum coefficients is explained by a circumstance that squaring formulas with these nets set the unsatiated algorithms of numeral integration on classes E_s^α $(\alpha > 1)$ (см. [1], [12]).

By the quantitative measure of quality of set of coefficients of a_0, a_1,…, a_s of parallelepiped net is named size

$$H(p, \vec{a}) = \frac{3^{s+1}}{p} \sum_{k=0}^{p-1} \prod_{j=0}^{s} \left(1 - 2 \cdot \left\{ \frac{a_j \cdot k}{p} \right\} \right)^2, \tag{1}$$

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which is equal to the close value of integral from a periodic function

$$h(\vec{x}) = \frac{3^{s+1}}{p} \prod_{j=0}^{s} (1 - 2\{x_j\})^2$$

on a squaring formula with a parallelepiped net

$$1 = \int_0^1 \dots \int_0^1 h(\vec{x}) d\vec{x} = \frac{3^{s+1}}{p} \sum_{k=0}^{p-1} \prod_{j=0}^{s} \left(1 - 2 \cdot \left\{\frac{a_j \cdot k}{p}\right\}\right)^2 - R_p[h],$$

where $R_p[h]$ — is an error of close integration.

Choice of function $h(\vec{x})$ and sizes $H(p, \vec{a})$ related to that function $h(\vec{x})$ it is the border function of class $E_s^\alpha\left(\cdot, \frac{\pi^2}{6}\right)$ (details of see [11]).

The quantitative measure of quality of optimum coefficients plays an important role in modern researches on the theory of oai?aoeei-?eneiaiai method of close analysis (см. [2] — [4], [6] — [11], [13] — [17]).

We will put $p_1 = \left[\frac{p-1}{2}\right]$, $p_2 = \left[\frac{p}{2}\right]$. A next lemma is just.

Lemma 1. *Equality is just*

$$3\left(1 - 2\left\{\frac{x}{p}\right\}\right)^2 = 1 + \frac{2}{p^2} + \sum_{m=-p_1}^{p_2}{}' \frac{6}{p^2 \sin^2 \pi \frac{m}{n}} e^{2\pi i \dfrac{mx}{p}}.$$

From which a theorem follows about the eventual row of Fur'e for the quantitative measure of quality of optimum coefficients.

Theorem 1. *Equality is just*

$$H(p, \vec{a}) = \left(1 + \frac{2}{p^2}\right)^s + \sum_{m_1, \dots, m_s = -p_1}^{p_2}{}' \frac{\delta_p()}{\psi(m_1) \dots \psi(m_s)},$$

где

$$\psi(m) = \begin{cases} \dfrac{p^2}{p^2 + 2}, & \textit{npu } m = 0; \\ \dfrac{p^2 \sin^2 \pi \frac{m}{p}}{6}, & \textit{npu } m \neq 0. \end{cases}$$

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ESTIMATES OF VOLUMES OF THE BINARY CODE NEIGHBORHOODS IN TERMS OF ITS SPECTRA[1]

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Two-sided estimates of the number of elements belonging to the r-neighborhood of a binary code in terms of its weight or distance spectra are obtained. These estimates are specified for first and second order Reed-Muller codes.

Let $\mathbb{F}_2$ be the field of two elements. For an arbitrary natural number n denote by $\mathbb{F}_2^n$ the n-dimensional space over $\mathbb{F}_2$. On the space $\mathbb{F}_2^n$ we introduce *Hamming metric*: distance $\mathrm{d}(x, y)$ between vectors $x, y \in \mathbb{F}_2^n$ is equal to the number of non-zero coordinates of their difference.

Arbitrary subset of the binary space $\mathbb{F}_2^n$ is called *binary code* $\mathcal{C}$ (see, e.g., [1]), and the code elements — *codewords*. The number n is called *length* of codewords and the code itself.

Distance from an arbitrary $x \in \mathbb{F}_2^n$ to the zero vector is called *Hamming weight* (or *weight*) $\mathrm{wt}(x)$ of the vector x. The value $\mathrm{d}(x, \mathcal{C}) = \min_{y \in \mathcal{C}} \mathrm{d}(x, y)$ is called the *distance* from $x \in \mathbb{F}_2^n$ to the code $\mathcal{C}$.

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Code distance $d = \mathrm{d}(\mathcal{C})$ is defined as minimal distance between two different elements of the code $\mathcal{C}$:

$$\mathrm{d}\,(\mathcal{C}) = \min_{\substack{a,b\in\mathcal{C}\\ a\neq b}} \mathrm{d}(a,b).$$

Binary *distance invariant* code is a subset $\mathcal{C}$ of the space $\mathbb{F}_2^n$ such that for all $x \in \mathcal{C}$ the multisets of Hamming distances $\rho(x,y)$, $y \in \mathcal{C}$, are the same. Reed-Muller and perfect codes are particular cases of the distance invariant codes (see, e.g., [2], [3]).

Split the space $\mathbb{F}_2^n$ on layers $\mathbb{F}_2^n(i) = \{x \in \mathbb{F}_2^n \colon \mathrm{wt}(x) = i\}$ and let

$$N_n^{(2)}(i,r) \stackrel{\mathrm{def}}{=} \left|\{x \in \mathbb{F}_2^n \colon \max\{\mathrm{d}\,(x,0), \mathrm{d}\,(x,c)\} \leqslant r, \mathrm{wt}(c) = i\}\right|.$$

Theorem 1. *Let $\mathcal{C} \subset \mathbb{F}_2^n$ be the binary distance invariant code of length n with minimal distance d and*

$$W_i = \mathbb{F}_2^n(i) \cap \mathcal{C}, \quad i \in \{0,1,\ldots,n\},$$

be the sets of codewords of the same weight, $\{|W_i|\}_{i=0}^n$ be weight spectrum of the code $\mathcal{C}$. Then

$$|\mathbb{F}_2(\mathcal{C},r)| = (1 - q(n,r))|\mathcal{C}| \sum_{m=0}^{r} C_n^m,$$

where $q(n,r) = 0$ for $0 \leqslant r < d/2$ and

$$0 \leqslant q(n,r) \leqslant \left(2 \sum\nolimits_{m=0}^{r} C_n^m\right)^{-1} \sum_{i=1}^{n} |W_i| N_n^{(2)}(i,r), \quad d/2 \leqslant r \leqslant n.$$

Several explicit estimates for $\sum\limits_{m=0}^{r} C_n^m$ and $N_n^{(2)}(i,r)$ are given below in the statements 1 and 2.

Theorem 2. *Let $\mathcal{C} \subset \mathbb{F}_2^n$ be the binary code of length n with minimal distance d,*

$$W_i(\mathcal{C}) = \{(c_1,c_2) : \, c_1,c_2 \in \mathcal{C}, \mathrm{d}(c_1,c_2) = i\}, \quad i \in \{0,1,\ldots,n\},$$

be sets of ordered codeword pairs with a given distance between them and $\mathbb{F}_2(\mathcal{C},r) = \{x \in \mathbb{F}_2^n \colon \mathrm{d}\,(x,\mathcal{C}) \leqslant r\}$ be r-neighborhood of the code $\mathcal{C}$. Then

$$|\mathbb{F}_2(\mathcal{C},r)| = (1 - q(n,r))|\mathcal{C}| \sum_{m=0}^{r} C_n^m,$$

where $q(n,r) = 0$ for $0 \leqslant r < d/2$ and

$$0 \leqslant q(n,r) \leqslant \left(2|\mathcal{C}| \sum\nolimits_{m=0}^{r} C_n^m\right)^{-1} \sum_{i=1}^{n} |W_i(\mathcal{C})| N_n^{(2)}(i,r), \quad d/2 \leqslant r \leqslant n.$$

Statement 1 ([4]). *For $r \leqslant n/2$ the estimates are true*

$$2^n\Phi\left(-\sqrt{nV\left(1-\frac{2r}{n}\right)}\right) \leqslant \sum_{m=0}^{r} C_n^m \leqslant 2^n\Phi\left(-\sqrt{nV\left(1-\frac{2(r+1)}{n}\right)}\right), \qquad (1)$$

where $V(z)=(1-z)\ln(1-z)+(1+z)\ln(1+z)=\sum_{s=1}^{\infty}\frac{z^{2s}}{s(2s-1)} \geqslant z^2$ *for* $|z|<1$.

Statement 2. *If* $0 \leqslant r \leqslant [n/2]$ *and* $0 \leqslant i \leqslant n$ *then*

$$N_n^{(2)}(i,r) \leqslant C_i^{i/2} C_{n-i}^{r-i/2}\frac{1+q_i}{(1-q_i)^2} \text{ for even } i;$$
$$N_n^{(2)}(i,r) \leqslant C_i^{[i/2]} C_{n-i}^{r-[i/2]}\frac{2}{(1-q_i)^2} \text{ for odd } i,$$

where $q_i=\frac{r-[i/2]}{n-i+[i/2]-r+1} \leqslant q=\frac{r}{n-r+1}$.

In [4] we have obtained explicit two-sided estimates for r-neighborhood of the binary first order Reed-Muller code, which are a consequence of theorem 1. Similar estimates for the second order Reed-Muller code are obtained by the author in [5].

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DIAGONAL RANKS OF REES SEMIGROUPS

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We say that semigroup S acts on a set X from the right, and that X is *right S-act* via $\alpha : X \times S \to X$, $(\alpha : (x,s) \mapsto xs \in X)$ if $x(st) = (xs)t$ for all $x \in X$, $s,t \in S$. The *diagonal right act of S* is the set $S \times S$ on which S acts via $(x,y)s = (xs, ys)$. *Right diagonal rank of semigroup S* (is denoted by $rdrS$) is called the smallest cardinality of generating set this act $(S \times S)_S$.

Diagonal rank of finite semigroup is interesting feature of this semigroup reflecting its properties. For example, if G is a group of n elements then $rdrS = n$. The purpose of this paper to compute right diagonal ranks of Rees matrix semigroups.

Теорема 1. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be Rees semigroup of matrix type over the group G with sandwich matrix P (see. [1], §3.1) and $|I|, |\Lambda|, |G| < \infty$. Then the right diagonal rank of semigroup S does not depend on the matrix elements P (depends only on its size) and if Λ consists of one element the right diagonal rank of semigroup S is equal to $|I|^2|G|$, $|I|^2|G|^2|\Lambda|(|\Lambda| - 1)$ otherwise.*

Теорема 2. *Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be Rees semigroup of matrix type over the group with zero $G \cup \{0\}$. Let $|I| = k$, $|G| = t$, $|\Lambda| = l$. Then for $l \geq 2$*

(i) if P has no zeros, then $rdrS = k^2t^2(l^2 - l) + 2k$,

(ii) if P has non-zero elements, but has no column with two or more non-zero elements, then $rdrS = k^2t^2(l^2 - l) + k^2t$,

(iii) in other cases $rdrS = k^2t^2(l^2 - l)$.

If $l = 1$, then $rdrS = k^2t + 2k$.

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THE DISPOSITION OF PRIMARY AND SECONDARY IDEMPOTENTS IN GREEN'S CLASSES ON THE SET OF BOOLEAN MATRICES

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We consider the set of finite matrices over an arbitrary Boolean algebra. The dual partial operations of multiplication $\sqcap$ and $\sqcup$ are defined naturally. The $(m \times k)$-matrix $C = A \sqcap B$ with the elements $c_{ij} = \bigcup_{t=1}^{n}(a_{it} \bigcap b_{tj})$ is called *the conjunctive* *product* of an $(m \times s)$-matrix $A = (a_{ij})$ and an $(s \times k)$-matrix $B = (b_{ij})$ (m, s, k are positive integers). We dually define *the disjoint product* $A \sqcup B$: $A \sqcup B = (A' \sqcap B')'$.

We consider the idempotents of the partial finite matrix semigroups under the introduced operations of multiplication $\sqcap$ and $\sqcup$. All of the idempotent matrices are divided into primary and secondary idempotents (secondary idempotents are obtained by a special procedure, which uses the operations of multiplication, transposition, and closure operation). Let $i(A) = A \sqcup A'^{\top}$. It is easy to prove that for any $(m \times n)$-matrix A $i(A)$ is an $(m \times m)$-idempotent. It is known [1], [2] that secondary idempotents are concerned with solvability of matrix equations and with Green's relations on partial semigroups of Boolean finite matrices. It is of interest to consider what $\mathcal{D}$-classes correspond to a certain $\mathcal{D}$-class under the action of $i(\cdot)$. We got some results about this question. Also, we obtained the following results:

THEOREM 1. *Each regular $\mathcal{D}$-class, which does not consist of null matrices, contains a secondary idempotent.*

THEOREM 2. *Each regular $\mathcal{D}$-class contains a primary idempotent.*

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ON THE DEFINITION OF n-ARY GROUP

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An n-ary groupoid $\langle G, f\rangle$ is called n-ary group if it satisfies the generalized associative law

$$f(f(a_1^n), a_{n+1}^{2n-1}) = f(a_1^i, f(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}), \tag{1}$$

for all $i = 1, \ldots, n-1$ and uniquely solvable equations

$$f(a_1^{i-1}, x_i, a_{i+1}^n) = b$$

with respect to variable x_i, wherein $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b$ — any elements of G, $i = 1, \ldots, n$ [1]. When $n = 2$ obtain the usual group. We are interested in the case when $n > 2$.

