Elementary polycycles

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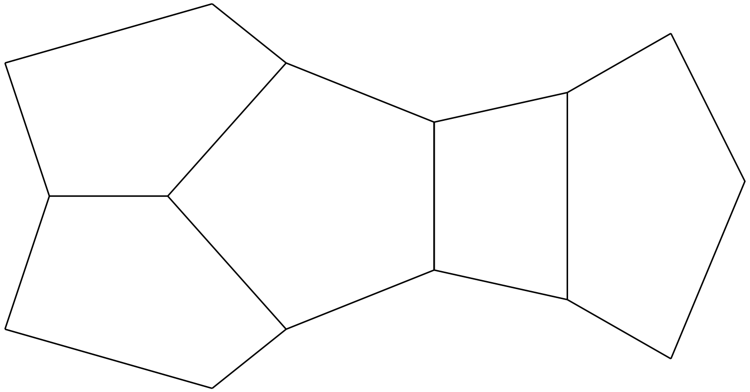
I. (R, q)-polycycles

Definition

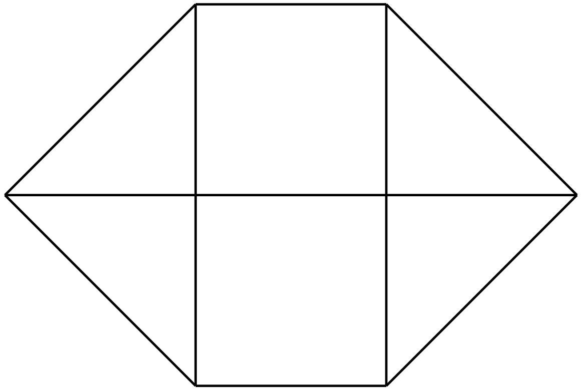
Given $q \in \mathbb{N}$ and $R \subset \mathbb{N}$, a (R, q)-polycycle is a non-empty 2-connected plane, locally finite graph G with faces partitionned in two sets F_1 and F_2 (F_1 is non-empty), so that:

- all elements of F_1 (called proper faces) are combinatorial i-gons with $i \in R$;
- all elements of F_2 (called holes) are pair-wisely disjoint, i.e. have no common vertices;
- all vertices have degree within $\{2, \ldots, q\}$ and all interior vertices are q-valent.

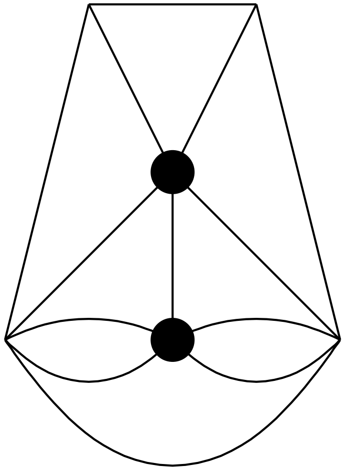
Examples with one hole



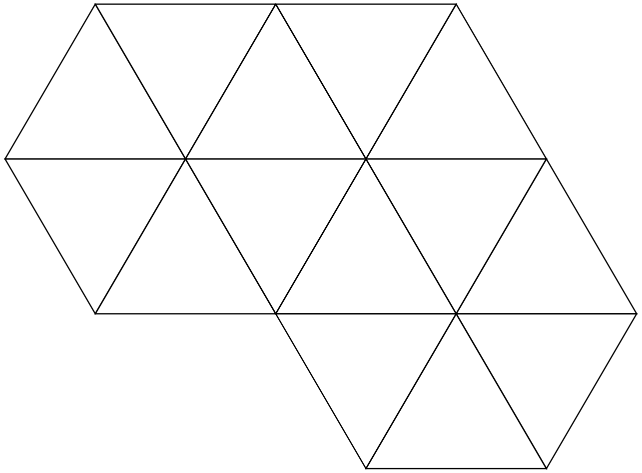
A $(\{4,5\},3)$-polycycle



A $(\{3,4,5\},4)$-polycycle

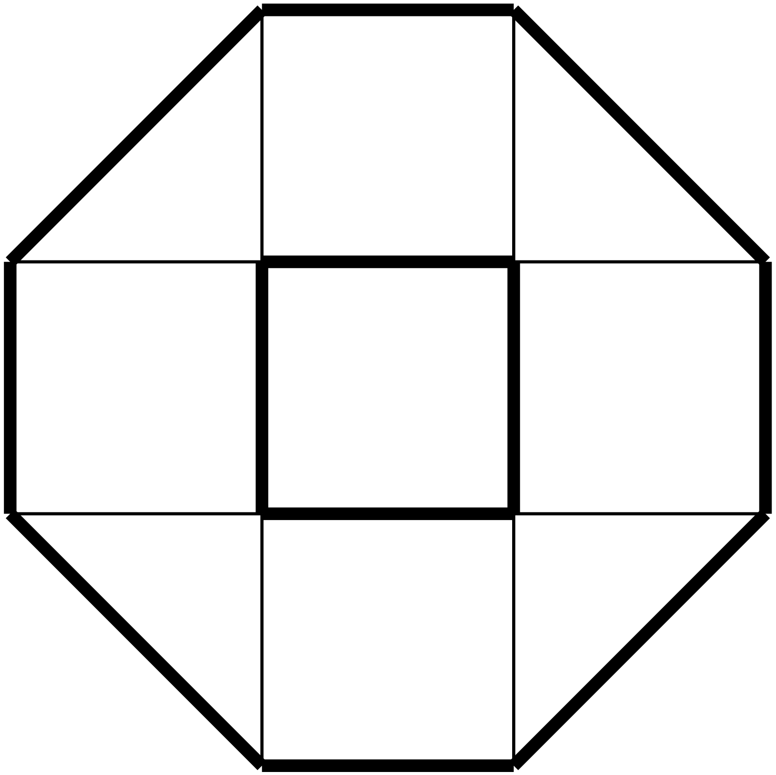


A $(\{2,3\},5)$-polycycle

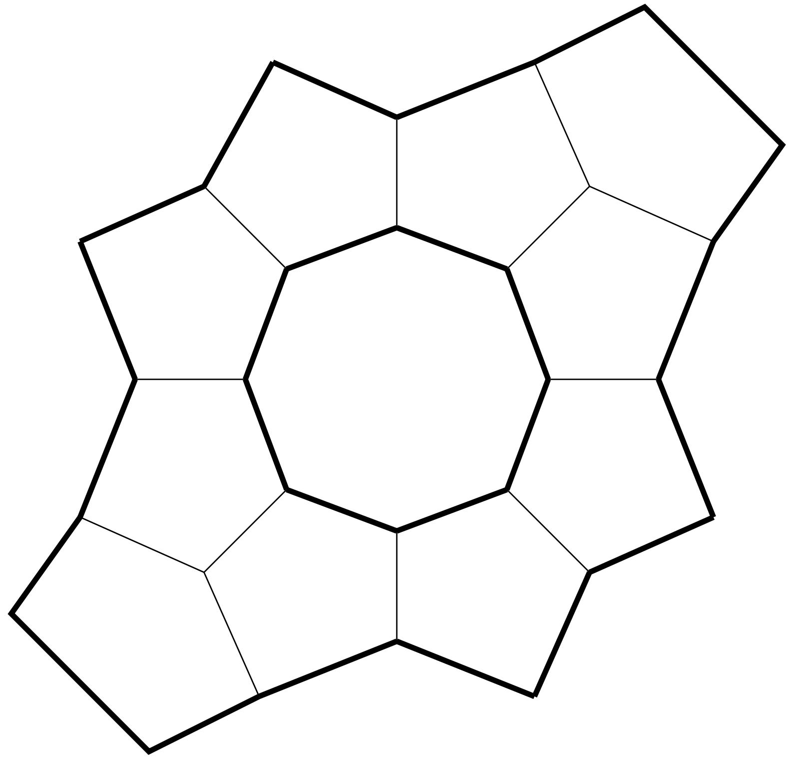


A $(\{3\},6)$-polycycle

Examples with two holes or more



A $(\{3,4\},4)$-polycycle

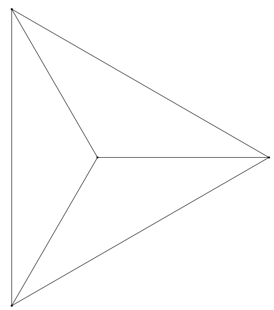


A $(\{5\},3)$-polycycle

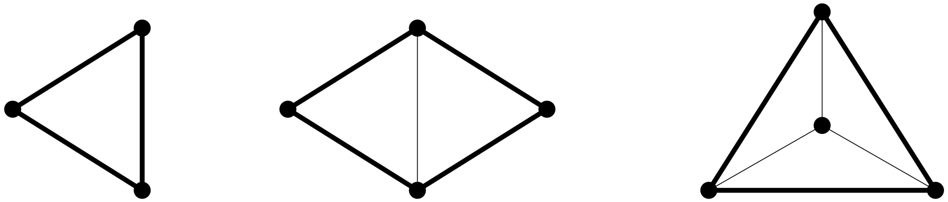
$(\{3\}, 3)$-polycycles

Any $(\{3\}, 3)$-polycycle is one of the following

- Tetrahedron (with no hole):



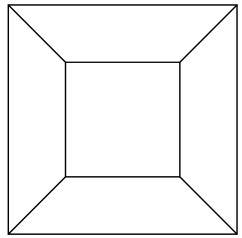
- 3 following polycycles (with one hole):



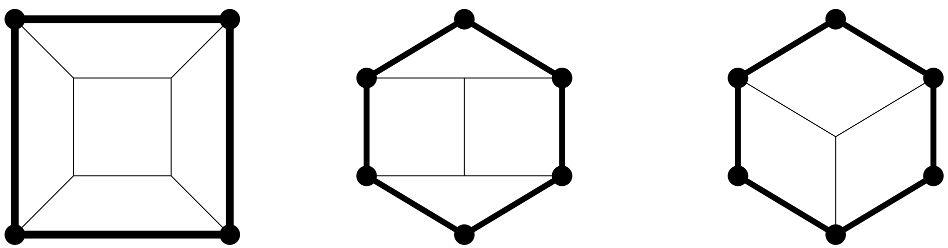
$(\{4\}, 3)$-polycycles

Any $(\{4\}, 3)$-polycycle is one of the following

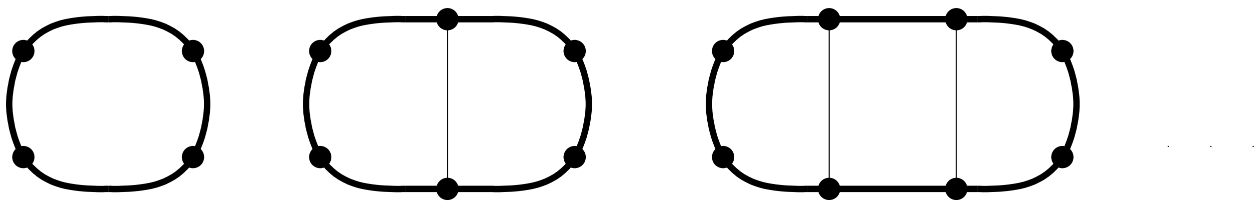
- Cube (with no hole):



- 3 following polycycles (with one hole)

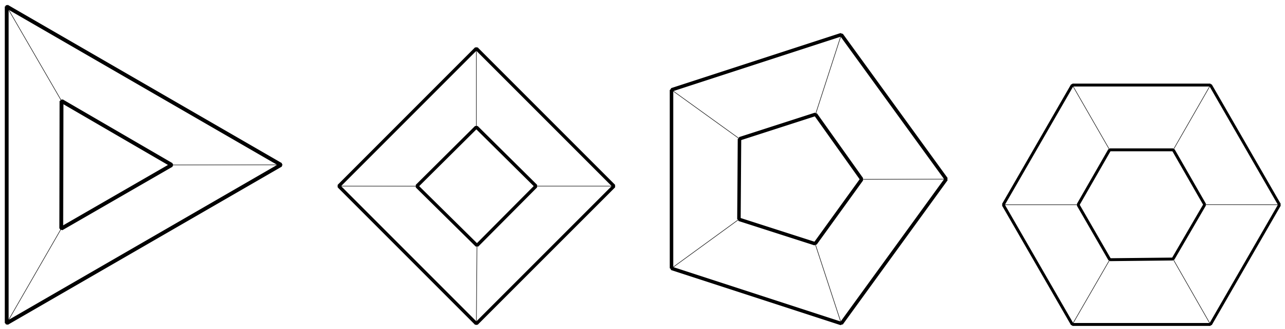


- Following infinite family (with one hole):

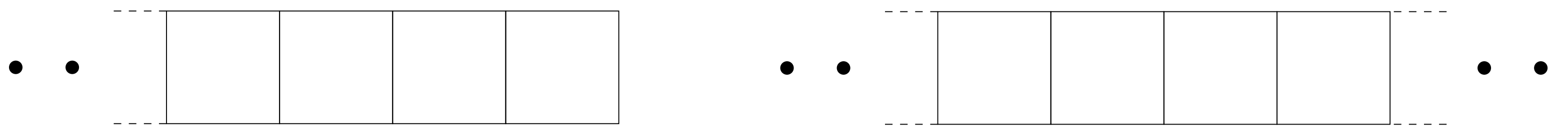


$(\{4\},3)$-polycycles

- The infinite family $Prism_n$ (with two holes)



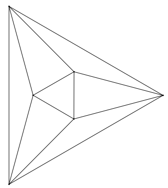
- Following two infinite $(\{4\},3)$-polycycles:



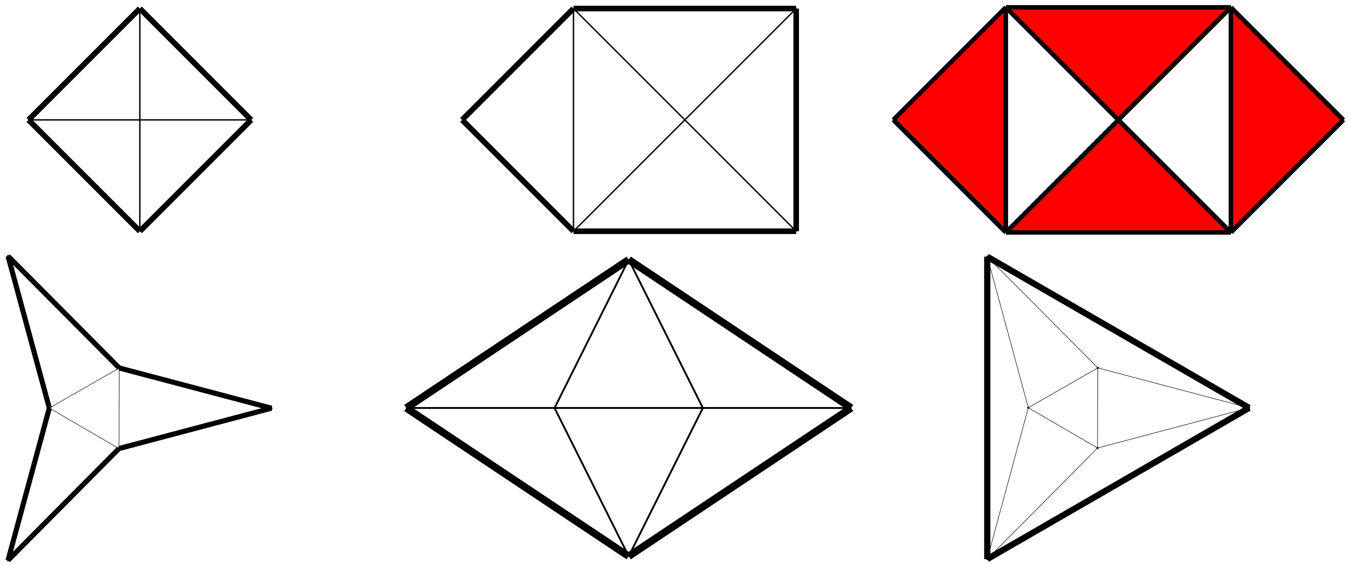
singly infinite polycycle   doubly infinite polycycle

$(\{3\}, 4)$-polycycles

- Octahedron (with no hole):

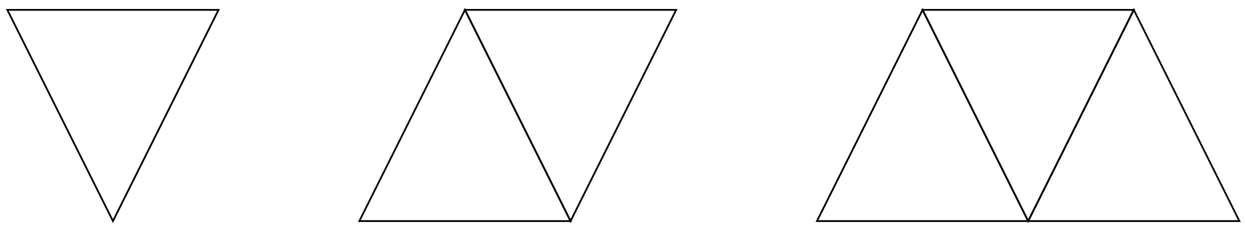


- Following polycycles (with one hole)

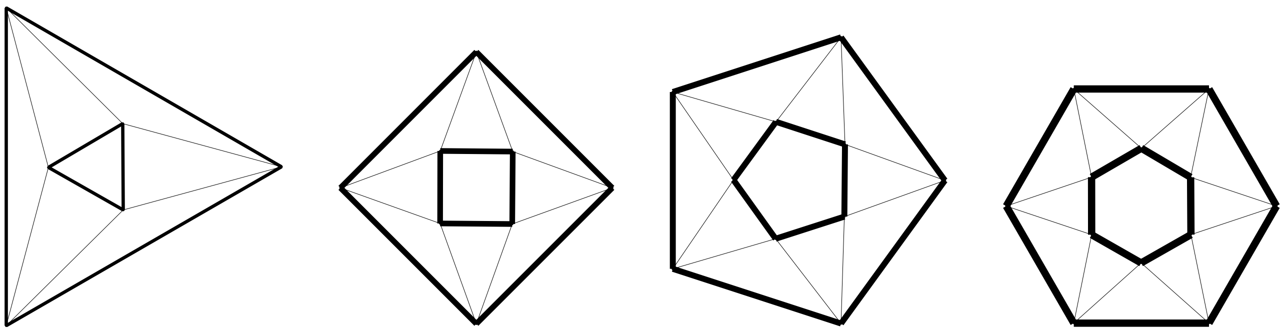


$(\{3\},4)$-polycycles

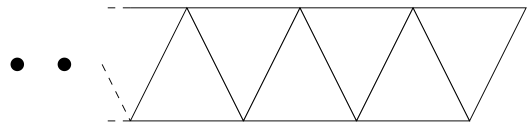
- Following infinite family (with one hole):



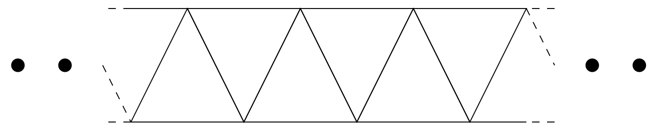
- The infinite family $APrism_n$ (with two holes)



- Following two infinite $(\{3\},4)$-polycycles:



singly infinite polycycle



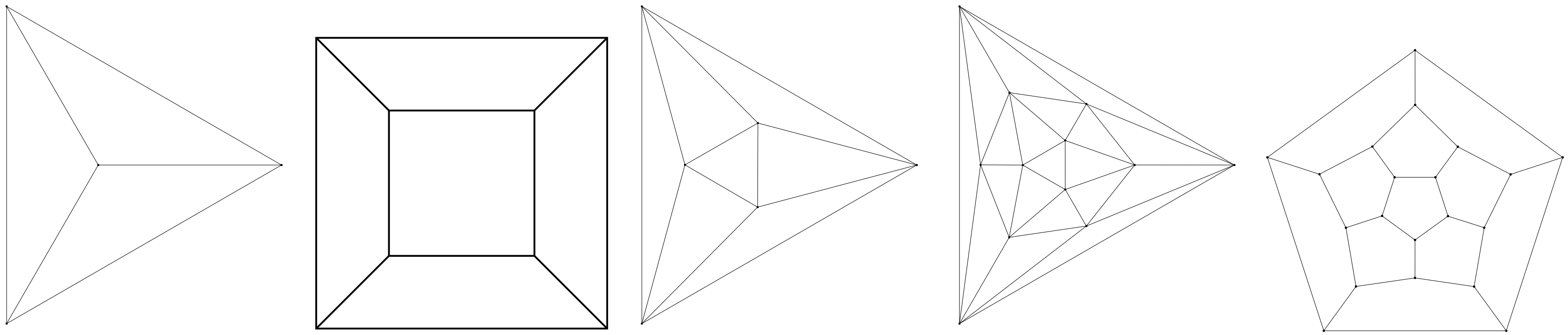
doubly infinite polycycle

Curvature conditions

- A (R, q)-polycycle is called elliptic, parabolic or hyperbolic if $\frac{1}{q} + \frac{1}{max_{i \in R} i} - \frac{1}{2}$ is positive, zero or negative, respectively.
- Elliptic cases:
  - $q = 3$ and R with $\max_{i \in R} i \leq 5$
  - $q = 4$ and R with $\max_{i \in R} i \leq 3$
  - $q = 5$ and R with $\max_{i \in R} i \leq 3$
- Parabolic cases:
  - $q = 3$ and R with $\max_{i \in R} i = 6$
  - $q = 4$ and R with $\max_{i \in R} i = 4$
  - $q = 6$ and R with $\max_{i \in R} i = 6$
- All other cases are hyperbolic.

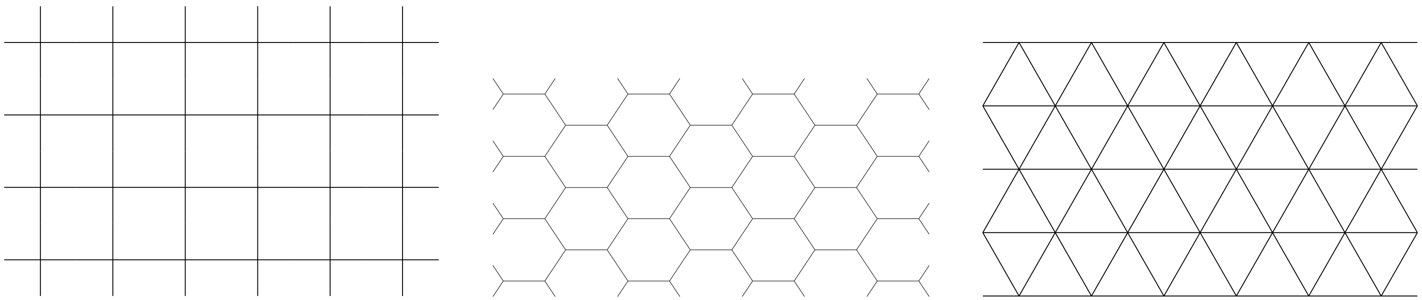
Limit case $F_2 = \emptyset$, $R = \{r\}$

- Elliptic $(\{r\}, q)$-polycycles: 5 Platonic solids



Tetra-hedron | Cube | Octa-hedron | Icosa-hedron | Dodeca-hedron

- Parabolic $(\{r\}, q)$-polycycles: 3 regular plane tilings



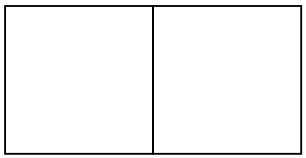
- Hyperbolic $(\{r\}, q)$-polycycles: infinity

Generalization and (r, q)-polycycles

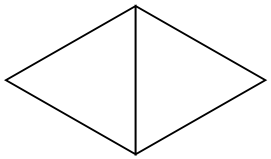
- A generalization of (R, q)-polycycle is (R, Q)-polycycles: the valency of interior vertices belong to a set Q. All the theory extends to this case.
- A (r, q)-polycycle is a $(\{r\}, q)$-polycycle with only one hole (the exterior one). Their theory has additional features:
  - There exist a canonical model of them in the form of (r^q) regular partition.
  - For any (r, q)-polycycle P, simple connectedness of P ensures the existence of a canonical map from P to (r^q).

Main examples of (r, q)-polycycles

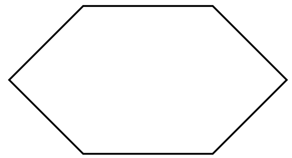
	Elliptic	Parabolic	Hyperbolic
(r, q)	$(3,3), (3,4), (4,3)$ $(5,3), (3,5)$	$(4,4)$ $(3,6), (6,3)$	all others
Exp. reg.part	α_3, β_3, γ_3, Do, Ico of sphere S^2	$(4^4), (6^3), (3^6)$ of Euclidean plane $\mathbb{R}^2$	(r^q) of hyperbolic plane $\mathbb{H}^2$



domino



diamond



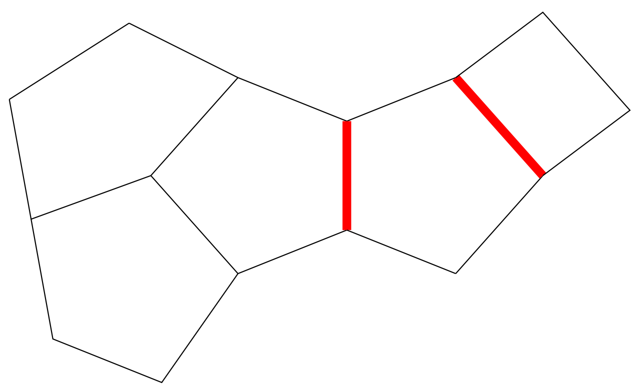
hexagon

Polyominoes: Conway, Penrose, Colomb (games, tilers of $\mathbb{R}^2$, etc.), enumeration (in Physics, Statistical Mechanics).
Polyhexes: application in Organic Chemistry.

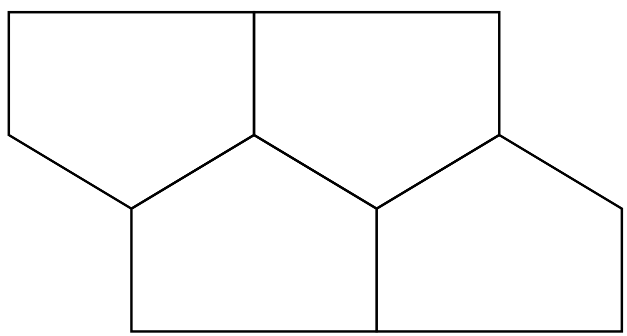
II. Decomposition into elementary polycycles

Elementary polycycles

- A bridge of a (R, q)-polycycle is an edge, which is not on a boundary and goes from a hole to a hole (possibly, the same).
- An elementary (R, q)-polycycle is one without bridges.
- Examples:



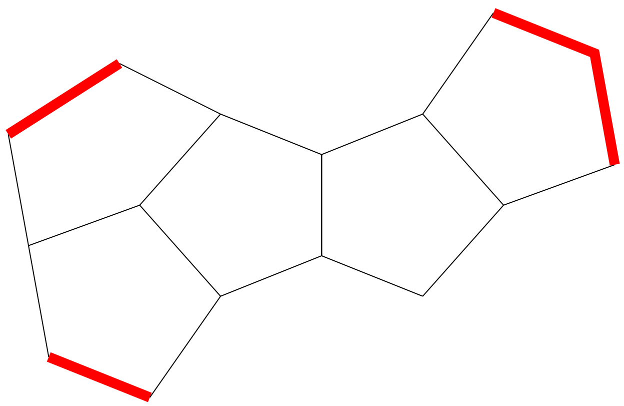
A non-elementary $(\{4, 5\}, 3)$-polycycle

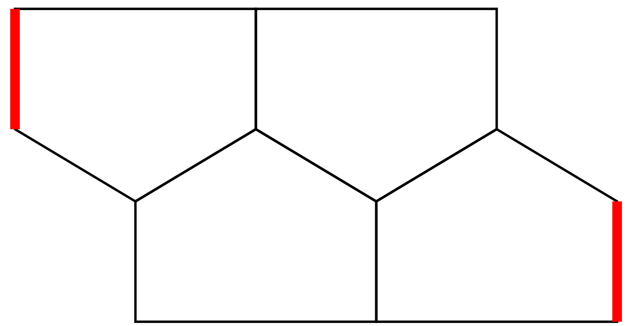


An elementary $(\{5\}, 3)$-polycycle

Open edges

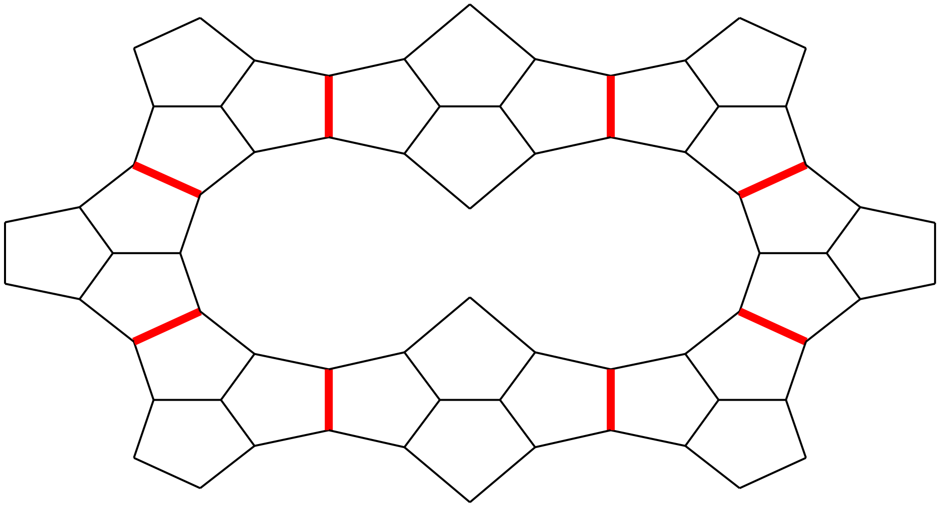
- An open edge of an (R, q)-polycycle is an edge on a boundary such that each of its end-vertices have degree less than q.
- Examples





Decomposition theorem

- Theorem: Any (R, q)-polycycle is uniquely decomposed into elementary (R, q)-polycycles along its bridges.
- In other words, any (R, q)-polycycle is obtained by gluing some elementary (R, q)-polycycles along open edges.