There are other definitions of n-ary group, equivalent to the above (cf. [2], [3], [4]).

It is proved that n-ary group $\langle G, f\rangle$ has $(n-2)$-ary operation t, which satisfies the identities

$$f(x_{n-1}, x_1^{n-2}, t(x_1^{n-2})) = x_{n-1}, \tag{2}$$

$$f(t(x_1^{n-2}), x_1^{n-1}) = x_{n-1}, \tag{3}$$

and $(n-1)$-ary operation g, which satisfies the identities

$$f(x_{n-1}, x_1^{n-2}, g(x_1^{n-1})) = t(x_1^{n-2}), \tag{4}$$

$$f(g(x_1^{n-1}), x_1^{n-1}) = t(x_1^{n-2}). \tag{5}$$

Binary group can be defined by the left (right) unit and the left (right) inverse element. Generalizing this definition, we obtain

THEOREM 1. *n-Ary groupoid $\langle G, f\rangle$ is the n-ary group if and only if it is true (1) and exist $(n-2)$-ary operation t, satisfying the identity (3) ((2)) and $(n-1)$-ary operation g, satisfying identity (5) ((4)).*

Binary group can be defined by the two-sided unit, and two-sided inverse element. Generalizing this definition, we obtain

THEOREM 2. *n-Ary groupoid $\langle G, f\rangle$ is the n-ary group if and only if it is true (1) and exist $(n-2)$-ary operation t, satisfying the identities (3), (2) and $(n-1)$-ary operation g, satisfying identities (5), (4).*

Similar to those given in Theorems 1, 2 definitions of n-ary group can be found in [5].

In the n-ary group $\langle G, f\rangle$ for any $a \in G$ solution of the equation

$$f(\overset{(n-1)}{a}, x) = a$$

is denoted by $\bar{a}$ and called an skew element to the element a. There are properties, which bind definition of skew element with $(n-2)$-ary operation t.

PROPOSITION 1. *In the n-ary group is true identities*

$$t(\overset{(n-2)}{x}) = \bar{x},\ t(\overset{(i)}{x}, \bar{x}, \overset{(n-i-3)}{x}) = x \quad \text{and}$$

$$t(x_1^{n-2}) = f_{(n-3)}(\overset{(n-3)}{x_{n-2}}, \bar{x}_{n-2}, \ldots, \overset{(n-3)}{x_1}, \bar{x}_1),$$

where $i = 0, 1, \ldots, n-3$.

Analogue of inverse element of group is $(n-1)$-ary operation g.

Note that, operations properties are generalization of properties inverse element of group: $(a^{-1})^{-1} = a$ and $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

PROPOSITION 2. *In the n-ary group holds identities*

$$g(x_1^{n-2}, g(x_1^{n-1})) = x_{n-1},$$

$$g(x_1^{n-2}, f(x_{n-1}, x_1^{n-2}, x_n)) = f(g(x_1^{n-2}, x_n), x_1^{n-2}, g(x_1^{n-2}, x_{n-1})).$$

With aid of operations t, g we obtain criteria for n-ary subgroups:

THEOREM 3. *Subset of H n-ary group $\langle G, f\rangle$ is n-ary subgroup if and only if H is closed for action n-ary operation f, $(n-2)$-ary operation t and $(n-1)$-ary operation g.*

THEOREM 4. *Subset of H n-ary group $\langle G, f\rangle$ is n-ary subgroup if and only if for any $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{n-1}$ of H true $f(g(a_1^{n-1}), b_1^{n-1}) \in H$.*

An n-ary subgroup N of n-ary group $\langle G, f\rangle$ is called invariant if

$$f(x, \overset{(n-1)}{H}) = f(\overset{(i-1)}{H}, x, \overset{(n-i)}{H})$$

for any $x \in G$ and all $i = 2, \ldots, n$.

THEOREM 5. *n-Ary subgroup N of n-ary group $\langle G, f\rangle$ is invariant if and only if for any $a_1, \ldots, a_{n-2} \in G$ and any $h \in N$ true $f(a_1^{n-2}, h, t(a_1^{n-2})) \in N$.*

An n-ary subgroup N of n-ary group $\langle G, f\rangle$ called semi-invariant if

$$f(x, \overset{(n-1)}{H}) = f(\overset{(n-1)}{H}, x)$$

for any $x \in G$.

THEOREM 6. *n-Ary subgroup N of n-ary group $\langle G, f\rangle$ is a semi-invariant if and only if if for any $a \in G$ and any $h_1, \ldots, h_{n-1} \in N$ true*

$$f(f(a, h_1^{n-1}), h_2^{n-1}, g(h_2^{n-1}, a)) \in N.$$

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ON LEXICOGRAPHIC EXTENSIONS OF PARTIALLY ORDERED GROUPS

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Let G be a partially ordered group, and $G^+ = \{x \in G | \, e \leqslant x\}$.

A subgroup M of G is said to be *convex* if the inequalities $a \leqslant g \leqslant b$ imply $g \in M$ for any $a, b \in M$ and $g \in G$. Recall that an *o-ideal* is a convex directed normal subgroup of partially ordered group.

A partially ordered group G is called a *lexicographic extension of a convex normal subgroup M by the partially ordered group G/M*, if the inequality $m < a$ holds for any elements $a \in G^+ \setminus M$ and $m \in M$. The notion of extension is important in the study of partially ordered groups.

Let us consider a series of results on lexicographic extensions of partially ordered groups.

Elements a and $b \in G^+$ are said to be *almost orthogonal* if the inequalities $c \leqslant a, b$ imply $c^n \leqslant a, b$ for any $c \in G$ and any integer $n > 0$. A partially ordered group G is an *$\mathcal{AO}$-group* if each $g \in G$ has a representation $g = ab^{-1}$ for some almost orthogonal elements a and b of G^+.

Theorem 1. *Suppose an $\mathcal{AO}$-group G is the lexicographic extension of its o-ideal M by the partially ordered group G/M, and H is a convex directed subgroup of G; then either $H \subseteq M$ or $M \subset H$.*

In fact, if $H \nsubseteq M$, then there exists $h \in H \setminus M$, where $h = ab^{-1}$ for some almost orthogonal elements a and b of G^+. According to Lemma 2 [1], $a, b \in H$. If $a \in M$, then $b \notin M$, otherwise $h \in M$.

Now assume that $h \in H^+$. If $m \in M^+$, then $m < h$. Hence, $M^+ \subset H$. The inclusion $M \subset H$ holds because M is a directed group.

Theorem 2. *Let an $\mathcal{AO}$-group G be the lexicographic extension of its o-ideal M by the partially ordered group G/M, and let T be the set-theoretic intersection of all convex directed subgroups H of G, where $M \subset H$, then T is a convex directed subgroup of G. If $t \in T \setminus M$, then $t = ab^{-1}$ for some almost orthogonal elements a and b of G^+. Furthermore; there exist integers $k > 0$ and $l > 0$ for which either the inequality $a \leqslant b^k$ or the inequality $b \leqslant a^l$ holds.*

Really, from Theorem 1 [1] it follows that T is a convex directed subgroup of G.

Since G is an $\mathcal{AO}$-group, then for $t \in T \setminus M$ we have a representation $t = ab^{-1}$ for some almost orthogonal elements a and b of G^+. According to Lemma 2 [1], $a, b \in T$.

If $a \in M$, then $b \notin M$, otherwise $t \in M$. According to Theorem 1, this implies that $M \subset [b]$. Hence, $[b] = T$ and $a \in [b]$. By the definition of $[b]$, there exists an integer $k > 0$ for which the inequality $a \leqslant b^k$ holds.

If $b \in M$, then $a \notin M$. According to Theorem 1, this implies that $M \subset [a]$. Hence, $b \in [a]$, and there exists an integer $l > 0$ for which the inequality $b \leqslant a^l$ holds.

If $a \notin M$ and $b \notin M$, then $[a] = T = [b]$. It remains to use the definitions of subgroups $[a]$ and $[b]$.

A partially ordered group G is an *interpolation group* if whenever $a_1, a_2, b_1, b_2 \in G$ and $a_1, a_2 \leqslant b_1, b_2$, then there exists $c \in G$ such that $a_1, a_2 \leqslant c \leqslant b_1, b_2$. An interpolation $\mathcal{AO}$-group is called a *pl-group*.

In a partially ordered group G, for any $a \in G^+ \setminus \{e\}$ there exists the convex directed subgroup $[a]$, where for each $x \in [a]^+$ the inequality $x \leqslant a^k$ holds for some integer $k > 0$.

Theorem 3. *Suppose a pl-group G is the lexicographic extension of a convex normal subgroup M, and T is the set-theoretic intersection of all convex directed subgroups H of G, where $H \nsubseteq M$; then the following assertions hold:*

1. T is an o-ideal of G;

2. G is the lexicographic extension of T by the partially ordered group G/T;

3. if $T \neq M$, then the set $T \setminus M$ is a chain.

In fact, from Lemma 33 [2] it follows that M is an o-ideal of G.

According to Theorem 2, to prove the statement 1 it is sufficient to show that T is a normal subgroup of G.

Let us assume that $T \neq M$, and $t \in T^+ \setminus M$. If $x \in G$, then $v = x^{-1}tx \notin M$.

By Lemma 2 [3], this implies that $T \subseteq [v] = [xtx^{-1}] = x[t]x^{-1}$.

Hence, $t = xux^{-1}$ for some $u \in [t]$. Thus, $x^{-1}tx \in [t]$, i.e., $x^{-1}T^+x \subset T$ for any $x \in G$.

The inclusion $x^{-1}Tx \subset T$ holds because $x^{-1}Tx$ is a directed group.

If $a \in G^+ \setminus T$ and $t \in M$, then $t < a$.

If $t \in T^+ \setminus M$, then $at^{-1} = xy^{-1}$ for some almost orthogonal elements x and y of G^+. This implies that $y \leqslant t^2$. It follows that $y \in T$. Hence, $x \notin T$, otherwise $a \in T$.

Thus, $[x] \nsubseteq T$, i.e., $T \subseteq [x]$. This follows that $y \in [x]$. According to Theorem 4 and Lemma 4 [3], this implies that $y = e$. Therefore, $at^{-1} = x > e$, i.e., $t < a$. This means that the assertion 2 holds.

According to Theorem 2, if $t \in T \setminus M$, then t has a representation $g = ab^{-1}$ for some almost orthogonal elements a and b of G^+, and there exist integers $k > 0$ and $l > 0$ for which either the inequality $a \leqslant b^k$ or the inequality $b \leqslant a^l$ holds.

According to Theorem 4 and Lemma 4 [3], this implies that either $a = e$ or $b = e$. This means that the assertion 3 holds.

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FAREY FRACTIONS AND PERMUTATIONS GENERATED BY FRACTIONAL PARTS

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Let $\alpha \in (0;1)$ be an irrational number. Let $\pi_{\alpha,n}$ be a permutation that orders fractional parts $\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$, i.e.

$$0 < \{\pi_{\alpha,n}(1)\alpha\} < \{\pi_{\alpha,n}(2)\alpha\} < \ldots < \{\pi_{\alpha,n}(n)\alpha\} < 1.$$

Permutations $\pi_{\alpha,n}$ are studied in [1].

The set of irreducible rational fractions $\frac{a}{b}$ with denominators $0 < b \leq n$, that belongs to $[0;1]$ and arranged in ascending order is called Farey sequence of level n. Denote by F^n a tiling of $[0;1]$ generated by the points of Farey sequence of level n. Denote by F_i^n intervals of this tiling.

It is proved the following theorem.

Theorem 1. *Permutations $\pi_{\alpha,n}$ and $\pi_{\beta,n}$ coincide if and only if $\alpha, \beta \in F_i^n$ for some i.*

Corollary 1. *Let $\pi(n)$ be a number of different permutations $\pi_{\alpha,n}$ for some n. Then*

$$\pi(n) = 1 + \sum_{k=2}^{n} \varphi(k),$$

where $\varphi(k)$ is the Euler function.

Also it is proved the following result.

Theorem 2. *Permutation $\pi_{\alpha,n}$ uniquely determines permutations $\pi_{\alpha,m}$ with $n < m < \pi_{\alpha,n}(1) + \pi_{\alpha,n}(n)$ but does not uniquely determines permutation $\pi_{\alpha,m}$ with $m = \pi_{\alpha,n}(1) + \pi_{\alpha,n}(n)$.*

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ON SOME CONDITIONS OF FINITENESS OF COLENGTH OF LEIBNIZ ALGEBRAS VARIETY

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Let K – a field with zero characteristic. Leibniz algebra above field K is defined as nonassociative algebra with bilinear product, in which Leibniz identity

$$(xy)z \equiv (xz)y + x(yz)$$

takes place. We shall agree that brackets in left-normed elements we will omit, i.e., for example, $x_1x_2x_3\ldots x_n = (((x_1x_2)x_3)\ldots x_n)$. Put in designation Y for the operator of multiplication on right on the element y, it is conveniently for record of the element of the form $x\underbrace{y\ldots y}_{n} = xY^n$, as record, for example, xy^2 will incorrect, as it means $x(yy)$, but we have $(xy)y$.

Design by $\mathbf{V}$ Leibniz algebras variety. Let $P_n(\mathbf{V})$ be a space, generated by multilinear elements with degree n from free generators $x_1, \ldots, x_n$ in relatively free algebra variety $\mathbf{V}$. We consider the space $P_n(\mathbf{V})$ as KS_n-module, giving operation of symmetric group as usually. Decompose $P_n(\mathbf{V})$ into the sum of irreducible modules и write out the character in the form of the sum of irreducible characters

$$\chi(P_n(\mathbf{V})) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\lambda \vdash n$ – is partition of the number n, χ_λ – character, a m_λ – multiplicity of corresponding λ irreducible module.

Colength of the variety $\mathbf{V}$ is defined on the formula $l_n(\mathbf{V}) = \sum_{\lambda \vdash n} m_\lambda$. If it exists such constant C, not dependent from n, that for any n inequality $l_n(\mathbf{V}) \leq \mathbf{C}$ is executed, then the colength of the variety $\mathbf{V}$ is called finite. The variety $\mathbf{V}$ has almost finite colength, if any its own subvariety has finite colength and colength of the variety is not finite.

Remind, that $\widetilde{\mathbf{N}_s\mathbf{A}}$– is Leibniz algebras variety, which is defined by the identity

$$(x_1x_2)(x_3x_4)\dots(x_{2s+1}x_{2s+2}) \equiv 0. \tag{1}$$

In the work [1] was received the result about one necessary condition of finiteness Leibniz algebras variety colength. In particular, it is proved, that any variety $\mathbf{V}$ with finite colength is a subvariety of the variety $\mathbf{N}_s\mathbf{A}$, for some suitable natural number s, i.e. in the variety $\mathbf{V}$ identity (1) takes place.

On the other hand in the paper [2] it was found one enough condition of finiteness of Leibniz algebras variety. Formulate it in the form of the theorem.

Theorem 1. . *Let* $\mathbf{V}$ *– a subvariety of the variety* $\widetilde{\mathbf{N}_s\mathbf{A}}$, *in which for some natural numbers* k, m, $k \leqslant m$, *and* $\alpha_1, \dots, \alpha_k \in K$ *the next identity takes place*

$$xY^kzY^{m-k} \equiv \sum_{i=1}^{k} \alpha_i xY^{k-i}zY^{m-k+i}.$$

Then variety $\mathbf{V}$ *has finite colength.*

Let us formulate a new result which was received in this year.

Theorem 2. *Let the Leibniz algebras variety* $\mathbf{V}$ *has finite colength, then the next identity takes place in it*

$$xY^kzY^{m-k-2} \equiv \sum_{i=1}^{k} \alpha_i xY^{k-i}zY^{m-2-k+i}.$$

So, for Leibniz algebras varieties is received necessary and enough condition for finiteness of the colength in the terms of identical relations:

Theorem 3. *Leibniz algebras variety* $\mathbf{V}$ *has finite colength in only case when (1) and for some natural numbers* k, m, $k \leqslant m$, *and* $\alpha_1, \dots, \alpha_k \in K$ *the identity of the form*

$$xY^kzY^{m-k-2} \equiv \sum_{i=1}^{k} \alpha_i xY^{k-i}zY^{m-2-k+i}$$

are the identities of the variety $\mathbf{V}$.

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UDC 511.3

ON THE NUMBER OF ZEROS OF SOME ANALYTIC FUNCTIONS

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Let $s = \sigma + it$ be a complex variable, α, $0 < \alpha \leq 1$, be a fixed parameter, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be the two periodic sequences of complex numbers with minimal periods $k \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively. We consider value-distribution of zeta-functions with periodic coefficients which are generalizations of the Riemann zeta-function and the Hurwitz zeta-function. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

The periodicity of the sequence of $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s; \mathfrak{a}) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta\left(s, \tfrac{m}{k}\right),$$

where $\zeta(s, \beta)$, $0 < \beta \leq 1$, is the classical Hurwitz zeta-function. Therefore, the latter equality gives for the function $\zeta(s; \mathfrak{a})$ analytic continuation to the whole complex plane, except for a possible simple pole at the point $s = 1$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$ is given, for $\sigma > 1$, by the Dirichlet series

$$\zeta(a, \alpha, \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

Similarly as in the case of $\zeta(s; \mathfrak{a})$ by the periodicity of the sequence $\mathfrak{b}$, we have that, for $\sigma > 1$,

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{l^s} \sum_{m=0}^{l-1} b_m \zeta\left(s, \tfrac{m+\alpha}{l}\right).$$

Thus the function $\zeta(s, \alpha; \mathfrak{b})$ also has analytic continuation to the whole complex plane, except for a possible simple pole at the point $s = 1$.

The zero distribution of the function $\zeta(s; \mathfrak{a})$ was considered in [3], of the function $\zeta(s, \alpha; \mathfrak{b})$ in [1]. In the report, we discuss the universality and zero distribution of some combinations of the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$.

Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. We say that the operator $F: H^2(D) \to H(D)$ belongs to the class $Lip(\beta_1, \beta_2)$, $\beta_1, \beta_2 > 0$, if the following hypotheses are satisfied:

1° For each polynomial $p = p(s)$, and any compact subset $K \subset D$ with connected complement, there exists an element $(g_1, g_2) \in F^{-1}\{p\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K;

2° For any compact subset $K \subset D$ with connected complement, there exist a positive constant c, and compact subsets K_1, K_2 of D with connected complements such that

$$\sup_{s \in K} |F(g_{11}(s), g_{12}(s)) - F(g_{21}(s), g_{22}(s))| \leq c \sup_{1 \leq j \leq 2} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{r1}, g_{r2}) \in H^2(D)$, $r = 1, 2$.

The joint universality of the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$ has been obtained in [2]. From this, we obtain the following assertion.

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, for each prime p,*

$$\sum_{m=1}^{\infty} \frac{|a_{p^m}|}{p^{\frac{m}{2}}} \leq c < 1,$$

and that the number α is transcendental. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha; \mathfrak{a}, \mathfrak{b}) > 0$ such that, for sufficiently large T, the function

$$c_1\zeta(s; \mathfrak{a}) + c_2\zeta(s, \alpha; \mathfrak{b}), \qquad c_1, c_2 \in \mathbb{C} \setminus \{0\},$$

has more than cT zeros in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.

Our main result is based on the universality theorem for the function $F(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$ with $F \in Lip(\beta_1, \beta_2)$.

Theorem 2. *Suppose that the sequence $\mathfrak{a}$ and the number α are as in Theorem 1 and that the operator $F \in Lip(\beta_1, \beta_2)$. Let $K \subset D$ be a compact subset with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 2 implies an analogue of Theorem 1.

Theorem 3. *Suppose that the sequence $\mathfrak{a}$ and the number α are as in Theorem* 1 *and that the operator $F \in Lip(\beta_1, \beta_2)$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha; \mathfrak{a}, \mathfrak{b}, F) > 0$ such that, for sufficiently large T, the function $F(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$ has more than cT zeros in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.*

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ABOUT GROWTH OF VARIETY OF LEIBNIZ ALGEBRAS, WICH IS CONNECTED WITH INFINITELY DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF THE HEISENBERG ALGEBRA[1]

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The characteristic of basic field Φ is equal zero. All undefined concepts can be found in the monograph [1]. Generalization of the concept of Lie algebra is a Leibniz algebra, which is determined by the identity $(xy)z \equiv (xz)y + x(yz)$ and, probably, first appeared in [2].

Let $F(X, \mathbf{V})$ be a relatively free algebra of countable set of free generators X of some variety $\mathbf{V}$. In 1949 A.I.Maltsev proved that in the case when the basic field has zero characteristic, every identity is equivalent to the system of multilinear identities. Therefore, in this case, all the information about the variety contained in the space of multilinear elements of degree n from the variables $x_1, x_2, \ldots x_n$,, the so-called multilinear components of relatively free algebra of variety and denoted $P_n(\mathbf{V})$. The growth of variety is the asymptotic behavior of the sequence of dimensions $c_n(\mathbf{V}) = dim P_n(\mathbf{V})$. In the case of the existence of such numbers $C_1 > 0, C_2 > 0$, $d_1 > 1, d_2 > 1$, that for all numbers n hold inequalities $C_1 d_1^n < c_n(\mathbf{V}) < C_2 d_2^n$, then

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we say that the growth of variety is exponential. If the upper and lower limits of this sequence are the same, then their values are called the exponent of variety.

Let $T = \Phi[t]$ is ring of polynomials from the variable t. Consider the three-dimensional Heisenberg algebra H with basis $\{a, b, c\}$ and multiplying $ba = -ab = c$, product of the remaining basis elements equal to zero. Transform the ring of polynomials T in the right module of algebra H, in which the basic elements of algebra H act on the polynomial f from T follows: $fa = f', fb = tf, fc = f$, where f' is partial derivative of a polynomial f in the variable t. Consider the direct sum of vector spaces H and T with multiplication: $(x + f)(y + g) = xy + fy$, where x, y are from H; f, g are from T. Thus constructed algebras generates the variety $\widetilde{\mathbf{V}}_3$ of Leibniz algebras. This variety is an analog to well known varieties $\mathbf{V}_3$ of Lie algebras. Earlier in article [3] for a variety $\widetilde{\mathbf{V}}_3$ has been proven that it has almost polynomial growth, in article [4] have been identified its multiplicity and colength, and in article [5] it was proved that it has integer exponent.

Theorem 1. *The exponent of variety $\widetilde{\mathbf{V}}_3$ of Leibniz algebras is equal 3.*

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STRONG UNIFORM DISTRIBUTION OF THE VAN DER CORPUT—HAMMERSLI FUNCTIONS[1]

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Consider the following function for the fixed natural $p > 1$

$$p(t) = \sum t_\nu p^{-\nu-1}, \quad t_\nu \in A(p)$$

for the any natural number $t = \sum\limits_{\nu=0}^{h-1} t_\nu p^\nu \geq 0$ — is he van der Corput—Hammersli where for the arbitrary T set

$$A(T) = \{0, 1, ..., T-1\}.$$

For the $p = 2$ this function is considered the van der Corput in [12] and K. Roth used this function in his seminal work on the square deviation[15].

In 1960 Hemmersli for construction of multidimensional quadrature formulas introduced multidimensional grid, which is now called Hemmersli grids:

$$X(N) = \left\{ \left(p_1(n), ..., p_s(n), \frac{n}{N} \right) \mid n = 0, 1, ..., N-1 \right\},$$

where $p_1, \ldots, p_s$ — is different pairwise coprime natural numbers greater than 1 and $p_j(n)$ — the van der Corput—Hammersli functions for $p = p_j$ $(j = 1, 2, \ldots, s)$.

Known to (see [13], [14], [9]), if $p_1, ..., p_s$ — is pairwise relatively prime, then Hemmersli grids is uniformly distributed and deviation $D(X(N))$ grid is of the order [2]

$$D(X(N)) = O\left(\ln^s N\right). \tag{1}$$

In [8] (стр. 174) introduced the definition of the uniformly distributed modulo 1.

Let $s \geq 1$ — a fixed natural number, $\gamma_1, ..., \gamma_s$ — arbitrary positive numbers not greater than 1, и $f_1(x), ... , f_s(x)$ — it functions defined for natural values of x. Let $N_T(\gamma_1, ..., \gamma_s)$ number of solutions system of inequalities

$$\begin{cases} \{f_1(x)\} < \gamma_1 \\ \ldots\ldots\ldots\ldots\ldots \\ \{f_s(x)\} < \gamma_s \end{cases} \quad x = 1, 2, ..., T.$$

[2]Deviation $D(X)$ of an arbitrary grid $X = \{\vec{x}_k | 1 \leqslant k \leqslant N\}$ s the quantity

$$D(X) = \sup_{0 \leqslant \gamma_1, \ldots, \gamma_s \leqslant 1} |D(X, \vec{\gamma})|, \quad D(X, \vec{\gamma}) = N(\vec{\gamma}) - N\gamma_1 \ldots \gamma_s,$$

where $N(\vec{\gamma})$ — number of grid points X, included in area $\Pi(\vec{\gamma}) = [0; \gamma_1) \times \ldots \times [0; \gamma_s)$, $D(X, \vec{\gamma})$ — local deviation grid X.

Definition 1. *System of the function is* **uniformly distributed** *in s-dimensional unit cube, if*

$$\lim_{T\to\infty}\frac{1}{T}N_T\left(\gamma_1,...,\gamma_s\right)=\gamma_1...\gamma_s,$$

or, is the same,

$$N_T\left(\gamma_1,...,\gamma_s\right)=\gamma_1...\gamma_s T+o(T)$$

for the any $0\leqslant\gamma_1,...,\gamma_s\leqslant 1$.

Of evaluation for (1) follows that the system functions $p_1(t),...,p_s(t)$ is uniformly distributed modulo 1.

Goal of this article - to prove the following theorem

Theorem 1. *Van der Corput–Hammersli functions* $p_1(t),...,p_s(t)$ *is a strong uniform distribution, that is, for any set of non-negative integers* $(n_1,...,n_s)$ *Van der Corput–Hammersli functions* $p_1(t+n_1),...\ ,p_s(t+n_s)$ *is uniformly distributed modulo 1.*

For the proof we need information about the estimates of the deviation of the modified Hemmersli — Roth grids, submitted by Nikolay Dobrovolsky in 1983 [2] or effective proof of Roth's theorem on quadratic dispersion[16]. Various aspects of the theory of modified Hemmersli — Roth grids considered in [1] — [7]. Various generalizations of Roth's theorem on quadratic dispersion are devoted [10] — [11].