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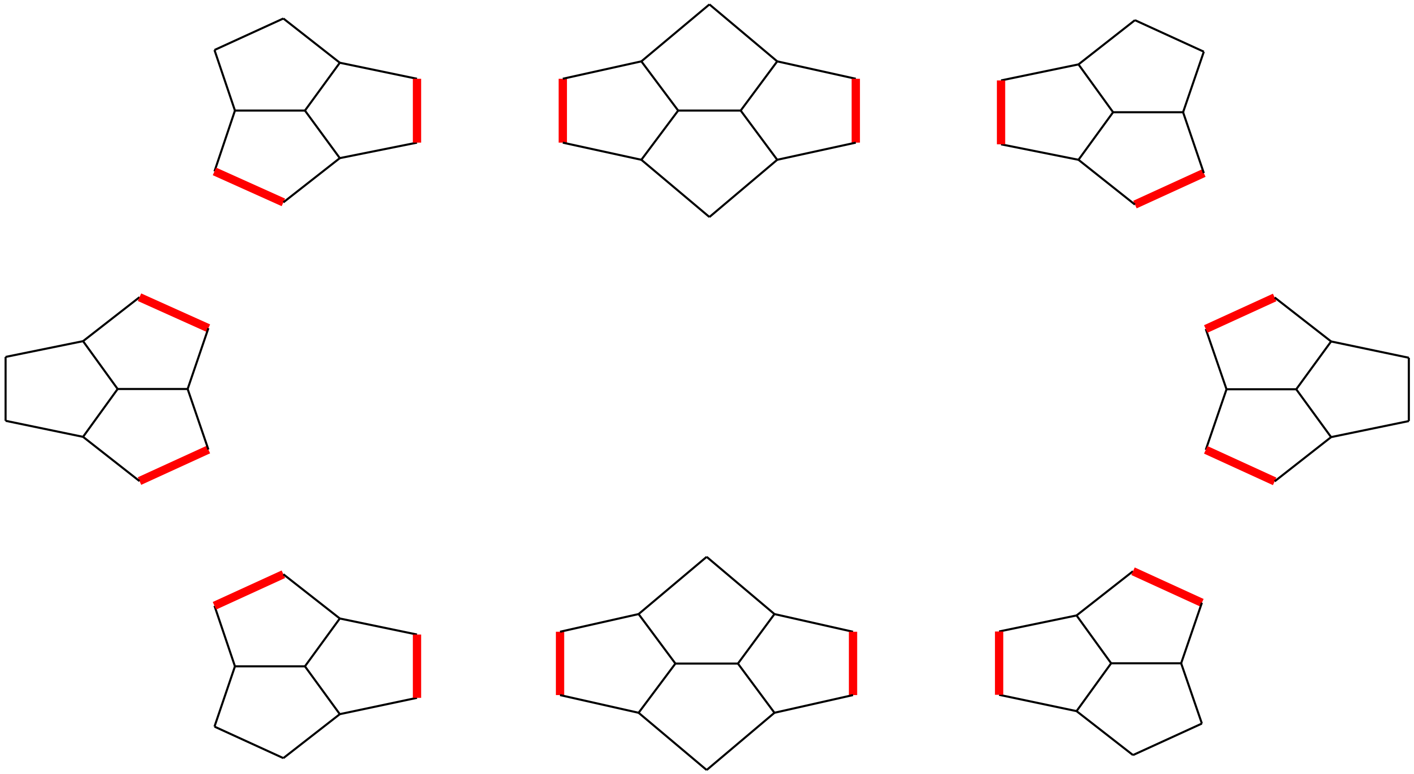
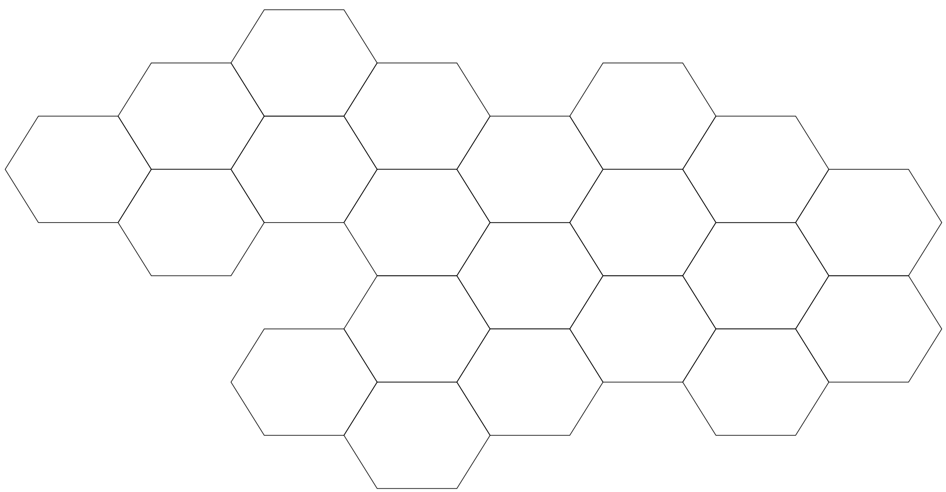


Summary

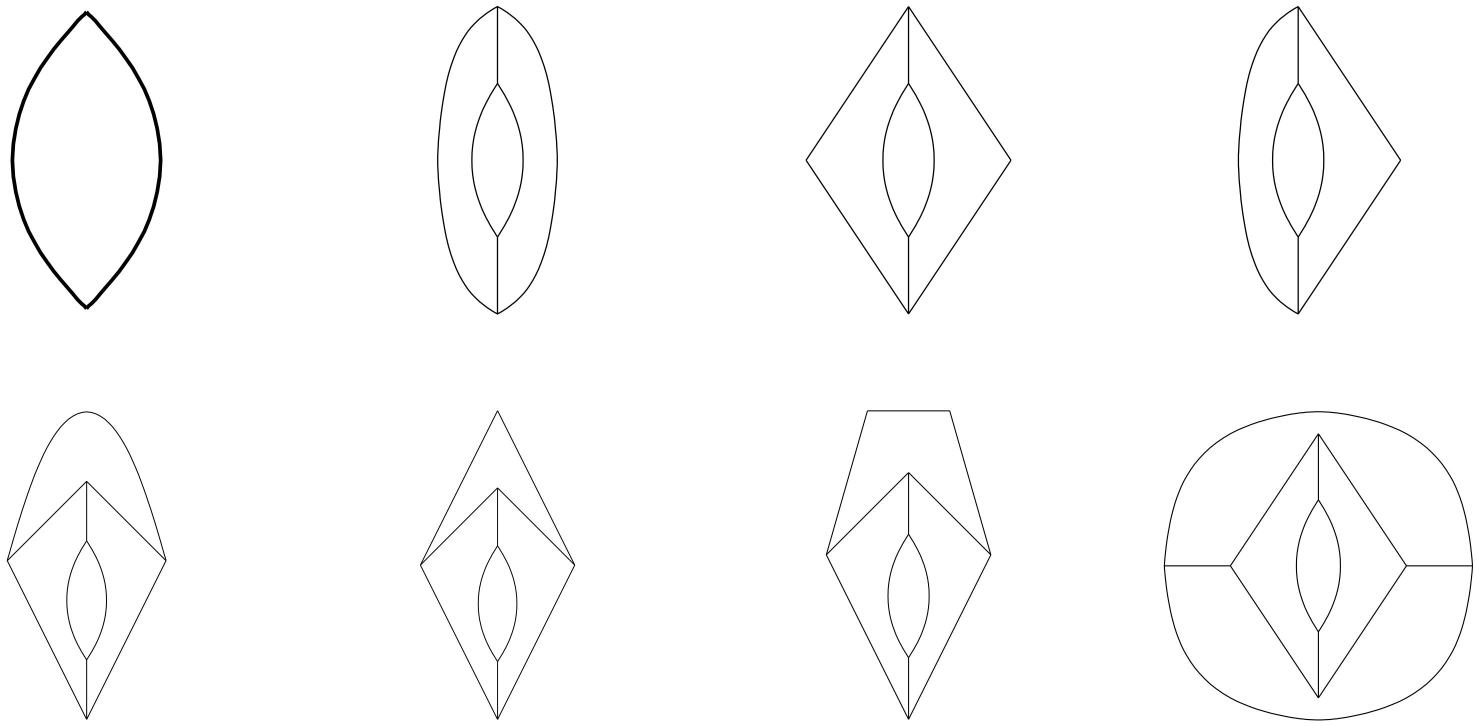
- Elementary (R, q)-polycycles provide a decomposition of (R, q)-polycycles.
- In order for this to be useful, we have to classify the elementary (R, q)-polycycles.
- For non-elliptic cases, there is no hope of classification (there is a continuum of elementary ones):



III. Classification of elementary $(\{2,3,4,5\},3)$-polycycles

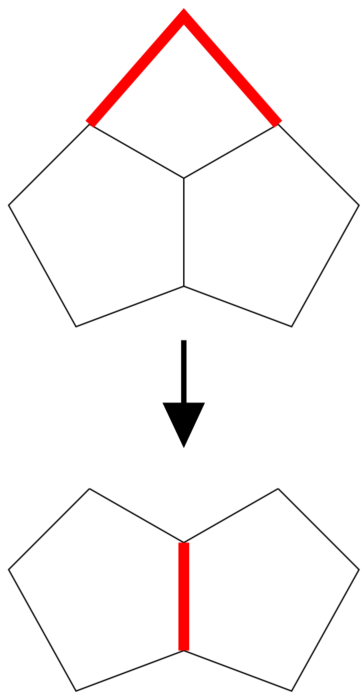
With at least one 2-gon

All elementary $(\{2,3,4,5\},3)$-polycycles, containing a 2-gon, are those eight ones:

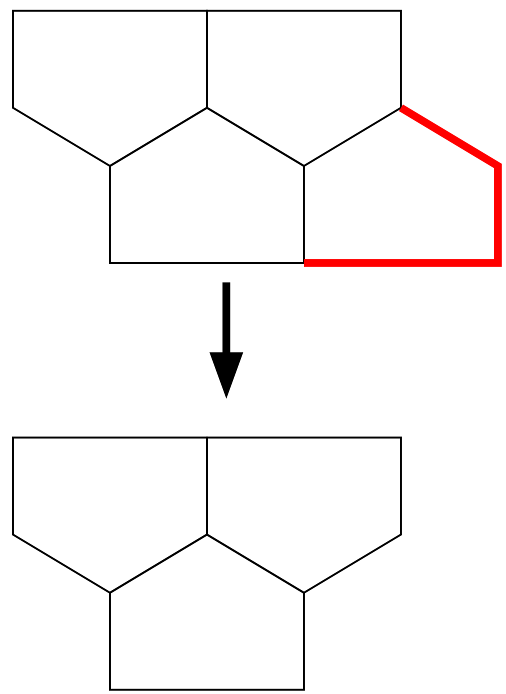


Totally elementary polycycle

- Call an elementary $(R,3)$-polycycle totally elementary if, after removing any face adjacent to a hole, one obtains a non-elementary $(R,3)$-polycycle.
- Examples:



A totally elementary polycycle

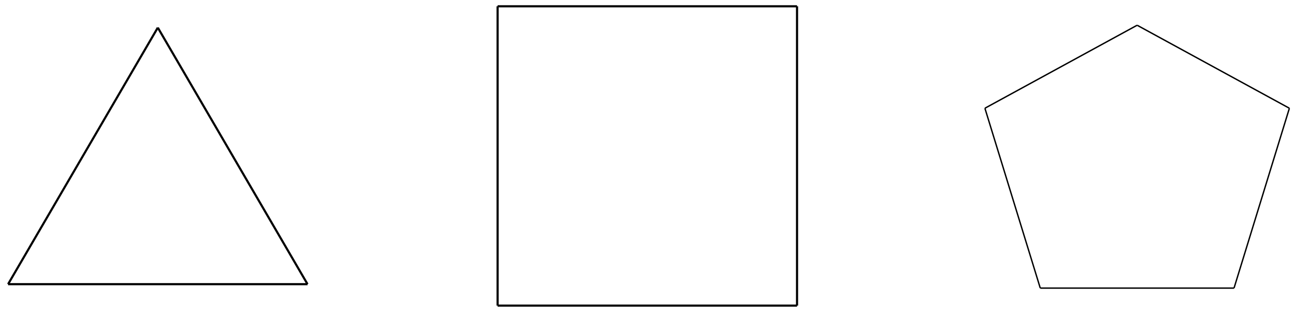


A non-totally elementary polycycle

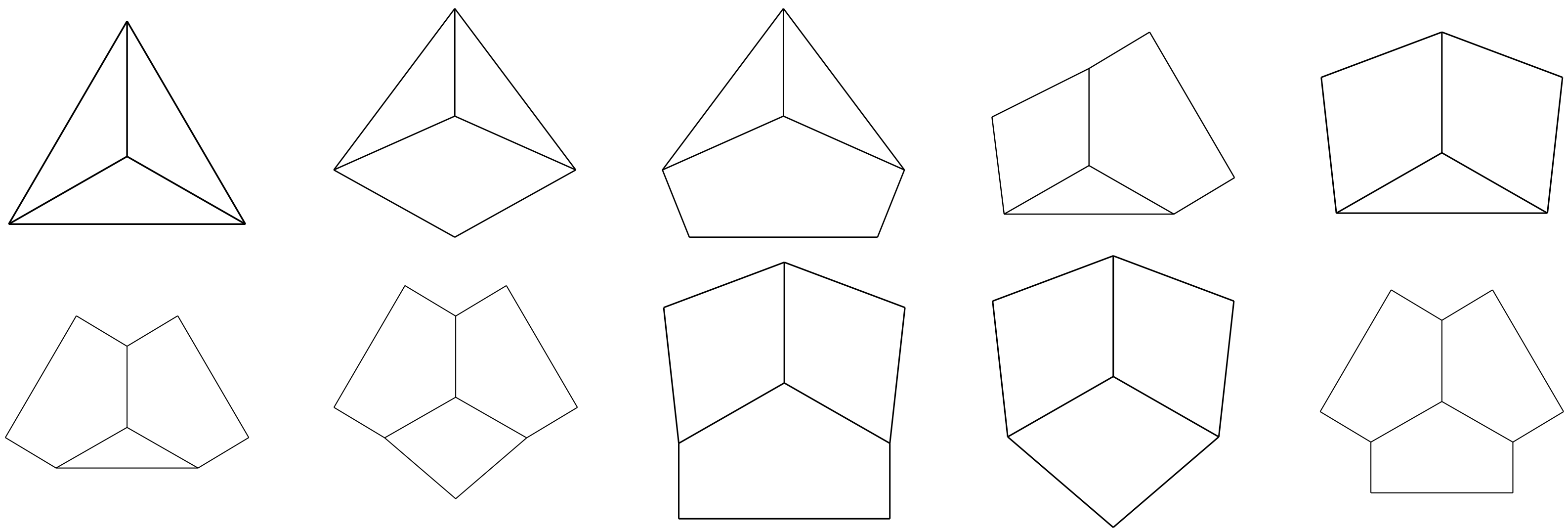
Classification of totally elementary

Any totally elementary $(\{3,4,5\},3)$-polycycle is one of:

- three isolated i-gons, $i \in \{3,4,5\}$:

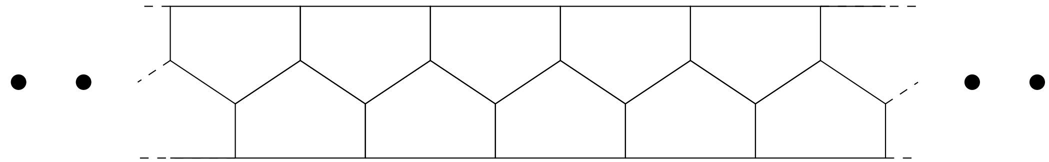


- all ten triples of i-gons, $i \in \{3,4,5\}$:

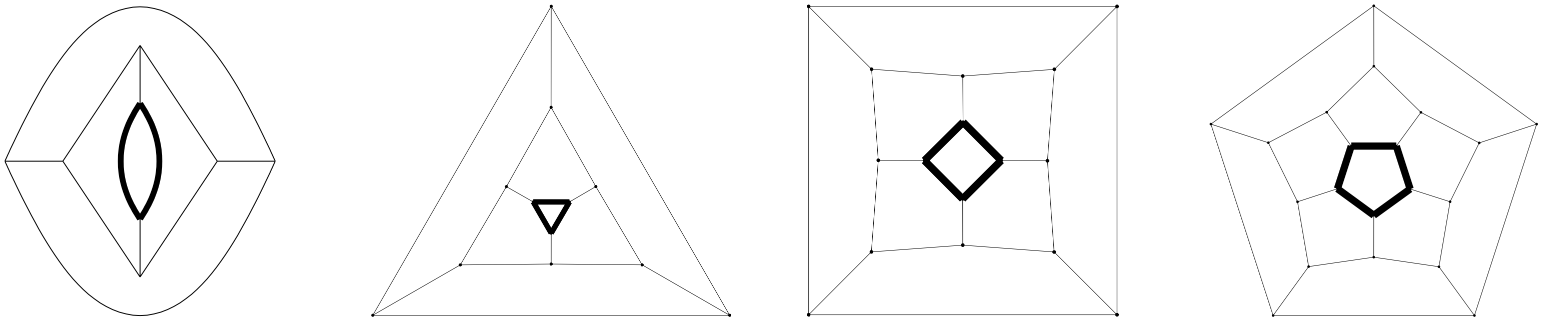


Classification of totally elementary

- the following doubly infinite $(\{5\}, 3)$-polycycle, denoted by $Barrel_\infty$:

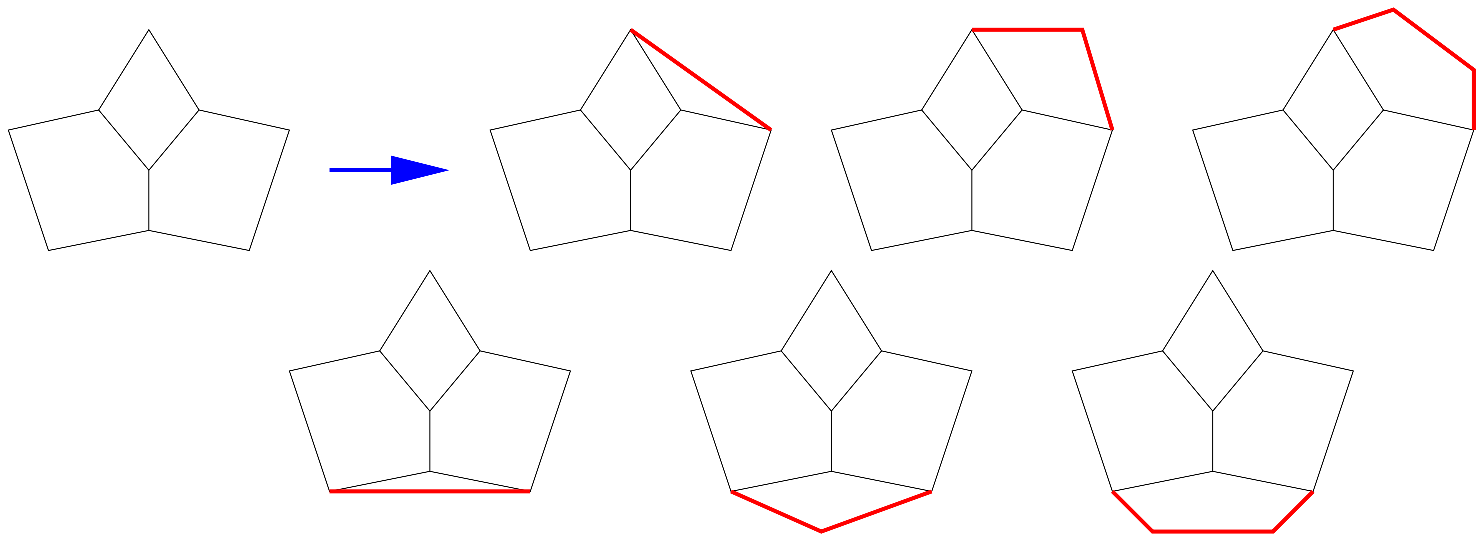


- the infinite series of $Barrel_m$, $m \geq 2$:



Classification methodology

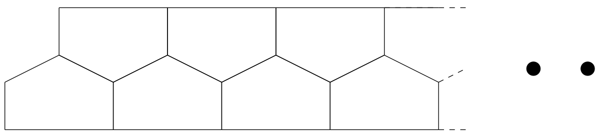
- If an elementary polycycle is not totally elementary, then it is obtained from another elementary one with one face less.
- So, from the list of elementary $(\{3,4,5\},3)$-polycycles with n faces, one gets the list of elementary $(\{3,4,5\},3)$-polycycles with $n+1$ faces.



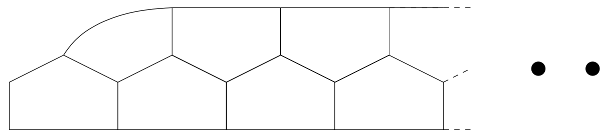
Full classification

Any elementary $(\{2,3,4,5\},3)$-polycycle is one of:

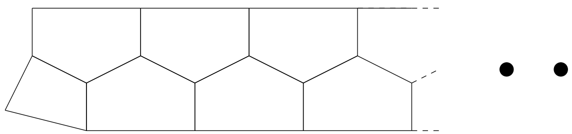
- eight such polycycles containing 2-gons
- totally elementary polycycles
- 204 sporadic polycycles with 4 to 11 proper faces
- six $(\{3,4,5\},3)$-polycycles, infinite in one direction:



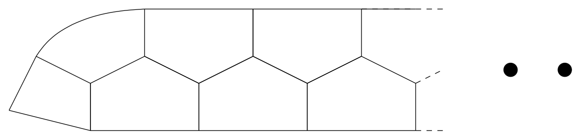
α



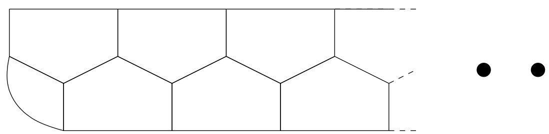
δ



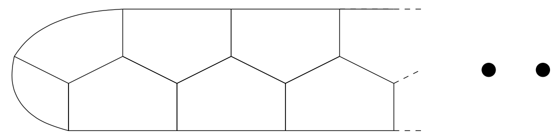
β



ε



γ

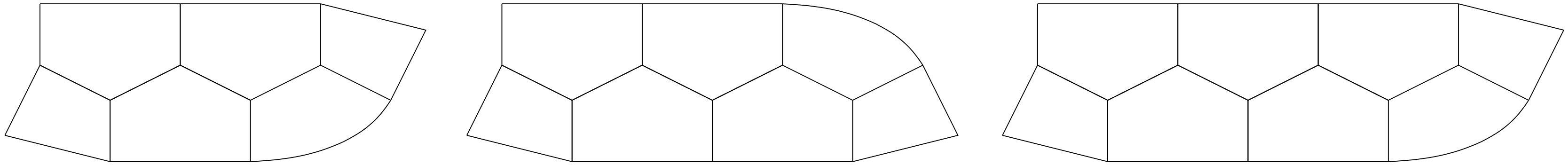


μ

Full classification

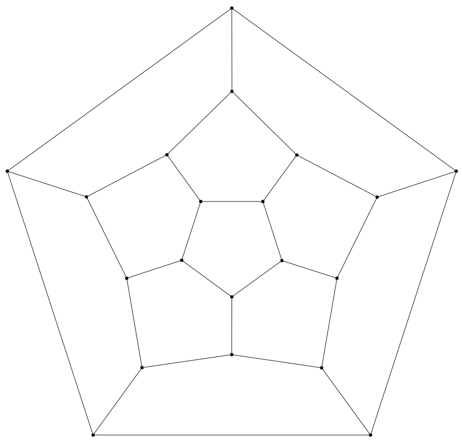
- $21 = \binom{6+1}{2}$ infinite series obtained by taking two endings of the above infinite polycycles and concatenating them.

  See below three examples in the infinite series $\beta\varepsilon$

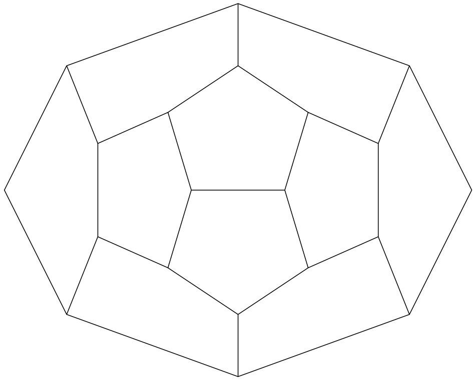


Subcase of $(\{5\},3)$-polycycles

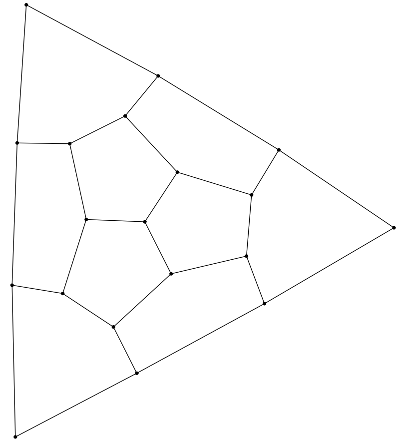
Sporadic elementary $(\{5\},3)$-polycycles:



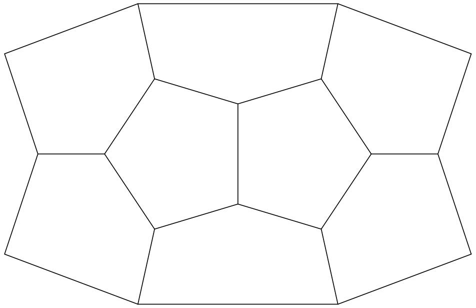
A_1



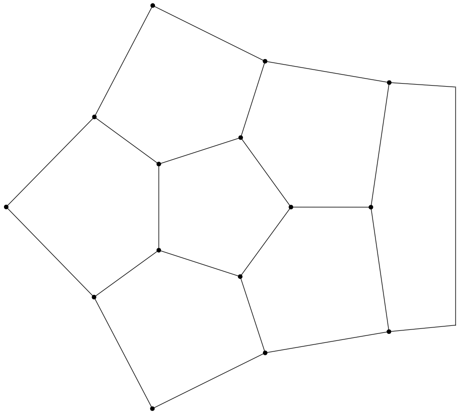
A_2



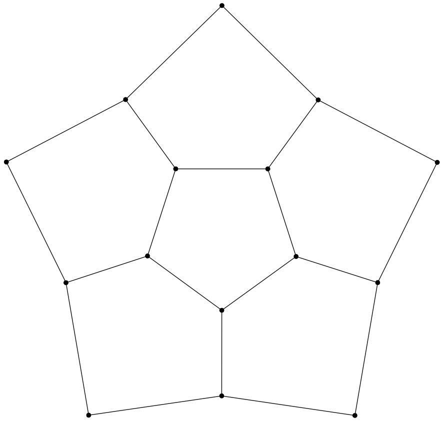
A_3



A_4

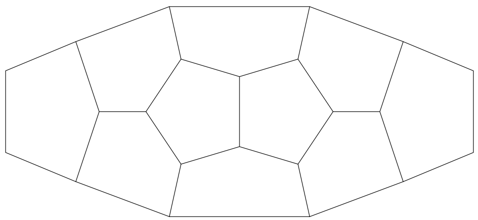


B_3

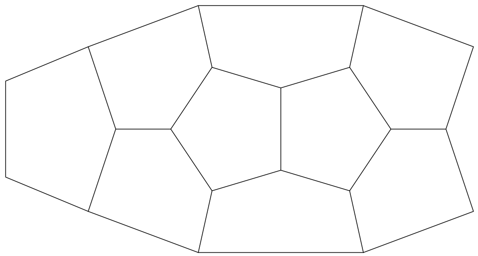


A_5

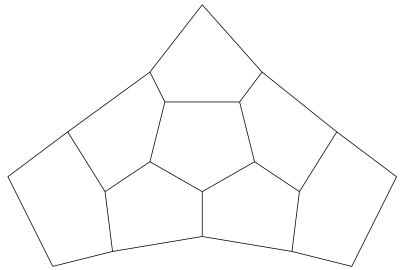
Subcase of $(\{5\}, 3)$-polycycles



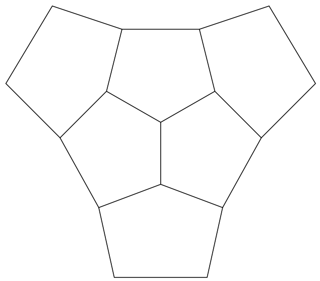
C_1



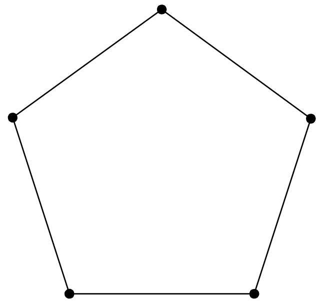
B_2



C_2



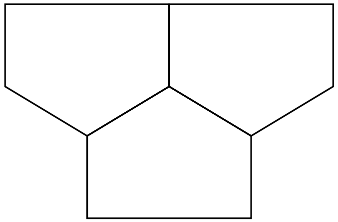
C_3



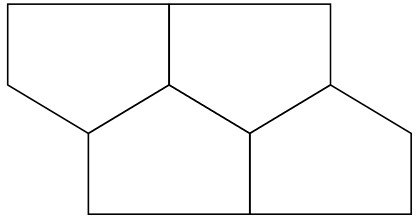
D

Subcase of $(\{5\},3)$-polycycles

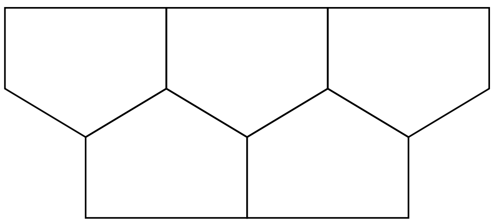
The infinite series of elementary $(\{5\},3)$-polycycles $\alpha\alpha$:



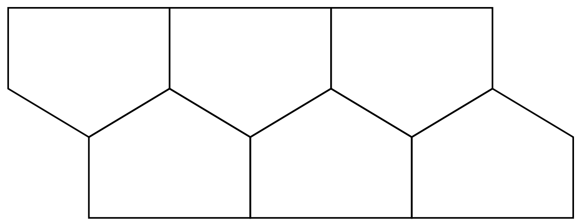
E_1



E_2



E_3

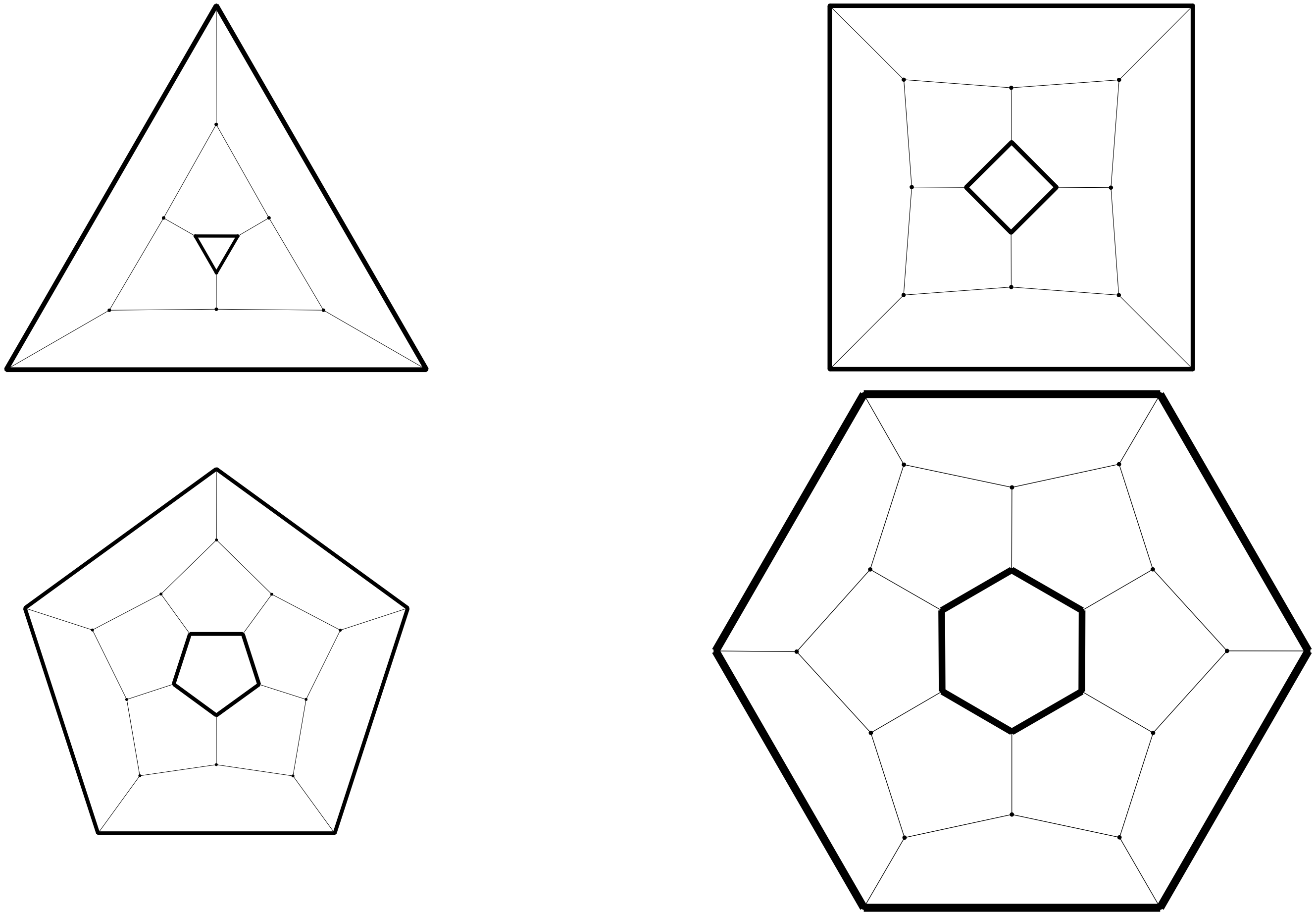


E_4

The only elementary infinite $(\{5\},3)$-polycycle are $Barrel_\infty$ and α.

Subcase of $(\{5\}, 3)$-polycycles

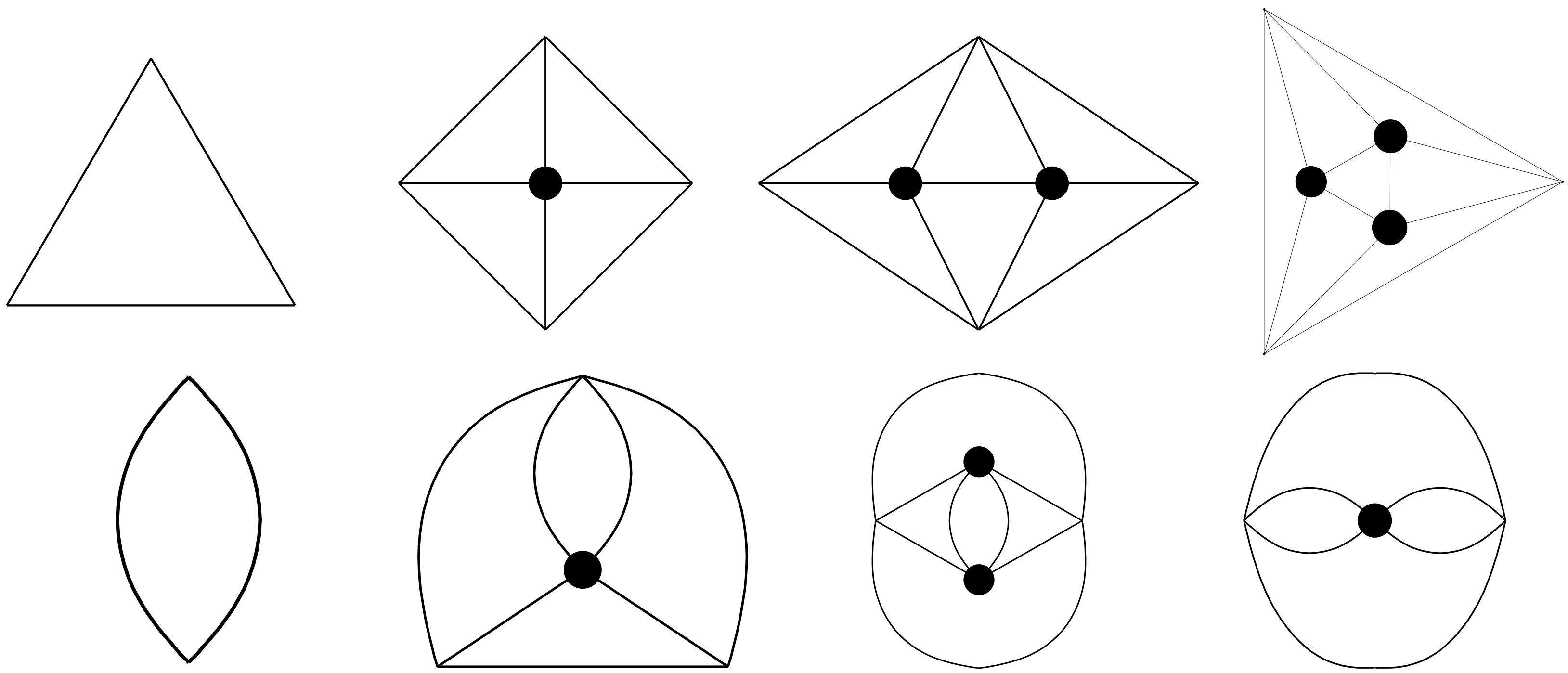
The infinite series of elementary $(\{5\}, 3)$-polycycles $Barrel_q$, $q \geq 3$:



IV. Classification of elementary $(\{2,3\},4)$-polycycles

The classification

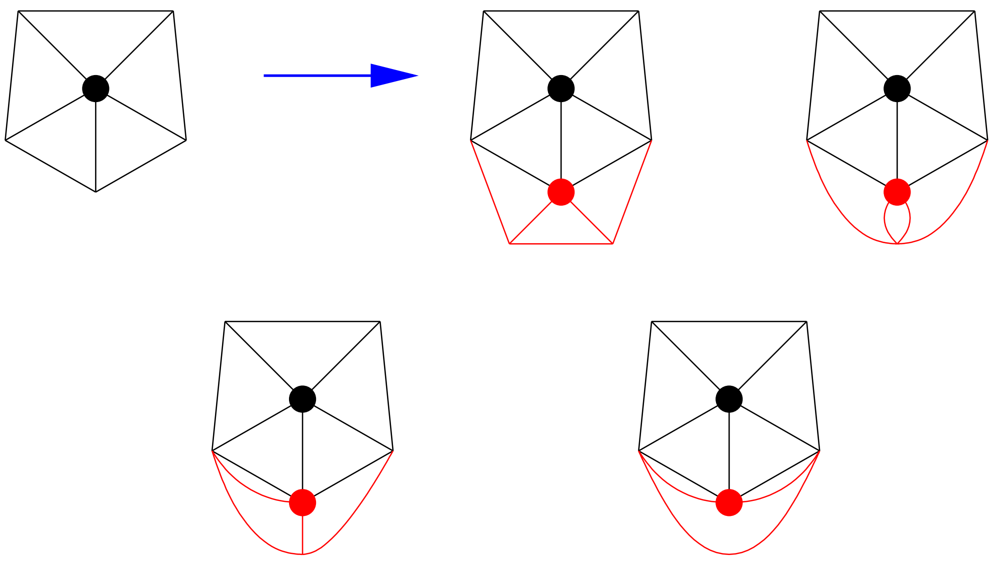
Any elementary $(\{2,3\},4)$-polycycle is one of the following eight:



V. Classification of elementary $(\{2,3\},5)$-polycycles

The technique

- Take an elementary $(\{2,3\},5)$-polycycle. If v is a vertex on the boundary, then we can consider all possible ways to make this vertex an interior vertex in an elementary $(\{2,3\},5)$-polycycle.
- From the list of elementary $(\{2,3\},5)$-polycycles with n interior vertices, one can obtain the list of elementary $(\{2,3\},5)$-polycycles with $n+1$ interior vertices.