For a fixed natural $p>1$, natural $h\geq 1,\ P=p^h$ *фan der Corput— Hemmersli function* $x(m)=x_{(h)}(m)$ *periodised by modulo* P on the set of integers is given by equalitiesv

$$x(m)=\begin{cases}\sum\limits_{\nu=0}^{h-1}m_\nu p^{-\nu-1} & \text{при } m\in A(P),\ m=\sum\limits_{\nu=0}^{h-1}m_\nu p^{\nu},\ m_\nu\in A(p)\\ x\left(P\left\{\frac{m}{P}\right\}\right) & \text{при } m\notin A(P).\end{cases}$$

Let $p_1,\ldots,p_s$ — various pairwise coprime natural numbers greater than 1. ДFor an arbitrary natural $N\ ge3$ define the quantities

$$\begin{gathered}h_j=[\ln N/\ln p_j]+1,\ P_j=p_j^{h_j}\ (j=1,\ldots,s);\\ M=P_1\ldots P_s;\ M_j=M/P_j\ (j=1,\ldots,s).\end{gathered}\tag{2}$$

Then the relations

$$N<P_j\leqslant Np_j,\ (M_j,P_j)=1\ (j=1,\ldots,s);\ N^s<M\leqslant N^s p_1\ldots p_s \tag{3}$$

Through $x_j(n)$ denote the function $x(n)$ with $p=p_j,\ h=h_j,\ P=P_j\ (j=1,\ldots,s)$. Let $\vec{t}=(t_1,\ldots,t_s)$ — arbitrary integer vector. For any integer n with $0\leqslant n\leqslant N-1$ assume

$$\vec{x}\left(n,\vec{t}\right)=\left(x_1(n+t_1),\ldots,x_s(n+t_s),\frac{n}{N}\right)\tag{4}$$

Definition 2. *Modified Hemmersli — Roth grid called*

$$XR\left(N,\vec{t}\right)=\left\{\vec{x}\left(n,\vec{t}\right)|n=0,\ldots,N-1\right\} \tag{5}$$

of N nodes.

Periodicity of $x_j(n)$ with period P_j follows that grid $XR\ left(N,\ vect\ right)$ depends periodically of t_j with period P_j $(j=1,\ ldots,s)$. This implies that for a given N there are exactly M various modified Hemmersli — Roth grids, that is about N^s different grids.

Arbitrary estimate for the deviation of the modified Hammer to li — Roth grid received in cite d35.

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ON MATRIX DECOMPOSITION OF ONE REDUCED CUBIC IRRATIONAL [1]

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In this work we considered the matrix decomposition of the cubic irrational α, satisfying the equation
$$x^3 - 4x^2 - 5x - 1 = 0.$$

For decomposition of the matrix
$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \prod_{k=0}^{\infty} \begin{pmatrix} 310941 \cdot k + 155427 & 156744 \cdot k + 78333 \\ 61578 \cdot k + 30882 & 31041 \cdot k + 15564 \end{pmatrix}$$
is constructed an algorithm of transition to regular continued fraction.

In figure 1 is given the text of the program of calculation of the incomplete private reduced cubic irrationalities $\alpha(p)$ on Lagrange's algorithm.

[1] Work is performed on a grant RFBR №11-01-00571a

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cfki1(p,n) := | a ← (−1  −p  −p + 1  1)^T, q_{floor((n−1)/40), 39} ← 0, q_{0,0} ← p, D^<0> ← a, D^<n> ← a
              | l0 ← 0, l1 ← 1, b ← p
              | for k ∈ 1..n − 1
              |   | r ← b, a ← (−a3  −a3·3·r − a2  −a3·3·r^2 − a2·2·r − a1  −a3·r^3 − a2·r^2 − a1·r − a0)^T
              |   | d ← −a0
              |   | for j ∈ 1..3
              |   |   | m ← d, l ← a_j
              |   |   | r ← m, m ← l, l ← r  if  m > l
              |   |   | while  m > 0
              |   |   |   r ← floor(l/m), r1 ← l − r·m, l ← m, m ← r1
              |   |   | d ← l
              |   | b ← 1, c ← 2, D^<k> ← a
              |   | while [(a3·c + a2)·c + a1]·c + a0 < 0
              |   |   b ← c, c ← 2·c
              |   | m ← b
              |   | while  m ≥ 1
              |   |   | r ← b + m, f ← [(a3·r + a2)·r + a1]·r + a0
              |   |   | b ← r  if  f < 0
              |   |   | c ← r  otherwise
              |   |   | m ← m/2
              |   | q_{l0,l1} ← b, l1 ← l1 + 1
              |   | l0 ← l0 + 1, l1 ← 0  if  l1 = 40
              | q
```

Figure 1

$$\mathrm{submatrix}(St,0,1,0,19) \rightarrow \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 6 & 484 & 196 & 114 & 75 & 51 & 44 & 30 & 17 & 20 & 11 & 11 & 14 & 9 & 6 & 10 & 5 & 10 & 5 & 2 \end{pmatrix}$$

$$\mathrm{submatrix}(St,0,1,20,39) \rightarrow \begin{pmatrix} 20 & 21 & 22 & 23 & 24 & 25 & 27 & 28 & 29 & 30 & 33 & 34 & 35 & 36 & 37 & 40 & 42 & 43 & 44 & 47 \\ 5 & 3 & 2 & 2 & 1 & 1 & 1 & 4 & 1 & 4 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

$$\mathrm{submatrix}(St,0,1,40,59) \rightarrow \begin{pmatrix} 54 & 55 & 60 & 63 & 68 & 78 & 79 & 82 & 84 & 87 & 93 & 95 & 97 & 111 & 120 & 123 & 128 & 129 & 134 & 140 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathrm{submatrix}(St,0,1,60,75) \rightarrow \begin{pmatrix} 141 & 154 & 164 & 176 & 180 & 201 & 228 & 234 & 244 & 255 & 288 & 425 & 467 & 1333 & 1813 & 3139 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 2.

Calculations $cfki(100)$ gives values 592 incomplete private, and $cfki(200)$ already — 1194 values. As results are presented in the form of the matrix, containing 40 elements in every line, the last elements of the last line can be zero. We will give distribution of values incomplete private taking into account the given zero values which aren't incomplete private.

This distribution is calculated by means of the program in figure 3.

```
R := cfki(200)
St := | t4000 ← 0
      | for k ∈ 0..29
      |   for j ∈ 0..39
      |     r ← R_k,j , t_r ← t_r + 1
      | j ← 0
      | for k ∈ 0..4000
      |   N_0,j ← k , N_1,j ← t_k , j ← j + 1  if  t_k > 0
      | N
```

Figure 3.

We will provide necessary data on matrix decomposition.

Let α is the given reduced cubic irrationality, that is $\alpha^{(1)} = \alpha > 1$, and the interfaced algebraic irrationalities satisfy to a ratio $-1 < \alpha^{(3)} < \alpha^{(2)} < 0$.

The concept of the given reduced cubic irrationality is generalization of the given quadratic irrationality.

It is easy to see that the positive root α the equations

$$x^3 - 4x^2 - 5x - 1 = 0$$

is the given reduced cubic irrationality.

Really, for a polynomial $f(x) = x^3 - 4x^2 - 5x - 1$ we have:

$$f(-1) = f(0) = f(5) = -1, \quad f(6) = 41, \quad f\left(-\frac{1}{2}\right) = \frac{3}{8},$$

therefore $\alpha = \alpha^{(1)} > 5$, $-1 < \alpha^{(3)} < -\frac{1}{2}$, $-\frac{1}{2} < \alpha^{(2)} < 0$.

In works [2] and [3] are considered matrix decomposition of algebraic irrationalities. In particular, for reduced cubic irrationality α, satisfying the equation

$$f(t) = t^3 + at^2 + bt + c, \quad f(\alpha) = 0$$

is given matrix decomposition

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \prod_{k=0}^{\infty} \left(\begin{pmatrix} t & -at^2 - 2bt - 3c \\ 1 & 3t^2 + 2at + b \end{pmatrix} \begin{pmatrix} 3k+2 & 0 \\ 0 & 3k+1 \end{pmatrix} \right).$$

$$\cdot \begin{pmatrix} 3t^2 + 2at + b & -at^2 - 2bt - 3c \\ 1 & t \end{pmatrix} \begin{pmatrix} ab - 9c & 2b^2 - 6ac \\ 2a^2 - 6b & ab - 9c \end{pmatrix} \Bigg) \quad (1)$$

also it is claimed that it meets at for t, which the difference $|t - \alpha|$ is small.

In the program in figure 4 is realized the algorithm of transition from matrix decomposition $\alpha(5)$ to usual continuous fraction.

```
cfki(n) := | M ← (1 0; 0 1), j ← 0, l ← 0
           | for k ∈ 0..n
           |   | M ← M·(103647·k + 51809   52248·k + 26111; 20526·k + 10294   10347·k + 5188)
           |   | A ← M[0,0], B ← M[0,1], C ← M[1,0], D ← M[1,1]
           |   | r ← floor(A/C)
           |   | while r = floor(B/D)
           |   |   | r1 ← A − C·r, A ← C, C ← r1
           |   |   | r1 ← B − D·r, B ← D, D ← r1
           |   |   | q[l,j] ← r, j ← j + 1
           |   |   | l ← l + 1, j ← 0  if j = 40
           |   |   | r ← floor(A/C), M ← (A B; C D)
           |   | a ← (A B C D)^T
           |   | d ← a[0]
           |   | for kk ∈ 1..3
           |   |   | p ← d, nn ← a[kk]
           |   |   | r ← p, p ← nn, nn ← r  if p > nn
           |   |   | while p > 0
           |   |   |    r ← floor(nn/p), r1 ← nn − r·p, nn ← p, p ← r1
           |   |   | d ← nn
           |   | M ← M·(1/d)
           | q
```

Figure 4.

General determination of convergence of matrix decomposition are the following.

Definition 1. *They say that matrix decomposition*

$$\prod_{k=0}^{\infty} \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

meets to number α, if for matrixes

$$M_n = \prod_{k=0}^{n} \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

the ratio is carried out

$$\lim_{n\to\infty}\frac{A_n}{C_n}=\lim_{n\to\infty}\frac{B_n}{D_n}=\alpha.$$

In this case it is written

$$\begin{pmatrix}\alpha\\ 1\end{pmatrix}=\prod_{k=0}^{\infty}\begin{pmatrix}a_k & b_k\\ c_k & d_k\end{pmatrix}.$$

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THE RESIDUAL π-FINITENESS OF CERTAIN FREE CONSTRUCTIONS OF GROUPS

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Let $\mathcal{C}$ be a class of groups. Recall that, in accordance with the common definition, a group X is said to be residually a $\mathcal{C}$-group if, for every nonidentity element $x \in X$, there exists a homomorphism ρ of X onto a group of $\mathcal{C}$ (or, briefly, a $\mathcal{C}$-group) such that $x\rho \neq 1$. Let us recall also that a periodic group is said to be a π-group, where π is a set of primes, if all prime divisors of the orders of its elements belong to π.

Let P be the free product of finitely generated nilpotent groups A and B with finite subgroups $C \leqslant A$ and $D \leqslant B$ amalgamated according to an isomorphism $\varphi\colon C \to D$. Let also E be the HNN extension of a finitely generated nilpotent group G with finite associated subgroups $H \leqslant G$ and $K \leqslant G$ and an associated isomorphism $\psi\colon H \to K$. As is well known, the groups P and E are residually finite. Therefore the natural question arises which are conditions for P and E to be residually $\mathcal{C}$-groups, where $\mathcal{C}$ is some subclass of the class of all finite groups.

The criteria for the groups P and E to be residually p-groups were proved by D. N. Azarov [1] and D. I. Moldavanskii [2], respectively. Using these results, the author obtains their generalizations on the case when $\mathcal{C}$ is a certain subclass of the class of all finite π-groups, where π contains all prime divisors of the periodic parts of either the free factors A, B or the base group G (see Theorems 1 and 2 below). Before to formulate these statements, we introduce a number of definitions and recall some facts on nilpotent groups.

As is well known (cf., e. g., [3, § 4]), the set of all elements of finite order of a locally nilpotent group N forms a characteristic subgroup which is called the periodic part of N and is denoted by $\tau(N)$. If N is finitely generated and hence nilpotent, then $\tau(N)$ is finite and, by Burnside-Wielandt's theorem, is decomposed into the direct product of its Sylow's subgroups.

Recall that a normal series of a group is said to be a chief series if it has no trivial factors and no refinements to longer normal series. Since every finite p-group is nilpotent (cf., e. g., [3, Lemma 1.4]), a normal series of such a group is chief if and only if all its factors are of order p.

Let further $\theta = \{p_1, \ldots, p_m\}$ be the set of all prime divisors of the orders of the groups $\tau(A)$ and $\tau(B)$, and let $\sigma = \{q_1, \ldots, q_n\}$ be the set of all prime divisors of the order of the group $\tau(G)$. Let also $A_k \leqslant \tau(A)$ and $B_k \leqslant \tau(B)$ be the Sylow's subgroups of the groups $\tau(A)$ and $\tau(B)$ corresponding to the number p_k, $k \in \{1, \ldots, m\}$, and let $G_\ell \leqslant \tau(G)$ be the Sylow's subgroup of the group $\tau(G)$ corresponding to the number q_ℓ, $\ell \in \{1, \ldots, n\}$. Since p_k does not necessarily divide the orders of both the groups $\tau(A)$ and $\tau(B)$, then one of the subgroups A_k and B_k can be equal to 1.

If p is a prime and π is a set of primes, then we denote by $\mathcal{F}_p$ the class of all finite p-groups, by $\mathcal{F}_\pi$ the class of all finite π-groups, by $\mathcal{FN}_\pi$ the class of all finite nilpotent π-groups, and by $\mathcal{F}_p \cdot \mathcal{FN}_\pi$ the class of finite solvable groups containing any possible extension of an $\mathcal{F}_p$-group by an $\mathcal{FN}_\pi$-group.

Theorem 1. *The following statements are equivalent.*

1. *For every $k \in \{1, \ldots, m\}$, there exist chief series $\mathcal{R}_k$ and $\mathcal{S}_k$ of the groups A_k and B_k, respectively, satisfying the following two conditions:*

   (a) *the series $\mathcal{R}_k$ and $\mathcal{S}_k$ are (C, D, φ)-compatible, i. e. φ maps the set of the intersections of the terms of $\mathcal{R}_k$ with the subgroup C onto the set of the intersections of the terms of $\mathcal{S}_k$ with the subgroup D;*

   (b) *the terms of $\mathcal{R}_k$ and $\mathcal{S}_k$ are normal in A and B, respectively.*

2. *There exists a homomorphism of P onto an $\mathcal{FN}_\theta$-group which is injective on $\tau(A)$ and $\tau(B)$.*

3. *P is residually an $\mathcal{F}_p \cdot \mathcal{FN}_\theta$-group for every prime p.*

Theorem 2. *The following statements are equivalent.*

1. *For every $\ell \in \{1, \ldots, n\}$, there exists a chief series*

$$1 = G_{\ell,0} \leqslant G_{\ell,1} \leqslant \cdots \leqslant G_{\ell,r_\ell - 1} \leqslant G_{\ell,r_\ell} = G_\ell, \qquad (\mathcal{T}_\ell)$$

of the group G_ℓ satisfying the following conditions:

(a) *the series $\mathcal{T}_\ell$ is (H, K, ψ)-compatible, i. e. $(H \cap G_{\ell,i})\psi = K \cap G_{\ell,i}$ for any $i \in \{0, 1, \dots, r_\ell\}$;*

(b) *for every $i \in \{0, 1, \dots, r_\ell - 1\}$ and for every $h \in H \cap G_{\ell,i+1}$, the elements h and $h\psi$ are congruent modulo $G_{\ell,i}$;*

(c) *the terms of $\mathcal{T}_\ell$ are normal in G.*

2. *There exists a homomorphism of E onto an $\mathcal{FN}_\sigma$-group which is injective on $\tau(G)$.*

3. *E is residually an $\mathcal{F}_p \cdot \mathcal{FN}_\sigma$-group for every prime p.*

As is readily seen, all numbers of the sets θ and σ necessarily occur among the prime divisors of all possible finite homomorphic images of P and E, respectively. The first statements of Theorems 1 and 2 serve as sufficient conditions for these groups to be residually an $\mathcal{F}_\theta$-group and an $\mathcal{F}_\sigma$-group, respectively. However, there are the examples showing that these conditions are not necessary. Therefore the following problem can be formulated.

Problem. *Which conditions are necessary and sufficient for P and E to be residually an $\mathcal{F}_\theta$-group and an $\mathcal{F}_\sigma$-group, respectively?*

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COMMUTATIVITY OF AN EVEN COMPONENT OF $\mathbb{Z}_2$-GRADED ALGEBRA AND NON-GRADED CONSEQUENCES OF THE COMMUTATIVITY [1 2]

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An algebra A over a field is called $\mathbb{Z}_2$-*graded*, if it is a direct sum of subspaces A_0 and A_1 such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. Obviously, A_0 is a subalgebra of A. It is well known that an associative $\mathbb{Z}_2$-graded algebra A is a PI-algebra if its even component A_0 is PI-algebra (see, for instance, [1]).

We study $\mathbb{Z}_2$-graded associative algebras with commutative even components and prove that any such algebra satisfies both the identity

$$\sum_{\sigma}[[x_{\sigma(1)}, y_1], [x_{\sigma(2)}, y_2]], x_{\sigma(3)}] = 0 \tag{1}$$

where σ belongs to the set of all even permutations of $\{1, 2, 3\}$, and $[x, y]$ is the commutator $xy - yx$ of elements x and y, and the identity

$$[[x, y]^2, x] = 0. \tag{2}$$

Let M be the ideal of all non-graded identities which are satisfied by any $\mathbb{Z}_2$-graded associative algebra with a commutative even component. If the characteristic of the base field is not equal to 2 then M does not contain identities of degree 4. In the case of characteristic zero any identity of degree 5 from M is a consequence of the identities (1) and (2).

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DECOMPOSITION OF UNITARY LINEAR GROUPS INTO PRODUCTS OF FREE FACTORS

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In the papers [1] and [2], the group of invertible elements of a free product of algebras over a skewfield was studied. The complete description of such a group was given in the paper [2], where it was shown that this group is a free product of some groups. The important thing is that the later result provides negative answers to the following conjectures:

1. The elementary subgroup $E_n(R)$ (i.e., the subgroup generated by all transvections) of a general linear group $GL_n(R)$ is always normal in $GL_n(R)$.

2. Every automorphism of the group $GL_n(R)$ ($n \geqslant 3$) is standard, i.e., expressed in terms of certain automorphisms and anti-automorphisms of the matrix ring $M_n(R)$.

We will describe a counterexample to similar conjectures for unitary linear groups. The main result is the theorem, in which the unitary linear group over a special ring is decomposed into a non-trivial free product such that one of the free factors contains the elementary unitary subgroup (i.e., the subgroup generated by special *unitary* transvections).

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CERTAIN RESIDUALLY ROOT (CLASS OF GROUPS) GENERALIZED FREE PRODUCTS WITH NORMAL AMALGAMATION

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This work continues the author's investigations published in [1], where the definitions of the most important terms used further can be found.

We assume until the end of the paper that $\mathcal{K}$ is a root class of groups and G is the free product of groups A and B with normal subgroups $H \leqslant A$ and $K \leqslant B$ amalgamated according to an isomorphism $\varphi\colon H \to K$. Since the subgroup H is normal in G, then the restrictions on this subgroup of all inner automorphisms of G are automorphisms of H and form a subgroup $\mathrm{Aut}_G(H)$ of the group $\mathrm{Aut}\, H$.

Let us formulate the results of this paper.

Theorem 1. *Let A and B be $\mathcal{K}$-groups. If $A/H \in \mathcal{K}$, $B/K \in \mathcal{K}$, $\mathrm{Aut}_G(H) \in \mathcal{K}$, then there is a homomorphism of G onto a $\mathcal{K}$-group which is injective on the subgroups A, B and, in particular, G is residually a $\mathcal{K}$-group.*

From now on and until the end of the paper, let the class $\mathcal{K}$ be homomorphically closed. With this assumption, we get the following criterion from Theorem 1.

Corollary 1. *Let $A, B \in \mathcal{K}$. Then the following statements are equivalent and any of them implies that G is residually a $\mathcal{K}$-group.*

1. There is a homomorphism of G onto a $\mathcal{K}$-group which is injective on the subgroups A and B.

2. The group $\mathrm{Aut}_G(H)$ is a $\mathcal{K}$-group.

We note that the free factors A and B do not necessarily belong to $\mathcal{K}$ in all statements formulated below excepting Corollary 4.

For any group X, let us denote by $\mathcal{K}^*(X)$ the family of subgroups $\{Y \trianglelefteq X \mid X/Y \in \mathcal{K}\}$. Recall also that subgroups $R \leqslant A$ and $S \leqslant B$ are said to be (H, K, φ)-compatible if $(H \cap R)\varphi = K \cap S$.

Theorem 2. *Let $H \neq A$, and let $K \neq B$. Let also $\{(R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ be the family of all pairs of (H, K, φ)-compatible subgroups of the families $\mathcal{K}^*(A)$ and $\mathcal{K}^*(B)$, respectively. If $\mathrm{Aut}_G(H)$ is a finite group, then G is residually a $\mathcal{K}$-group if and only if the following conditions hold: $\mathrm{Aut}_G(H) \in \mathcal{K}$, $\bigcap_{\lambda \in \Lambda} R_\lambda = \bigcap_{\lambda \in \Lambda} S_\lambda = 1$, H is $\mathcal{K}$-separable in A, K is $\mathcal{K}$-separable in B.*

Corollary 2. *Let H and K be proper central subgroups of A and B, respectively, and let the family $\{(R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ be defined as in Theorem 2. G is residually a $\mathcal{K}$-group if and only if the following conditions hold: $\bigcap_{\lambda \in \Lambda} R_\lambda = \bigcap_{\lambda \in \Lambda} S_\lambda = 1$, H is $\mathcal{K}$-separable in A, K is $\mathcal{K}$-separable in B.*