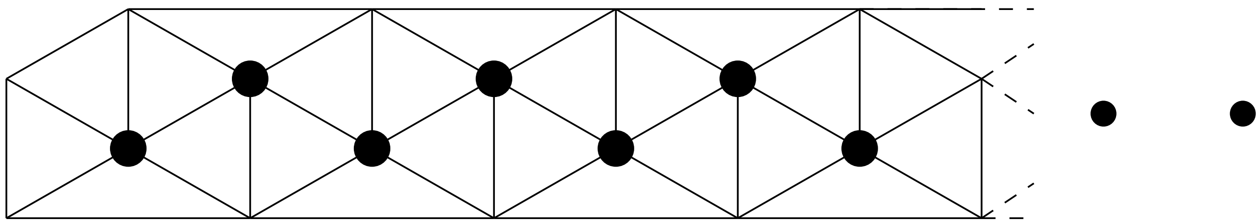


The classification

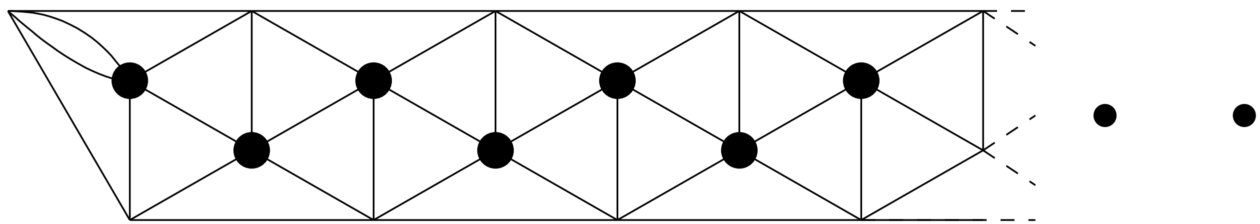
Any elementary $(\{2,3\},5)$-polycycle is one of:

- 57 sporadic $(\{2,3\},5)$-polycycles.
- three following infinite $(\{2,3\},5)$-polycycles:

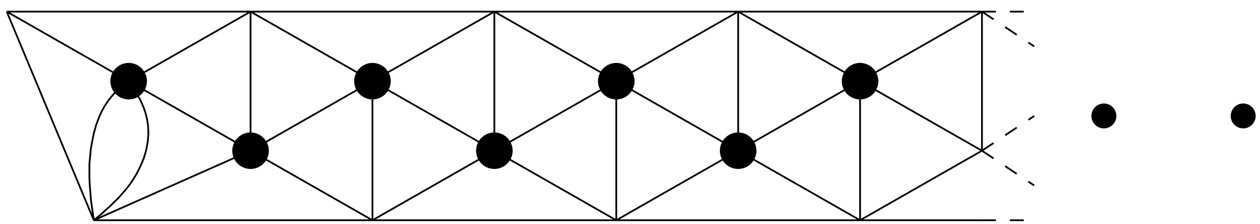
α:



β:

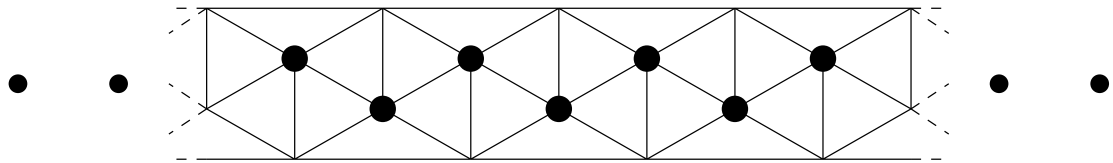


γ:

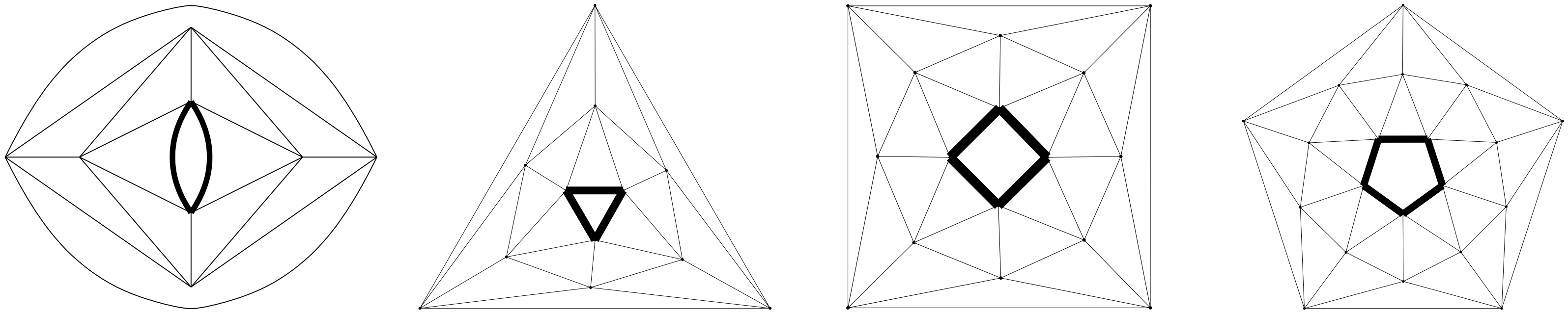


The classification

- the following 5-valent doubly infinite $(\{2,3\},5)$-polycycle, called snub ∞-antiprism:



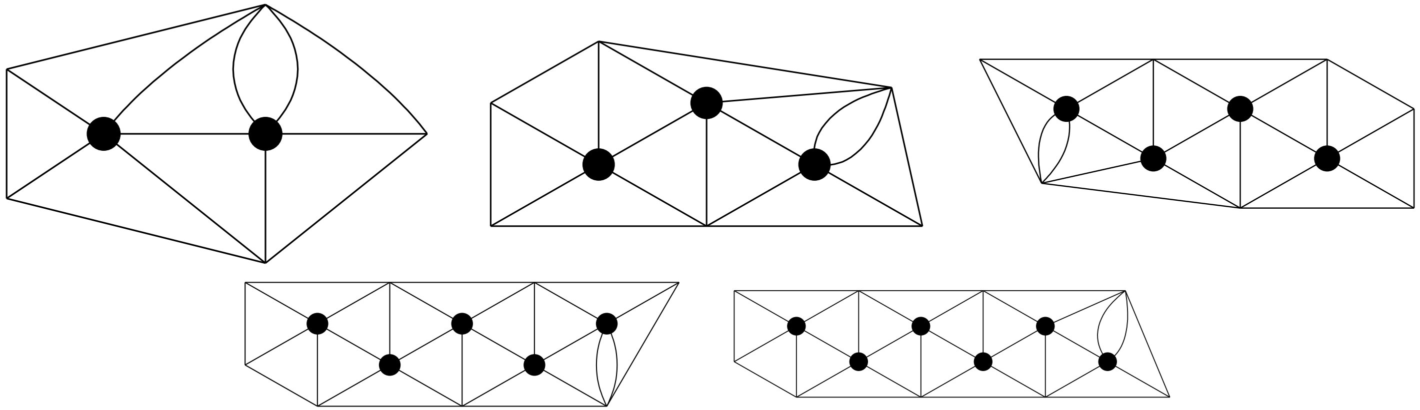
- the infinite series of snub m-antiprisms, $m \geq 2$ (two m-gonal holes):



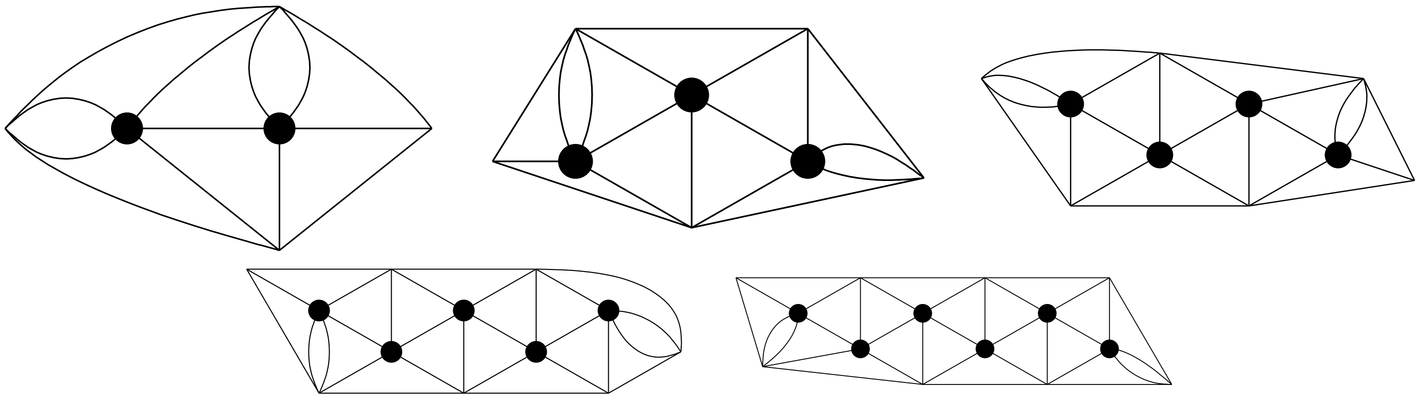
- six infinite series of $(\{2,3\},5)$-polycycles with one hole (they are obtained by concatenating endings α, β, γ)

The classification

Infinite series $\alpha\gamma$ of elementary $(\{2,3\},5)$-polycycles:

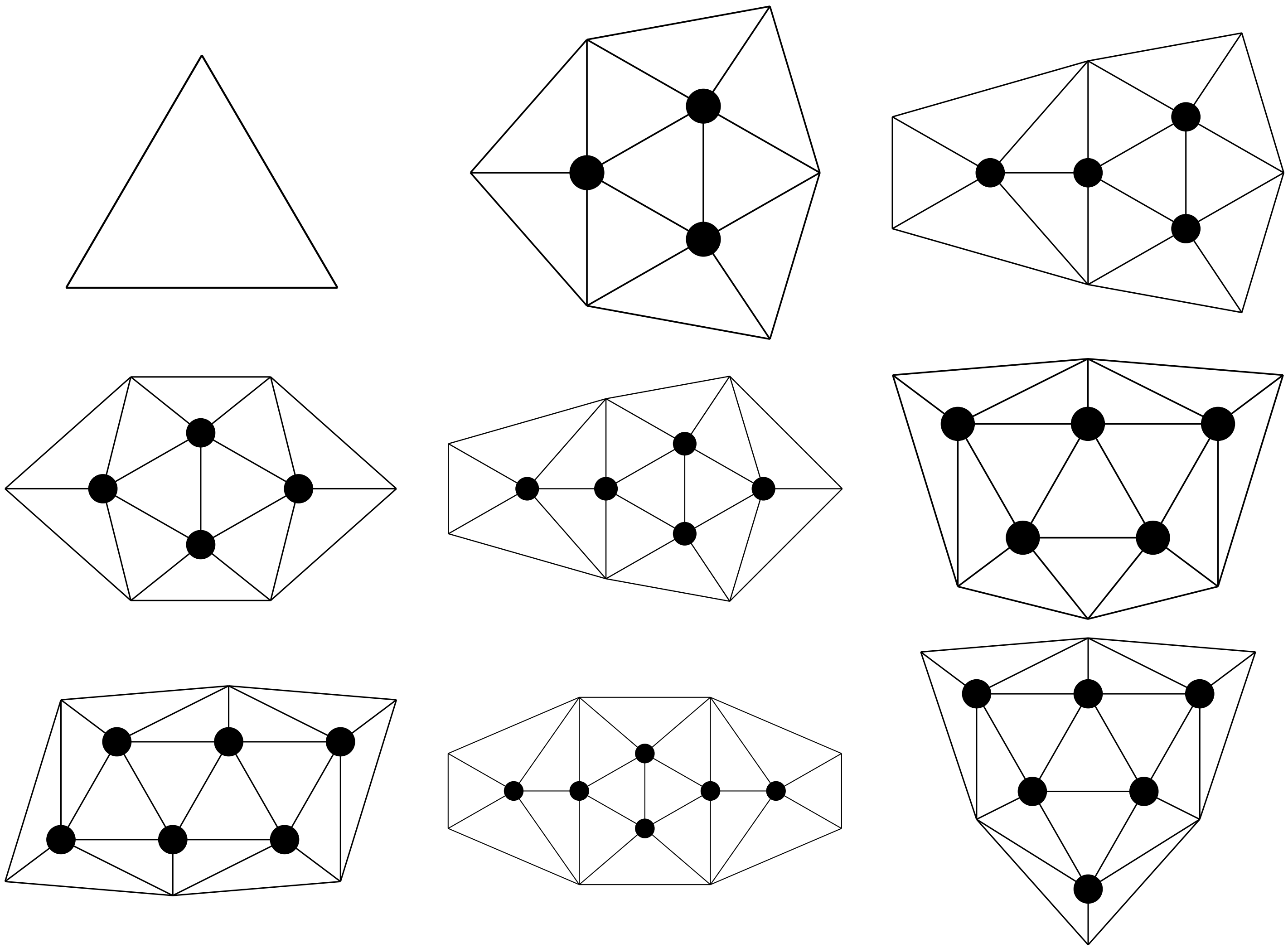


Infinite series $\beta\gamma$ of elementary $(\{2,3\},5)$-polycycles:

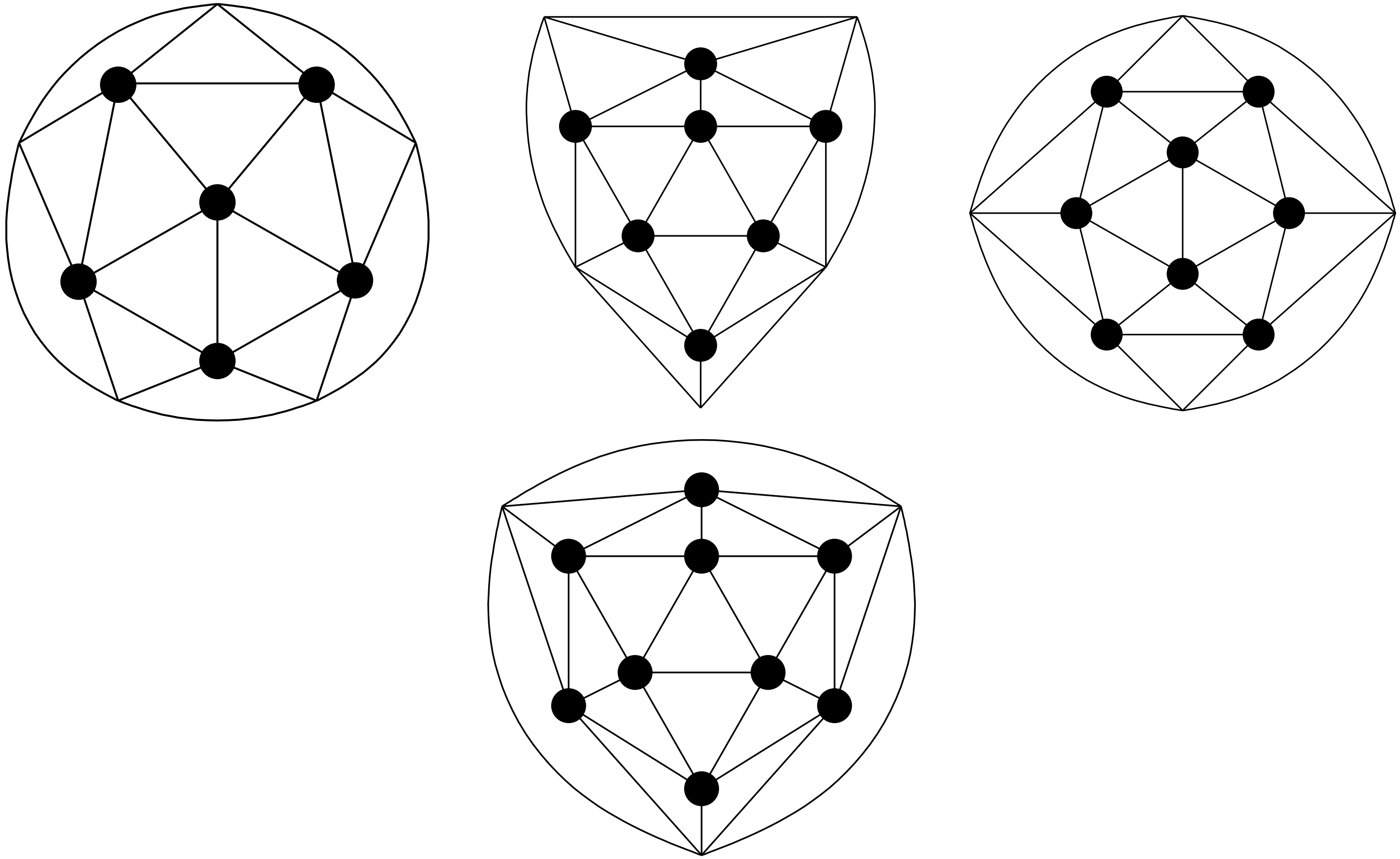


Subcase of $(\{3\}, 5)$-polycycles

- Sporadic elementary $(\{3\}, 5)$-polycycles:

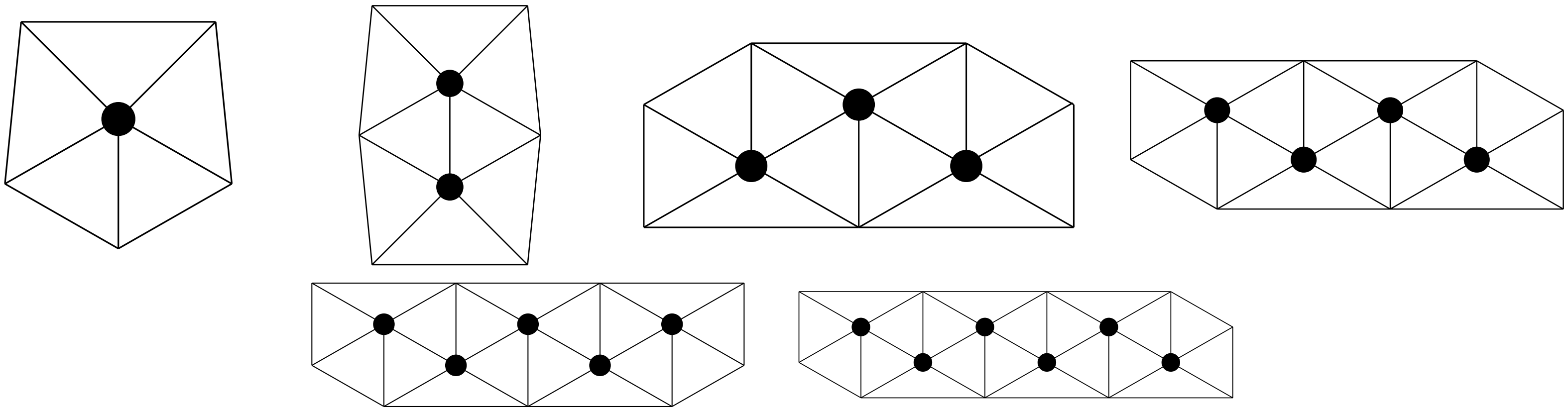


Subcase of $(\{3\}, 5)$-polycycles



Subcase of $(\{3\}, 5)$-polycycles

- The infinite series of elementary $(\{3\}, 5)$-polycycles $\alpha\alpha$:



- The only elementary infinite $(\{3\}, 5)$-polycycles are α and snub ∞-antiprism.
- The infinite series of elementary $(\{3\}, 5)$-polycycles snub m-antiprisms, $m \geq 2$:

VI. Application to extremal polycycles

Definition

- Given a finite (r, q)-polycycle P, denote by
  - $n_{int}(P)$ the number of interior vertices
  - and $f_1(P)$ the number of faces in F_1.
- Fix $x \in \mathbb{N}$. An (r, q)-polycycle with $f_1(P) = x$ is called extremal if it has maximal $n_{int}(P)$ among all (r, q)-polycycles with $f_1(P) = x$.
- Problem: to find $N_{r,q}(x)$, the maximal number of vertices.
- Fact: For fixed $r, q, f_1(P) = x$ extremal polycycle has also maximal $n_{int}(P)$, $e_{int}(P)$ (interior faces) and minimal n, l, $Perim = n_{ext}$
- For (r, q)=$(3, 3)$, $(4, 3)$, $(3, 4)$, the question is trivial. 8 authors, 1997: found $N_{5,3}(x)$ for $x < 12$ (unique, partial subgraph of Dodecahedron).

Use of elementary polycycles

- If a (r,q)-polycycle P is decomposed into elementary (r,q)-polycycles $(EP_i)_{i\in I}$ appearing x_i times, then one has:

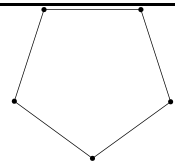
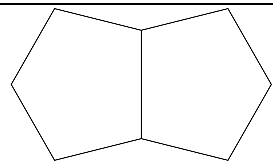
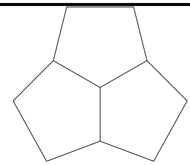
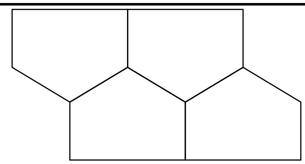
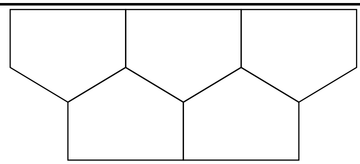
$$\left\{\begin{array}{rcl} n_{int}(P) & = & \sum_{i\in I} x_i n_{int}(EP_i) \\ f_1(P) & = & \sum_{i\in I} x_i f_1(EP_i) \end{array}\right.$$

- If one solves the Linear Programming problem

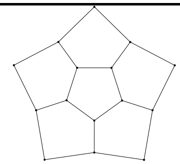
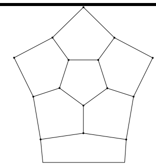
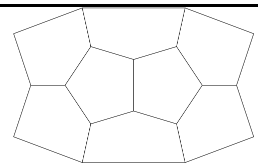
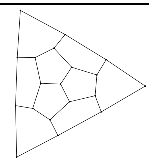
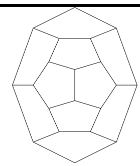
$$\begin{array}{rl} \text{maximize} & \sum_{i\in I} x_i n_{int}(EP_i) \\ \text{with} & x = \sum_{i\in I} x_i f_1(EP_i) \\ & \text{and } \ x_i \in \mathbb{N} \end{array}$$

and if $(x_i)_{i\in I}$ realizing the maximum can be realized as (r,q)-polycycle, then $N_{r,q}(x)$ can be found.

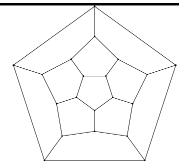
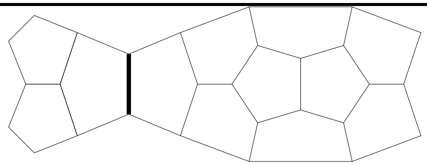
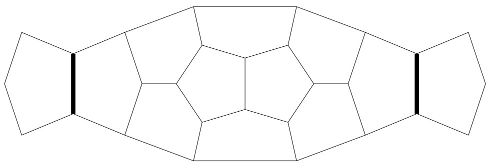
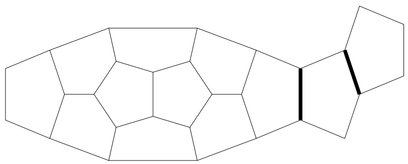
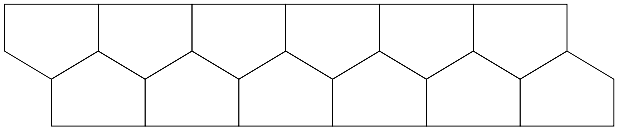
Small extremal $(5,3)$-polycycles

x	$N_{5,3}(x)$	extremal	components
1	0		D
2	0		D, D
3	1		E_1
4	2		E_2
5	3		E_3

Small extremal $(5,3)$-polycycles

x	$N_{5,3}(x)$	extremal	components
6	5		A_5
7	6		B_3
8	8		A_4
9	10		A_3
10	12		A_2

Small extremal $(5,3)$-polycycles

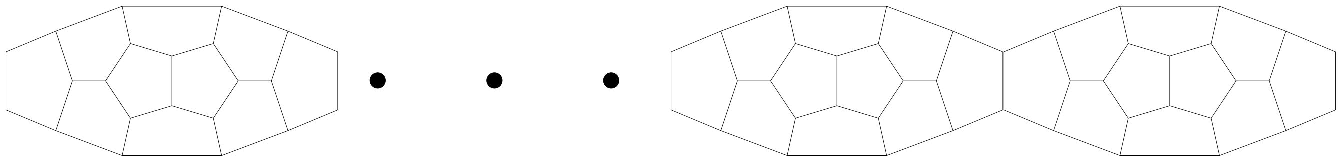
x	$N_{5,3}(x)$	extremal	components
11	15		A_1
12	10		E_1, B_2
			D, C_1, D
			C_1, D, D
			E_{10}

Extremal $(5,3)$-polycycles

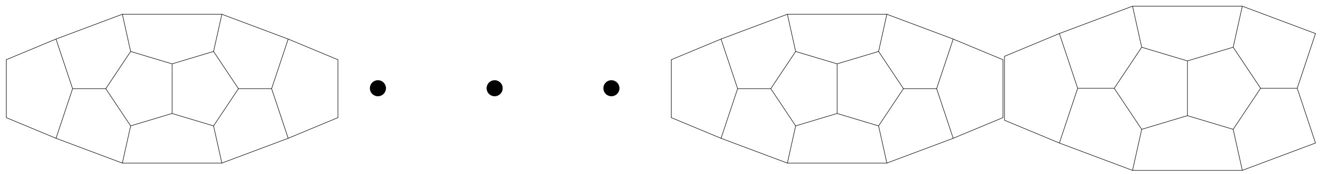
- Theorem: For any $x \geq 12$, one has

$$N_{5,3}(x) = \begin{cases} x & if \quad x \equiv 0,8,9 \pmod{10}, \\ x-1 & if \quad x \equiv 6,7 \pmod{10}, \\ x-2 & if \quad x \equiv 1,2,3,4,5 \pmod{10}. \end{cases}$$

- Extremal polycycle realizing the extremum:
  - If $x \equiv 0 \pmod{10}$ (unique):

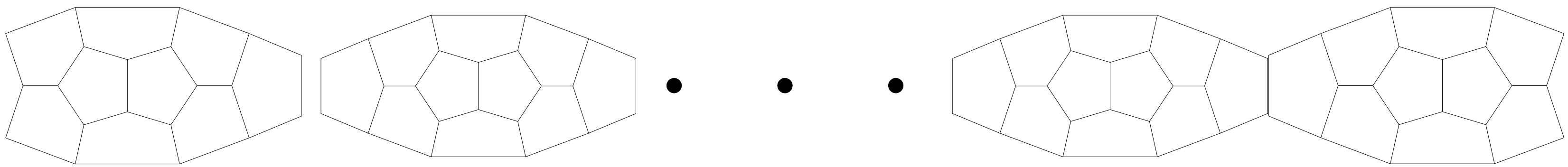


  - If $x \equiv 9 \pmod{10}$ (unique):

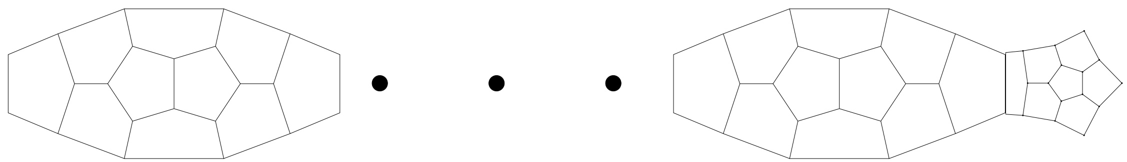


Extremal $(5,3)$-polycycles

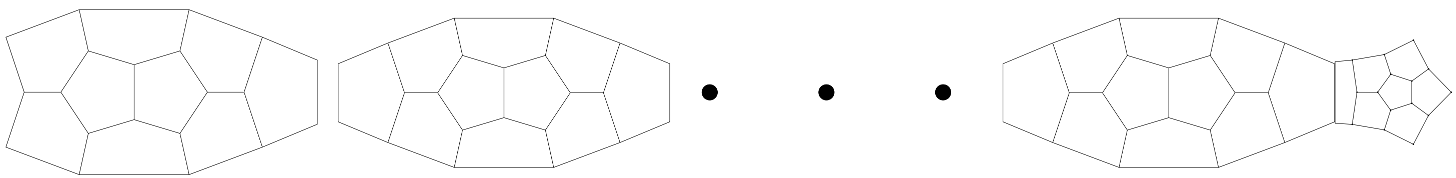
- Extremal polycycle realizing the extremum:
  - If $x \equiv 8 \pmod{10}$ (unique):



  - If $x \equiv 7 \pmod{10}$ (non-unique):



  - If $x \equiv 6 \pmod{10}$ (non-unique):



  - Otherwise (non-unique): E_n

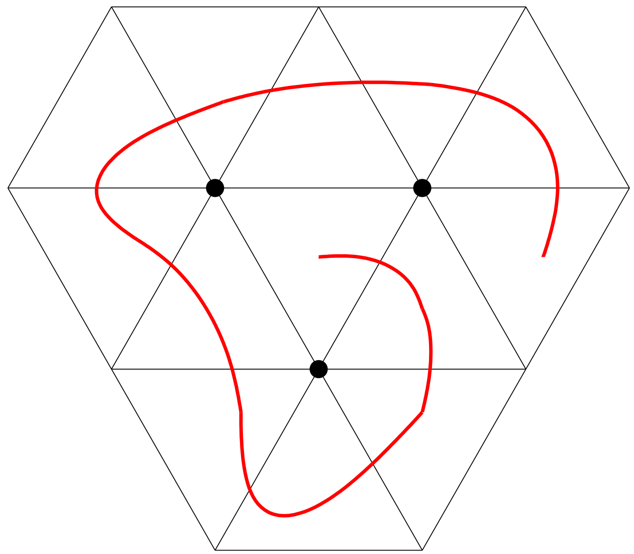
Extremal $(3,5)$-polycycles

Theorem

- $N_{3,5}(x) = \lfloor \frac{x}{3} \rfloor + 1$ for $x \equiv 14, 16, 17 \pmod{18}$,
- $N_{3,5}(x) = \lfloor \frac{x}{3} \rfloor - 1$ for $x \equiv 3, 4, 6, 7, 9, 11 \pmod{18}$, and
- $N_{3,5}(x) = \lfloor \frac{x}{3} \rfloor$, otherwise,
- but with 5 exceptions: above value plus 1 for $x = 11, 15, 17$ and $N_{3,5}(x) = x - 10$ for $16 \leq x \leq 19$.