Theorem 3. *Let A and B be residually $\mathcal{K}$-groups, and let H and K be finite subgroups. Then the following statements are equivalent.*

1. G is residually a $\mathcal{K}$-group.

2. There is a homomorphism of G onto a $\mathcal{K}$-group which is injective on the subgroup H.

3. The group $\mathrm{Aut}_G(H)$ is a $\mathcal{K}$-group.

We note, that Theorem 3 generalizes and extends Corollary 2 from [2] which serves as a criterion of the residual p-finiteness of a free product of two finite p-groups with normal amalgamated subgroups.

Corollary 3. *Let A and B be residually $\mathcal{K}$-groups, and let H and K be finite subgroups. If at least one of the subgroups H and K lies in the center of the corresponding free factor or the group $\mathrm{Aut}_G(H)$ is Abelian, then G is residually a $\mathcal{K}$-group.*

Theorem 4. *Let A and B be residually $\mathcal{K}$-groups. And let H and K be cyclic, or let K be central in B.*

1. If $H \neq A$, $K \neq B$ and G is residually a $\mathcal{K}$-group, then H is $\mathcal{K}$-separable in A, K is $\mathcal{K}$-separable in B.

2. Let H be $\mathcal{K}$-separable in A, let K be $\mathcal{K}$-separable in B, and let, for any subgroup $M \in \mathcal{K}^*(K)$, exist a subgroup $N \in \mathcal{K}^*(B)$ such that $N \cap K = M$. Then G is residually a $\mathcal{K}$-group.

Corollary 4. *Let A be residually a $\mathcal{K}$-group, let B be a $\mathcal{K}$-group, let $H \neq A$, and let $K \neq B$. Let also H and K be cyclic, or let K be central in B. G is residually a $\mathcal{K}$-group if and only if H is $\mathcal{K}$-separable in A.*

Corollary 5. *Let A be residually a $\mathcal{K}$-group, let B be a finitely generated nilpotent residually $\mathcal{K}$-group, let $H \neq A$, and let $K \neq B$. Let also H and K be cyclic, or let K be central in B. G is residually a $\mathcal{K}$-group if and only if H is $\mathcal{K}$-separable in A, K is $\mathcal{K}$-separable in B.*

A group is said to be of finite Hirsch-Zaicev rank r if it possesses a finite subnormal series whose factors are either infinite cyclic or periodic and if the number of the infinite cyclic factors is exactly r.

Theorem 5. *Let A and B be residually torsion-free $\mathcal{K}$-groups, and let H and K be of finite Hirsch-Zaicev rank. Then the following statements are equivalent and any of them implies that G is residually a $\mathcal{K}$-group.*

1. The group $\mathrm{Aut}_G(H)$ is a $\mathcal{K}$-group.

2. There is a homomorphism of G onto a $\mathcal{K}$-group which is injective on the subgroup H.

Corollary 6. *Let A and B be residually torsion-free $\mathcal{K}$-groups, and let H and K be of finite Hirsch-Zaicev rank. If at least one of the subgroups H and K lies in the center of the corresponding free factor or the group $\mathrm{Aut}_G(H)$ is Abelian, then G is residually a $\mathcal{K}$-group.*

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ON HAMILTONIAN TERNARY ALGEBRAS WITH OPERATORS

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Universal algebra A is a *Hamiltonian algebra* if every subuniverse of A is the block of some congruence of the algebra A. The Hamilton property was introduced by B. Csákány [1] for universal algebras as natural generalization of Hamiltonian group.

Universal algebra is an *algebra with operators* if it has an additional set of unary operations acting as endomorphisms with respect to basic operations. In other words, the operators are permutable with basic operations.

An algebra with operators is *ternary* if it has exactly one basic operation and this operation is ternary.

Denote by C_h^t $(h > 0, t \geqslant 0)$ the unar $\langle a | f^t(a) = f^{h+t}(a)\rangle$.

PROPOSITION 1. *Let $\langle A, \Omega\rangle$ be an arbitrary algebra with an operator $f \in \Omega$. If $\langle A, f\rangle \cong C_1^t$ for some $t \in N \cup \{\infty\}$ or $\langle A, f\rangle \cong C_n^0$ for some $n \in N$ then algebra $\langle A, \Omega\rangle$ is Hamiltonian.*

An algebra with one ternary operation p that satisfies the Mal'tsev identities $p(x, y, y) = p(y, y, x) = x$ and has one unary operation permutable with p is an *unar with Mal'tsev operation* [2]. In [2] on any unar $\langle A, f\rangle$ a ternary operation p is defined so that the algebra $\langle A, p, f\rangle$ becomes an unar with Mal'tsev operation. The operation p is defined as follows.

Let $\langle A, f\rangle$ be an arbitrary unar and $x, y \in A$. By $f^n(x)$ we denote the result of f applied n times to an element x. In particular, $f^0(x) = x$. Assume $M_{x,y} = \{n \in \mathrm{N} \cup \{0\} \mid f^n(x) = f^n(y)\}$, and also $k(x, y) = \min\ M_{x,y}$, if $M_{x,y} \neq \emptyset$ and $k(x, y) = \infty$, if $M_{x,y} = \emptyset$. Let's assume further

$$p(x, y, z) \overset{def}{=} \begin{cases} z, & \text{if } k(x, y) \leqslant k(y, z) \\ x, & \text{if } k(x, y) > k(y, z). \end{cases} \tag{1}$$

THEOREM 1. *Let $\langle A, p, f\rangle$ be an unar with Mal'tsev operation p defined by the rule (1). The algebra $\langle A, p, f\rangle$ is Hamiltonian if and only if either $\langle A, f\rangle \cong C_1^t$ for some $t \in N \cup \{\infty\}$, or $\langle A, f\rangle \cong C_1^0 + C_1^0$, or $\langle A, f\rangle \cong C_n^0$ for some $n \in N$.*

In [4] we define a ternary operation w on an arbitrary unar $\langle A, f\rangle$ which permutes with f and satisfies the identities $w(x, x, y) = x$, $w(y, x, x) = x$, $w(x, y, x) = y$. We also define symmetric Mal'tsev operation (minority function) in [3] and ternary near-unanimity operation in [4] on $\langle A, f\rangle$. So we get three classes of ternary algebras with one operator.

For each of these classes we obtain necessary and sufficient conditions for their Hamiltonity. These conditions are similar to that given in the theorem 1.

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ON JACK'S CONNECTION COEFFICIENTS AND THEIR COMPUTATION

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This paper deals with the computation of Jack's connection coefficients that we define as a generalization of both the connection coefficients of the class algebra of the symmetric group and the connection coefficients of the double coset algebra. Using orthogonality properties of Jack symmetric functions and the Laplace Beltrami operator we yield explicit formulas for some of these coefficient that generalize a classical result of Dénes for the number of minimal factorizations of a long cycle into an ordered product of transpositions.

For any integer n we note S_n the symmetric group on n elements and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \vdash n$ an integer partition of $|\lambda| = n$ with $\ell(\lambda) = p$ parts sorted in decreasing order. If $m_i(\lambda)$ is the number of parts of λ that are equal to i, then we may write λ as $[1^{m_1(\lambda)}\,2^{m_2(\lambda)}\ldots]$ and define $Aut_\lambda = \prod_i m_i(\lambda)!$ and $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$. A partition λ is usually represented as a Young diagram of $|\lambda|$ boxes arranged in $\ell(\lambda)$ lines so that the i-th line contains λ_i boxes. Given a box s in the diagram of λ, let $l'(s), l(s), a(s), a'(s)$ be the number of boxes to the north, south, east, west of s respectively. These statistics are called **co-leglength, leglength, armlength, co-armlength** respectively. We note for some parameter α:

$$h_\lambda(\alpha) = \prod_{s\in\lambda}(\alpha a(s) + l(s) + 1), \qquad h'_\lambda(\alpha) = \prod_{s\in\lambda}(\alpha(1 + a(s)) + l(s)). \tag{1}$$

Finally, λ' is the conjugate of partition λ and for two integer partitions λ and μ, we note $\lambda > \mu$ if for all $i \geq 1$, $\lambda_1 + \lambda_2 + \ldots + \lambda_i > \mu_1 + \mu_2 + \ldots + \mu_i$.

Let Λ be the ring of symmetric functions, $m_\lambda(x)$ the monomial symmetric function indexed by λ on indeterminate x, $p_\lambda(x)$ and $s_\lambda(x)$ the power sum and Schur symmetric function respectively. Whenever the indeterminate is not relevant we shall simply write m_λ, p_λ and s_λ. We note $\langle\cdot\,,\cdot\rangle$ the scalar product on Λ such that the power sum symmetric functions verify $\langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda\mu}$ where $\delta_{\lambda\mu}$ is the Kronecker delta. The Schur symmetric functions are characterized by (a) the fact they form an orthogonal basis for $\langle \cdot , \cdot \rangle$ (they form even an orthonormal basis) and (b) the transition matrix between Schur and monomial symmetric functions is upper unitriangular. Using an additional parameter α one can define the **Jack's symmetric functions** J^α_λ as the set of symmetric function characterized by:

(a') The J^α_λ are orthogonal for the alternative scalar product $\langle \cdot , \cdot \rangle_\alpha$ that verifies:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu}, \tag{2}$$

(b') The transition matrix between the J^α_λ and the monomial symmetric functions is upper triangular and the coefficient in m_λ of the expansion of J^α_λ in the monomial basis is equal to $h_\lambda(\alpha)$. Formally it means that the J^α_λ may be expressed with the help of some scalar coefficients $u^\alpha_{\lambda\mu}$ as:

$$J^\alpha_\lambda = h_\lambda(\alpha) m_\lambda + \sum_{\mu < \lambda} u^\alpha_{\lambda\mu} m_\mu, \tag{3}$$

According to the above definition, J^1_λ is the normalized Schur symmetric function $h_\lambda(1) s_\lambda$ and J^2_λ is the zonal polynomial Z_λ. Let $\theta^\lambda_\mu(\alpha)$ denote the coefficient of p_μ in the expansion of J^α_λ in the power sum basis:

$$J^\alpha_\lambda = \sum_\mu \theta^\lambda_\mu(\alpha) p_\mu \tag{4}$$

In the case $\alpha = 1$, the $\theta^\lambda_\mu(1)$'s are equal to the irreducible characters of the symmetric group (up to a normalization factor). In the general case, with proper normalization, these coefficients are called the **Jack's characters** (see [2]). This paper is devoted to the computation of the numbers $a_{\lambda^1, \lambda^2, \ldots, \lambda^r}$ for integer r greater or equal to 1 and $\lambda^i \vdash n$ for $1 \leq i \leq r$ that we define by:

$$a_{\lambda^1, \lambda^2, \ldots, \lambda^r}(\alpha) = \sum_{\beta \vdash n} \frac{1}{h_\beta(\alpha) h'_\beta(\alpha)} \prod_i \theta^\beta_{\lambda^i}(\alpha) \tag{5}$$

We name these numbers the **Jack's connection coefficients**. In some cases we may be interested only in the number of parts of the λ^i. We note:

$$a_{n, p_1, p_2, \ldots, p_r}(\alpha) = \sum_{\lambda^i \vdash n,\ \ell(\lambda^i) = p_i} a_{\lambda^1, \lambda^2, \ldots, \lambda^r}(\alpha). \tag{6}$$

We pay a particular attention to the case when most of the λ^i are equal to $\rho = [1^{n-2}2]$. If we note $a^r_\lambda(\alpha) = a_{\lambda, \rho, \ldots, \rho}(\alpha)$ (with ρ appearing r times, $r \geq 0$), i.e:

$$a^r_\lambda(\alpha) = \sum_{\beta \vdash |\lambda|} \frac{1}{h_\beta(\alpha) h'_\beta(\alpha)} \theta^\beta_\lambda(\alpha) \left(\theta^\beta_\rho(\alpha)\right)^r. \tag{7}$$

We show the following result.

THEOREM 1. *Let $a_\lambda^r(\alpha)$ be defined as above. Then for any integer partition λ we have $a_\lambda^r(\alpha) = 0$ for $r < |\lambda| - \ell(\lambda)$ and*

$$a_\lambda^{|\lambda|-\ell(\lambda)}(\alpha) = \frac{(|\lambda| - \ell(\lambda))!}{\alpha^{\ell(\lambda)} Aut_\lambda \prod_i \lambda_i!} \prod_i \lambda_i^{\lambda_i - 2} \tag{8}$$

REMARK 1. *In the specific case $\lambda = (n)$, Equation 8 reads:*

$$a_{(n)}^{n-1}(\alpha) = \frac{1}{\alpha n} n^{n-2}. \tag{9}$$

We view this later formula as a generalization of the classical formula of Dénes for the number of minimal factorizations of a long cycle in the symmetric group into a product of transpositions.

We show the following additional theorems:

THEOREM 2. *Let $\lambda^i \vdash n$ for $1 \leq i \leq r$ and $\alpha \neq 0$. The Jack's connection coefficients for parameters α et $1/\alpha$ are linked by*

$$a_{\lambda^1,\ldots,\lambda^r}(\alpha^{-1}) = (-\alpha)^{(2-r)n+\sum_i \ell(\lambda^i)} a_{\lambda^1,\ldots,\lambda^r}(\alpha) \tag{10}$$

THEOREM 3. *Let $a_{n,p_1,p_2,\ldots,p_r}(\alpha)$ be the coefficients defined in equation 6 and X_i ($1 \leq i \leq r$) r scalar indeterminate. We have the following formula for any integer $r \geq 1$:*

$$\sum_{p_1,p_2,\ldots,p_r \geq 1} a_{n,p_1,p_2,\ldots,p_r}(\alpha) \prod_{1\leq i\leq r} X_i^{p_i} = \sum_{\beta \vdash n} \frac{1}{h_\beta(\alpha) h'_\beta(\alpha)} \prod_{1 \leq i \leq r} R_\beta^\alpha(X_i) \tag{11}$$

where :

$$R_\lambda^\alpha(k) = \prod_{s \in \lambda} (k + \alpha a'(s) - l'(s)) = J_\lambda^\alpha(I_k). \tag{12}$$

Theorem 3 is a generalization of the main formula in [3].

For indeterminate $x = (x_1, x_2, \ldots)$ define the **Laplace Beltrami Operator** by

$$D(\alpha) = \frac{\alpha}{2} \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_i \sum_{j \neq i} \frac{x_i x_j}{x_i - x_j} \frac{\partial}{\partial x_i} \tag{13}$$

THEOREM 4. *Let $a_\lambda^r(\alpha)$ be the Jack's connection coefficients defined by 7. We have the following equality:*

$$a_\lambda^r(\alpha) = \frac{1}{\alpha^n n!} [p_\lambda] D(\alpha)^r (p_1^n), \tag{14}$$

where $[p_\lambda] D(\alpha)^r (p_1^n)$ denotes the coefficient of p_λ in the power sum expansion of $D(\alpha)^r (p_1^n)$.

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ABOUT SEMIRINGS WITH SEMILATTICE MULTIPLICATION

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We continue studying of semirings with semilattice multiplication, namely semirings in which the multiplicative semigroups are commutative and idempotent (see [1]).

Semiring is an algebraic system $\langle S, +, \cdot \rangle$ such that $\langle S, + \rangle$ is a commutative semigroup, $\langle S, \cdot \rangle$ is a semigroup, the multiplication is distributive over the addition. If there is an element 0 in the semiring S, such that $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$, then we call S a *semiring with a zero* 0.

Semiring with identities $x + x = x, x + y = xy$ is called *mono-semiring*. We say that *constant addition* is held in the semiring, if the identity $x + y = u + v$ is true.

For any semiring S the zero element 0 can be externally attached . If semilattice multiplication is held in S , then a semiring $S \cup \{0\}$ is called semiring with semilattice multiplication.

Theorem 1. *Any subdirectly irreducible semiring S with semilattice multiplication satisfies the following properties:*

1) *there is the* 1 *in S; $S \setminus \{1\}$ is the ideal in S with its own unity element e*;
2) *smallest nonzero congruence in S sticks together only the elements* 1 *and e;*
3) *in S equality $3 = 2$ is true or equalities $3 = 1$ and $e = 2$ are true.*

From the property 3 of the proposition 1 the next follows:

Theorem 2. *Arbitrary semiring with semilattice multiplication is a subdirect product semirings with semilattice multiplication with the identity $3x = x$ and semirings with semilattice multiplication with the identity $3x = 2x$.*

On an arbitrary semiring S let's consider binary relations $\sim$ and $\approx$ that are defined by the next way:

$$x \sim y \Leftrightarrow \exists s, t \in S\,(x + s = y, y + t = x);$$

$$x \approx y \Leftrightarrow 3x = 3y.$$

PROPOSITION 1. *For any semiring S with semilattice multiplication following statements hold:*

1) *relation $\sim$ is the smallest congruence on S on which faktorsemiring satisfies the identity $3x = 2x$;*

2) *relation $\sim$ is the smallest congruence on S on which faktorsemiring satisfies the identity $3x = x$;*

3) *intersection of congruences $\sim$ and $\approx$ is the relation equality in S.*

We denote by $\mathfrak{M}$ variety of all semirings with semilattice multiplication. For semirings $S_1, \dots, S_n$ we denote by $\mathfrak{M}(S_1, \dots, S_n)$ the variety of semirings that is defined by the set of identities performed on each of the semirings $S_1, \dots, S_n$. We denote by $\mathfrak{M}(f = g)$ the subvariety of $\mathfrak{M}$ that is generated by the semiring identity $f = g$.

It is obviously that to within an isomorphism there are four two-element commutative multiplicative idempotent semirings: two-element chain $\mathbb{B}$, two-element field $\mathbb{Z}_2$, two-element mono-semiring $\mathbb{D} = \{1, \infty\}$ with a identity 1, two-element semiring $\mathbb{T} = \{1, \infty\}$ with a identity 1 and constant addition.

Let $\mathfrak{N} = \mathfrak{M}(\mathbb{B}, \mathbb{Z}_2, \mathbb{D}, \mathbb{T})$.

PROPOSITION 2. *Lattice of subvarieties of $\mathfrak{M}$ has exactly 4 atoms $\mathfrak{M}(\mathbb{B})$, $\mathfrak{M}(\mathbb{Z}_2)$, $\mathfrak{M}(\mathbb{D})$, $\mathfrak{M}(\mathbb{T})$ and it is atomic.*

PROPOSITION 3. *For any semiring S following statements hold:*

1) $S \in \mathfrak{M}(\mathbb{B}, \mathbb{Z}_2, \mathbb{T}) \Leftrightarrow$ *S satisfies the identity $x + 2xy = 3x$;*

2) $S \in \mathfrak{M}(\mathbb{B}, \mathbb{D}, \mathbb{T}) \Leftrightarrow$ *in S the dual distributive law $x + yz = (x + y)(x + z)$ holds $\Leftrightarrow S \in \mathfrak{N}$ and the identity $3x = 2x$ is true;*

3) $S \in \mathfrak{M}(\mathbb{Z}_2, \mathbb{D}, \mathbb{T}) \Leftrightarrow$ *S satisfies the identity $x + 2xy = 3xy$;*

4) $S \in \mathfrak{M}(\mathbb{B}, \mathbb{Z}_2, \mathbb{D}) \Leftrightarrow S \in \mathfrak{N}$ *and S satisfies the identity $3x = x$.*

PROPOSITION 4. *In the lattice of subvarieties of $\mathfrak{M}$ following equalities hold:*

1) $\mathfrak{N} = \mathfrak{M}(x + 2xy + yz = x + 2xz + yz)$;

2) $\mathfrak{M} = \mathfrak{M}(3x = x) \vee \mathfrak{M}(3x = 2x)$;

3) $\mathfrak{M}(3x = 2x) \cap \mathfrak{M}(3x + y = x + y) = \mathfrak{M}(2x + y = x + y) = \mathfrak{M}(2x = x) \vee \mathfrak{M}(\mathbb{T})$;

4) $\mathfrak{M}(3x + y = x + y) = \mathfrak{M}(3x = x) \vee \mathfrak{M}(\mathbb{T})$.

PROPOSITION 5. *Lattice of subvarieties of $\mathfrak{N}$ is 16-element Boolean lattice.*

Semiring S is called *free* in the class $\mathbf{K}$ of semirings with the set X of free generators if for any semiring $T \in \mathbf{K}$ an arbitrary mapping $X \to T$ extends to a homomorphism $S \to T$. Thus, any semiring from the class $\mathbf{K}$ is a homomorphic image of the corresponding free semiring from $\mathbf{K}$.

Free semiring in the variety $\mathfrak{M}$ with the set X of free generators we denote by $FM(X)$ or $FM_{|X|}$ where $|X|$ is the cardinality of X. For example, FM_1 semiring with one free generator $a = 1$ has the form $FM_1 = \{1, 2, 3\}$ by the identity $4x = 2x$.

Let S is a semiring with zero 0 and unity element 1, and P is a multiplicative semigroup. Free (left) S-semimodule SP with a basis $\{p : p \in P\}$ is the set of finite (non-zero) formal sums

$$s = \sum_{\substack{p\in P \\ \text{almost all } s_p=0 \\ \text{some } s_p\neq 0}} s_p p \,(s_p \in S, p \in P),$$

with operations

$$\left(\sum s_p p\right) + \left(\sum t_p p\right) = \sum \left(s_p + t_p\right) p$$

and

$$s\left(\sum s_p p\right) = \sum s s_p p.$$

Semimodule SP becomes semiring, if the multiplication is defined by the formula:

$$\left(\sum s_p p\right)\left(\sum t_p p\right) = \sum_{p\in P}\left(\sum_{\substack{u,v\in P \\ uv=p}} s_u t_v\right) p.$$

Semiring SP is called *semiring semigroup* semigroup P over a semiring coefficients S. We identify $p \equiv 1p$. $0p$ means non-availability of corresponding term in the formal sum s.

We denote by T semiring with zero $FM_1 \cup \{0\}$. For a free semilattice $L(X)$ with the set $X = \{x_i : i \in I\}$ of free generators in the variety of all semilattices we consider semigroup semiring $TL(X)$. It is a semiring (nonzero) polynomials without constant term in the variables x_i with coefficients $0, 1, 2, 3 \in T$. We obtain that $FM(X) \cong TL(X)/\rho$, where ρ — the smallest congruence on which elements $u + v$ and $u + v + 2uv$ equivalent for any (disjoint) words $u, v \in L(X)$.

Proposition 6. *Following equalities hold:* $|FM_1| = 3, |FM_2| = 39, |FM_3| = 2289$.

We note that the free multiplicatively idempotent semiring with set X of free generators is finite if and only if the set X is finite. This follows from the known fact about the finiteness of finitely generated idempotent semigroups.

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IDEALS IN PARTIAL SEMIRINGS OF CONTINUOUS $[0,\infty]$-VALUED FUNCTIONS

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Partial semirings $\mathrm{C}^+_\infty(X)$ of continuous functions on topological spaces X with values in the semiring $[0,\infty]$ are studied [3]. The semiring $[0,\infty]$ is considered with the usual topology. Partial semiring of all continuous non-negative real-valued functions on topological space X with pointwise defined operations of addition and multiplication of functions is denoted $\mathrm{C}^+(X)$.

Partial semirings $\mathrm{C}^+_\infty(X)$ differ from semirings because product of some functions in partial semirings $\mathrm{C}^+_\infty(X)$ might be not defined (be discontinuous). Non-empty set $I \subseteq \mathrm{C}^+_\infty(X)$ is the ideal in partial semirings $\mathrm{C}^+_\infty(X)$ if for each $f, g \in I$ and $h \in \mathrm{C}^+_\infty(X)$ functions $f+g$ and fh belong to the ileal I, and fh is the continuous function.

The ideal $I \neq S$ of partial semiring S is called: *maximal* if in S there are no any ideals that differ from S and strictly contain I; *simple* if for each $a, b \in S$ condition $ab \in I$ implies $a \in I$ or $b \in I$; *strong* (*subtractive*), if $a+b \in I \Rightarrow a, b \in I$ $(a+b, a \in I \Rightarrow b \in I)$ for each $a, b \in S$.

Topological space is called *Tychonoff space* (*Hewitt space*) if it is homeomorphic to a (closed) subspace of some Tychonoff power of $\mathbf{R}$. Tychonoff space is called P-*space* if every its zero-set is open set in it (see [2]). For any function $f \in \mathrm{C}^+_\infty(X)$ we define *zero-set* $\mathrm{Z}(f) = \{x \in X \,:\, f(x) = 0\}$, closed set $\mathrm{H}(f) = \{x \in X : f(x) = \infty\}$ and open set $\mathrm{coz} f = X \setminus (\mathrm{Z}(f) \bigcup \mathrm{H}(f))$.

For any function $f \in \mathrm{C}^+_\infty(X)$ let's define functions $f^*, f_{(1)} \in \mathrm{C}^+_\infty(X)$ by the next way:

$$f^*(x) = \begin{cases} \infty, & \text{if } x \in \mathrm{Z}(f), \\ 0, & \text{if } x \in \mathrm{H}(f), \\ f^{-1}(x), & \text{if } x \in \mathrm{coz} f; \end{cases}$$

$$f_{(1)}(x) = \begin{cases} f(x), & \text{if } x \in f^{-1}[0;1], \\ 1, & \text{if } x \in f^{-1}[1;\infty]. \end{cases}$$

The product of functions ff^* equals 1 on $\mathrm{coz} f$ and equals 0 on $\mathrm{Z}(f) \cup \mathrm{H}(f)$ so ff^* has not to be a continuous function. The function ff^* is continuous if and only if the zero-set $\mathrm{Z}(f)$ and $\mathrm{H}(f) = \mathrm{Z}(f^*)$ are open. Therefore the partial semiring $\mathrm{C}^+_\infty(X)$ is semiring then and only then when X is P-space.

Let X be an arbitrary Tychonoff space and βX be its Stone-Cech compactification [1]. Extension of function $f \in \mathrm{C}^+_\infty(X)$ to βX is uniquely determined and denoted $f^\beta \in \mathrm{C}^+_\infty(\beta X)$. For every function $f \in \mathrm{C}^+_\infty(\beta X)$ we have $f = (f|_X)^\beta$. For any function $g \in \mathrm{C}^+_\infty(X)$ equality $g = g^\beta|_X$ holds.

Let $p \in \beta X$. Following sets are ideals in $\mathrm{C}^+_\infty(X)$:

$$M_p = \left\{ f \in \mathrm{C}^+_\infty(X) | p \in \overline{\mathrm{Z}(f)}^{\beta X} \right\},$$

$$M_{p,\infty} = \left\{ f \in \mathrm{C}^+_\infty(X) | p \in \overline{\mathrm{Z}(f) \cup \mathrm{H}(f)}^{\beta X} \right\},$$

$$O_p = \left\{ f \in \mathrm{C}^+_\infty(X) | p \in \left(\overline{\mathrm{Z}(f)}^{\beta X} \right)^0 \right\},$$

$$O_{p,\infty} = \left\{ f \in \mathrm{C}^+_\infty(X) | p \in \left(\overline{\mathrm{Z}(f)}^{\beta X} \right)^0 \text{ or } p \in \left(\overline{\mathrm{H}(f)}^{\beta X} \right)^0 \right\}.$$

Inclusions $O_p \subseteq M_p \subset M_{p,\infty}$ are obvious. In $\mathrm{C}^+_\infty(X)$ there are not any maximal ideals besides $M_{p,\infty}$ [3].

Lemma 1. *Let J be ideal of partial semiring $\mathrm{C}^+_\infty(X)$ such that $J \nsubseteq M_{p,\infty}$. Then there exists a function $f \in J \setminus M_{p,\infty}$ with values in the unit interval $[0, 1]$.*

Lemma 2. *For any two distinct points $p, q \in \beta X$ equality $O_p + O_q = \mathrm{C}^+_\infty(X)$ is true. Ideal O_p is contained in a unique maximal ideal $M_{p,\infty}$.*

Lemma 3. *In the partial semiring $C^+(X)$ for any functions $f, g \in \mathrm{C}^+_\infty(X)$ functions $f_{(1)} + g_{(1)}$ and $(f + g)_{(1)}$ divide each other.*

Lemma 4. *If P is a prime ideal in $\mathrm{C}^+_\infty(X)$ and $P \subseteq M_p, p \in \beta X$ then for any function $f \in \mathrm{C}^+_\infty(X)$ $f \in P$ then and only then when $f_{(1)} \in P$.*

Theorem 1. *For Tychonoff space X any prime ideal P of partial semiring $\mathrm{C}^+_\infty(X)$ satisfies the following properties:*

1) P contains the ideal O_p for some uniquely determined point $p \in \beta X$;

2) if $O_p \subseteq P$, where $p \in \beta X$, then $P \subseteq M_{p,\infty}$;

3) if $P \subseteq M_p$, where $p \in \beta X$, then P is the strong ideal;

4) if $O_p \subseteq P \nsubseteq M_p$ for some point $p \in \beta X$, then the ideal P is not subtractive and $O_{p,\infty} \subseteq P$.

Corollary 1. *Every prime ideal of partial semiring $\mathrm{C}^+_\infty(X)$ is contained in an unique maximal ideal.*

Corollary 2. *For any prime ideal P of partial semiring $\mathrm{C}^+_\infty(X)$ following conditions are equivalent: 1) P is strong ideal; 2) P is subtractive ideal; 3) $P \subseteq M_p$ for (unique) point $p \in \beta X$.*

Since ideals M_p are strong prime ideals then from the corollary 2 we obtain the next:

Theorem 2. *Let X be Tychonoff space. Ideals M_p, $p \in \beta X$ and only they are maximum ideals among all the strong (subtractive) prime ideals of partial semiring $\mathrm{C}^+_\infty(X)$.*

Note 1. *Like classical Gelfand-Kolmogorov theorem for rings* $\mathrm{C}(X)$ *[2, chapter 7], on a Tychonoff space X there exists a homeomorphism between spaces of all maximal ideals of partial semiring $\mathrm{C}^+_\infty(X)$ and compactification βX. Topological space* $\mathrm{MaxC}^+_\infty(X)$ *with Stone–Zariski topology is called maximal spectrum of partial semiring $\mathrm{C}^+_\infty(X)$. We have $\mathrm{MaxC}^+_\infty(X) \approx \beta X$.*

Lemma 5. *For any function $f \in \mathrm{C}^+_\infty(X)$ the following statements are true:*

1) $\mathrm{H}(f) = \varnothing \Leftrightarrow$ for any maximal ideal M of partial semiring $\mathrm{C}^+_\infty(X)$, if $f \in M$ then $f \in P$ for the maximal strong prime ideal $P \subset M$;

2) $\mathrm{Z}(f) = \varnothing \Leftrightarrow$ for any maximal ideal M of partial semiring $\mathrm{C}^+_\infty(X)$ if $f \in M$ then $f \notin P$ for the maximal strong prime ideal $P \subset M$.

According Lemma 5 conditions $\mathrm{H}(f) = \varnothing$ and $\mathrm{Z}(f) = \varnothing$ are expressed with the terms of the partial semiring $\mathrm{C}^+_\infty(X)$. Therefore the following statements is true:

Proposition 1. *For any homomorphism $\alpha : \mathrm{C}^+_\infty(X) \to \mathrm{C}^+_\infty(Y)$ of the partial semirings the equality $\alpha(\mathrm{C}^+(X)) = \mathrm{C}^+(Y)$ is true.*

From the proposition 1 and the monograph [4, Chapter 2] we obtain:

Theorem 3. *Any Hewitt space X isuniquely defined up to homeomorphism by the partial semirings $\mathrm{C}^+_\infty(X)$.*

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FREE PRODUCTS OF LEFT ZERO AND RIGHT ZERO DIMONOIDS

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Recall that a dimonoid [1] is an algebra with two binary associative operations satisfying some three axioms. A dimonoid $(D, \dashv, \vdash)$ is called a left zero and right zero dimonoid, if $(D, \dashv)$ is a left zero semigroup and $(D, \vdash)$ is a right zero semigroup.

The construction of a free product of arbitrary dimonoids was given in [2]. Here the free product of left zero and right zero dimonoids is characterized.

Let $\{(D_i, \dashv_i, \vdash_i)\}_{i\in X}$ be a family of arbitrary pairwise disjoint left zero and right zero dimonoids. Denote by $F[(D_i, \dashv_i)]_{i\in X}$ the free product of semigroups $(D_i, \dashv_i), i \in X$, and singleton semigroups $\{i\}, i \in X$.

Let

$$R(D_i)_{i\in X} = \{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in F[(D_i, \dashv_i)]_{i\in X} \times \mathbb{N} \mid k \geqslant m\},$$

$$R^\star(D_i)_{i\in X} = \{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in R(D_i)_{i\in X} \mid$$

$$s_{\gamma_l} \in \cup_{i\in X} D_i \Leftrightarrow l = m,\ 1 \leqslant l \leqslant k\},$$

$$\mu : (\cup_{i\in X} D_i) \cup X \to (\cup_{i\in X} D_i) \cup X : a \mapsto a\mu = \begin{cases} i, & a \in D_i, \\ a, & a \in X. \end{cases}$$

If $k = 1$, then the sequences $s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}$, $s_{\gamma_2}...s_{\gamma_l}...s_{\gamma_k}$ will be regarded empty.

Define operations $\dashv$ and $\vdash$ on $R^\star(D_i)_{i\in X}$ by

$$(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \dashv (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) =$$

$$= \begin{cases} (s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k} s_{\alpha_1}\mu...s_{\alpha_l}\mu...s_{\alpha_r}\mu, m), & s_{\gamma_k}\mu \neq s_{\alpha_1}\mu, \\ (s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k} s_{\alpha_2}\mu...s_{\alpha_l}\mu...s_{\alpha_r}\mu, m), & s_{\gamma_k}\mu = s_{\alpha_1}\mu, \end{cases}$$

$$(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \vdash (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) =$$

$$= \begin{cases} (s_{\gamma_1}\mu...s_{\gamma_l}\mu...s_{\gamma_k}\mu s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, k + t), & s_{\gamma_k}\mu \neq s_{\alpha_1}\mu, \\ \left(s_{\gamma_1}\mu...s_{\gamma_l}\mu...s_{\gamma_{k-1}}\mu s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, k + t - 1\right), & s_{\gamma_k}\mu = s_{\alpha_1}\mu \end{cases}$$

for all $(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m), (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) \in R^\star(D_i)_{i\in X}$. The algebra $(R^\star(D_i)_{i\in X}, \dashv, \vdash)$ will be denoted by $\breve{R}(D_i)_{i\in X}$.

Theorem 1. *$\breve{R}(D_i)_{i\in X}$ is the free product of left zero and right zero dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$.*

Structural properties of the dimonoid $\breve{R}(D_i)_{i\in X}$ are investigated.

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ENDOTOPISM SEMIGROUPS OF AN EQUIVALENCE

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Let X be an arbitrary nonempty set and $\rho \subseteq X \times X$. The ordered pair (φ, ψ) of transformations φ and ψ of a set X is called an *endotopism* [1] of ρ if for all $a, b \in X$ the condition $(a, b) \in \rho$ implies $(a\varphi, b\psi) \in \rho$. The set of all endotopisms of ρ is a semigroup with respect to the componentwise multiplication operation. This semigroup is called an *endotopism semigroup* of a relation ρ and is denoted by $Et(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *half-strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies the existence of preimages $a' \in a\varphi\varphi^{-1}$ and $b' \in b\psi\psi^{-1}$ such that $(a', b') \in \rho$. The set of all half-strong endotopisms of ρ we denote by $HEt(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *locally strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies that for every $a' \in a\varphi\varphi^{-1}$ there exists $b' \in b\psi\psi^{-1}$ such that $(a', b') \in \rho$ and analogously for all preimages of $b\psi$. We denote the set of all locally strong endotopisms of ρ by $LEt(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *quasi-strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies that there exists $a' \in a\varphi\varphi^{-1}$ such that for every $b' \in b\psi\psi^{-1}$ we have $(a', b') \in \rho$ and analogously for a suitable preimage of $b\psi$. By $QEt(\rho)$ we denote the set of all quasi-strong endotopisms of ρ.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies that $(a, b) \in \rho$. We denote the set of all strong endotopisms of ρ by $SEt(\rho)$. It is not hard to check that $SEt(\rho)$ is a subsemigroup of $Et(\rho)$.

The ordered pair (φ, ψ) of permutations φ and ψ of a set X is called an *autotopism* of $\rho \subseteq X \times X$ if $(a, b) \in \rho$ iff $(a\varphi, b\psi) \in \rho$ for all $a, b \in X$. The set of all autotopisms of ρ we denote by $At(\rho)$. Obviously, $At(\rho)$ is a subgroup of $Et(\rho)$ for any ρ on X.

For an arbitrary set X, the relations $i_X = \{(a, a) | a \in X\}$ and $\omega_X = X \times X$ are called *identity* and *universal* relations on X, respectively. A binary relation ρ on X is called *trivial* if $\rho = i_X$ or $\rho = \omega_X$.

We denote the set of all equivalence relations on a set X by $Eq(X)$ and the set of all equivalence relations on X with n classes of a cardinality ≥ 2 by $Eq^n(X)$.

Lemma 1. *The set* $LEt(\alpha)$, $\alpha \in Eq(X)$, *is a semigroup if and only if one of following conditions holds:*

(i) α *is an identity equivalence;*

(ii) there exists a unique class $A \in X/\alpha$ *such that* $|A| \geq 2$.

Lemma 2. *The set* $HEt(\alpha)$, $\alpha \in Eq(X)$, *is a semigroup if and only if* α *is a trivial equivalence.*

Lemma 3. *For all* $\alpha \in Eq(X)$, *we have* $QEt(\alpha) = SEt(\alpha)$.

A semigroup S is a called *regular* [2] if for all $a \in S$ there exists $x \in S$ such that $axa = a$.

The regularity of semigroups of all types of endotopisms of an arbitrary equivalence describes the following theorem.

Theorem 1. *(i) The semigroup* $Et\,(\alpha)$, $\alpha \in Eq(X)$, *is regular if and only if* α *is a trivial equivalence;*

(ii) The semigroup $HEt(\alpha)$, *where* α *is trivial, is regular;*

(iii) The semigroup $LEt(\alpha)$, *where* $\alpha \in Eq^1(X)$ *or* $\alpha = i_X$, *is regular;*

(iv) The semigroup $SEt(\alpha)$, $\alpha \in Eq(X)$, *is regular if and only if quotient-set* X/α *is finite;*

(v) The group $At(\alpha)$ *is regular for any* $\alpha \in Eq(X)$.

A semigroup S is called *coregular* [3] if for all $a \in S$ there exists $x \in S$ such that

$$axa = xax = a.$$

Theorem 2. *(i) The semigroup* $Et\,(\alpha)$, $\alpha \in Eq(X)$, *is coregular if and only if* $|X| \in \{1, 2\}$*;*

(ii) The semigroup $HEt(\alpha)$, *where* α *is trivial, is coregular if and only if* $Et\,(\alpha)$ *is coregular;*

(iii) The semigroup $SEt(\alpha), \alpha \in Eq(X)$, *is coregular if and only if* $|X| \in \{1, 2\}$ *or* $|X| = 3, \alpha \notin \{i_X, \omega_X\}$*;*

(iv) The semigroup $LEt(\alpha)$, *where* $\alpha \in Eq^1(X)$ *or* $\alpha = i_X$, *is coregular if and only if* $SEt\,(\alpha)$ *is coregular;*

(v) The group $At\,(\alpha)\,, \alpha \in Eq(X)$, *is coregular if and only if* $SEt\,(\alpha)$ *is coregular or* $|X| = 4, |X/\alpha| = 3$.

Note that for any $\rho \subseteq X \times X$, we have the following sequence of inclusions:

$$Et(\rho) \supseteq HEt(\rho) \supseteq LEt(\rho) \supseteq QEt(\rho) \supseteq SEt(\rho) \supseteq At(\rho).$$

With this sequence we associate the sequence of number $(s_1, s_2, s_3, s_4, s_5)$, where $s_i \in \{0, 1\}, i \in \{1, ..., 5\}$. Here 1 stands $\neq$ and 0 stands $=$ at the respective position in the above sequence for endotopisms. For example, $s_2 = 0$ indicates that $HEt(\rho) = LEt(\rho)$, and $s_5 = 1$ means that $SEt(\rho) \neq At(\rho)$. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the *endotype* (or the *endotopism type*) of ρ relative to its endotopisms and is denoted by $Ettype(X, \rho)$.

THEOREM 3. *For any equivalence α on a set X, we have*

$$Ettype(X,\alpha) = \begin{cases} 0, & |X| = 1, \\ 4, & 2 \le |X| < \infty,\ \alpha = i_X, \\ 16, & 2 \le |X|,\ \alpha = \omega_X, \\ 20, & |X| = \infty,\ \alpha = i_X, \\ 23, & \alpha \ne i_X, \alpha \ne \omega_X. \end{cases}$$

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THE NUMBER OF SOLUTIONS OF THE DIOPHANTINE EQUATION OF SHORT PERIODS WITH SEMISIMPLE NUMBERS OF SHORT PERIODS

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In 1940 I.M. Vinogradov [1] used the method of trigonometric sums proved the asymptotic formula for the number of primes in intervals of the form $[(2m)^2, (2m+1)^2)$, $m \in \mathbb{N}$.

In 1986 S.A. Gritsenko [2] proved the asymptotic formula for the number of primes p such $p \leqslant x$ and

$$p \in [(2m)^c, (2m+1)^c), \tag{1}$$

где $m \in \mathbb{N}$, и $c \in (1, 2]$. Here, intervals (1) have known as short intervals or Vinogradov‘s intervals.

In 1988 S.A. Gritsenko solved problems with primes of (1) ([3], [4]).

In 1992 A. Balog and K. J. Friedlander was also solved problems with primes from the intervals of the form [5].

Note that in the works [3]-[5] additive problems are ternary or solved by the scheme of the ternary problem.

There is interest in binaries additive problems with primes from the intervals of the form (1). There are no options theorem Bombieri-Vinogradova for Prime numbers of the intervals of the form (1), which is equal to the classical theorem of Bombieri-Vinogradova. Therefore, to solve the binary additive problem with primes from the interval of the form (1) is currently not possible.

The author of this work were solved some binary additive problem with semisimple numbers from the Vinogradov's intervals ([6], [7]). The report will be considered another binary additive problem with semisimple numbers of a special form. This problem is analogous to Hardy-Littlewood's problem with semisimple numbers. In 1965 Yu. V. Linnik has solved a similar problem using a dispersion method.

The paper considers Diophantine equation

$$xy + p_1 p_2 = n,$$

where p_1 and p_2 are primes, x and y are natural numbers, as well as conditions are met: numbers p_1p_2 are in the intervals of the form (1), $m \in \mathbb{N}$, $c \in (1, 2]$ and $p_i > \exp(\sqrt{\ln n}\ )$ $(i = 1, 2)$.

The number of solutions of this equation denote by $J_1(n)$. We used the method of trigonometric sums I.M. Vinogradov and proved the asymptotic formula:

$$J_1(n) = \frac{1}{2} J(n)(1 + O(\frac{\ln\ln\ln n}{\ln\ln n})),$$

where

$$J(n) \sim c_0 n \ln\ln n,$$

$$c_0 = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{r\varphi(r)}.$$

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CONTENTS

O. Shchekaturova, V. Yaroshevich The Disposition of Primary and Secondary Idempotents in Green's Classes on the Set of Boolean Matrices ... 167