Non-elliptic case

- For parabolic (r, q)-polycycles (i.e. (r, q)=$(4, 4)$, $(6, 3)$ or $(3, 6)$) the method of elementary polycycles fails (since there is no classification) but “extremal animals” of Harary-Harborth 1976 (proper ones, growing as a spiral) are extremal:

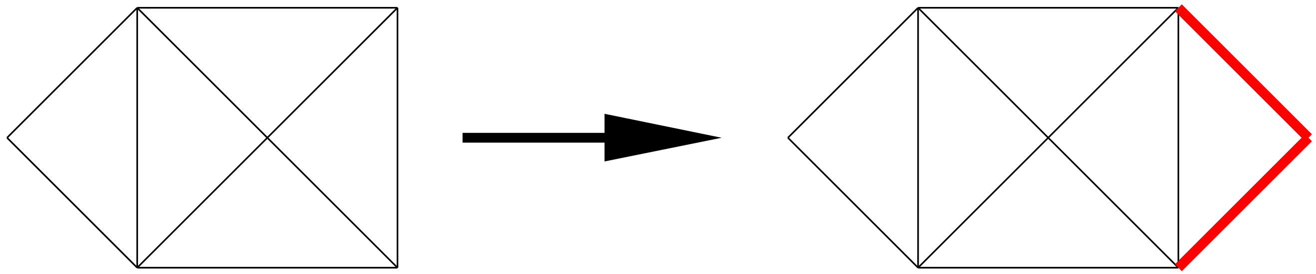


- Hyperbolic cases are very difficult.

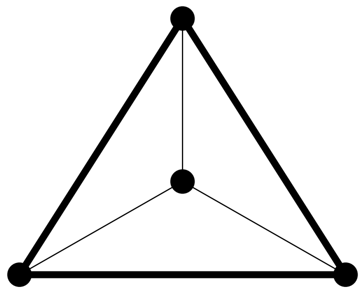
VII. Application to non-extendible polycycles

Definition

- A (r, q)-polycycle is called non-extendible if it is no proper subgraph of another (r, q)-polycycle. Examples:



Extendible $(3, 4)$-polycycle

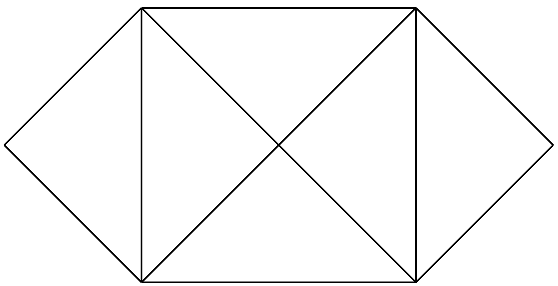


Non-extendible $(3, 3)$-polycycle

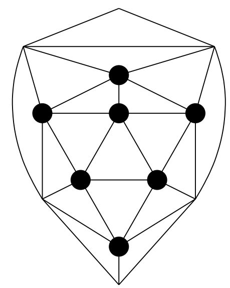
Classification

Theorem: All non-extendible (r, q)-polycycles are:

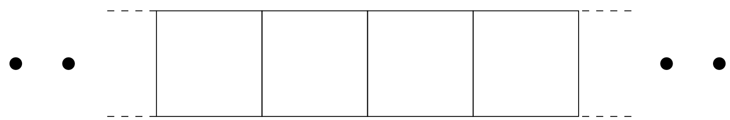
- Regular partitions (r^q)
- Four following examples:



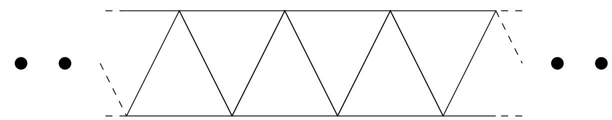
$(3, 4)$-polycycle



$(3, 5)$-polycycle



$(4, 3)$-polycycle

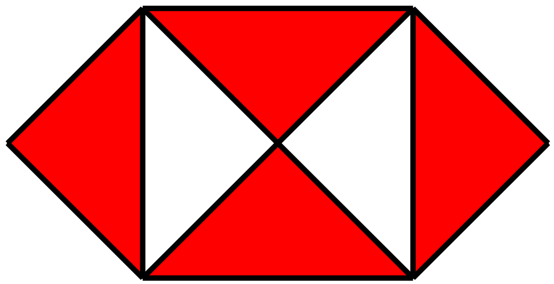


$(3, 4)$-polycycle

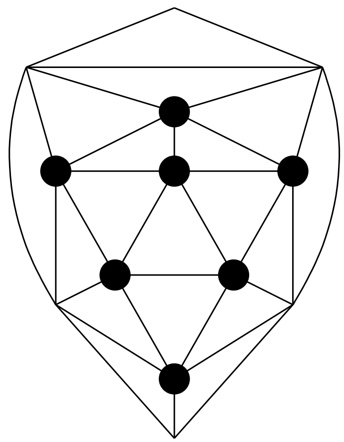
- For any $(r, q) \neq (3, 3)$, $(3, 4)$, $(4, 3)$ a continuum of infinite ones.

All finite non-extendible polycycles

So, the number of finite non-extendible (r, q)-polycycles is 7: five Platonic polyhedra and vertex-splits of two of them:



vertex-split Octahedron: from 1983, logo of HSBC, Hongkong and Shanghai Banking Corporation Ltd

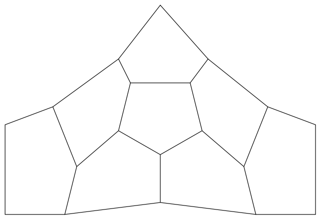


vertex-split Icosahedron: also looks OK

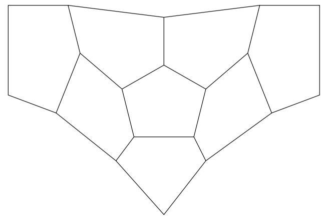
Above *Hexagon* was developed from bank's 19th century house flag: white rectangle divided diagonally to produce a red hourglass shape. This flag was derived from Scottish flag: *saltire* or *crux decussata* (heraldic symbol in the form of diagonal cross; Saint Andrew was crucified upon). 13th-century tradition states that the cross was X-shaped at

Infinite non-extendible polycycles

- Take the two elementary $(5,3)$-polycycles and



C_2



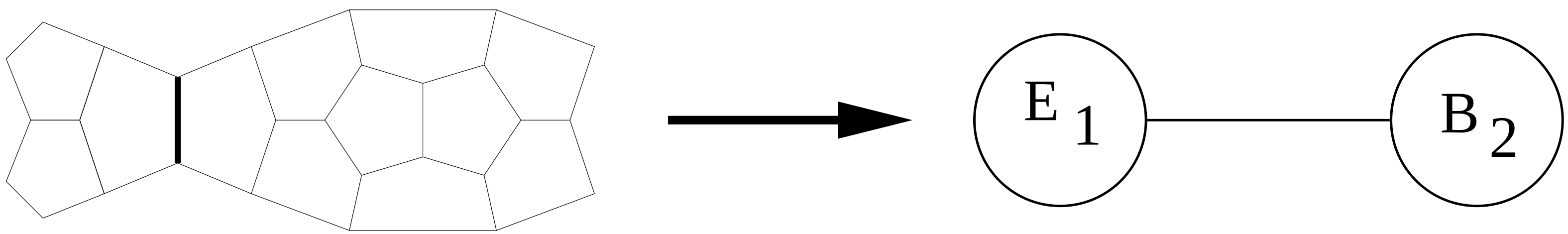
C_2'

form infinite word $\dots u_{-1}u_0u_1\dots$ with u_i being C_2 or C_2'. This gives a continuum of non-extendible $(5,3)$-polycycles.

- Similarly, one has a continuum of $(3,5)$-polycycles.
- For non-elliptic (r,q), one takes the infinite tiling (r^q), removes an infinity of r-gonal faces sharing no edges and takes the universal cover of this (r,q)-polycycle.

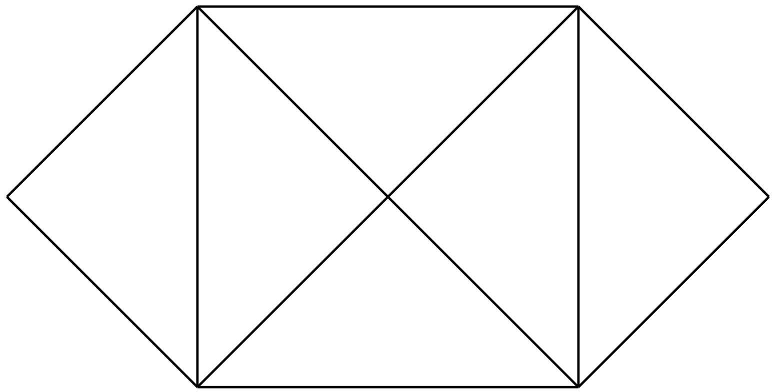
Finite non-extendible polycycles

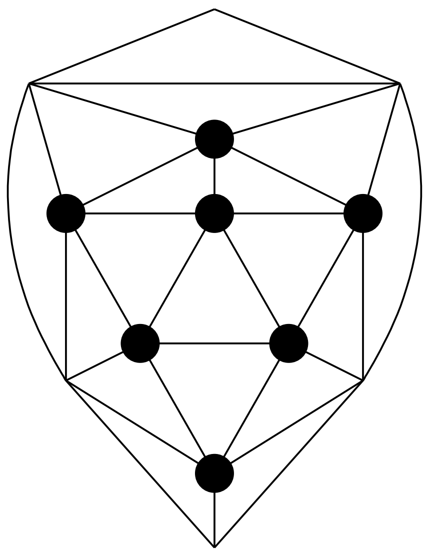
- Main lemma: all finite non-extendible (r, q)-polycycles are elliptic, i.e. $\frac{1}{q} + \frac{1}{r} > \frac{1}{2}$
- So, we can use decomposition of non-extendible (r, q)-polycycles into elementary (r, q)-polycycles and the classification of them.
- Given an (r, q)-polycycle P, the graph of its elementary components is denoted by $el(P)$; its vertices are its elementary (r, q)-polycycles with two elementary (r, q)-polycycles adjacent if they share an edge:



Finite non-extendible polycycles

- A finite $(\{r\}, q)$-polycycle P is a non-extendible (r, q)-polycycle if and only if $el(P)$ is a tree.
- Every tree is either an isolated vertex, or contains at least one vertex of degree 1.
- One checks on this vertex that there is only two possibilities:





VIII. 2-embeddable (r, q)-polycycles

2-embedding

- The Hamming distance on $\{0,1\}^n$ is defined by

$$d(x,y) = \#\{1 \leq i \leq n \mid x_i \neq y_i\}$$

- Given a connected graph G, denote by d_G the shortest path distance between vertices of G
- A graph G is said to be 2-embeddable if, for some n, there exists a mapping

$$\begin{array}{rcl} \psi : V(G) & \longrightarrow & \{0,1\}^S \\ v & \longmapsto & \psi(v) \end{array}$$

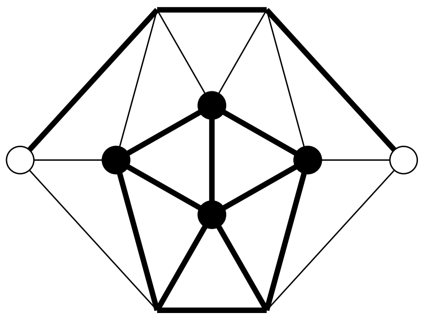
such that, for all vertices v, v' of G, one has $d(\psi(v), \psi(v')) = 2d_G(v, v')$

Alternating zones

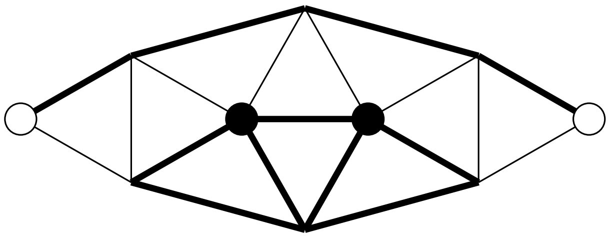
- In a plane graph G, an alternating zone, is a sequence of edges e_i such that e_i and e_{i+1} belong to a same face F_i and it holds:
  - If $|F_i|$ is even, e_i and e_{i+1} in opposition
  - If $|F_i|$ is odd, e_i and e_{i+1} are opposed. There are two possible choices for e_{i+1} given e_i and they are required to alternate.
- A subgraph H of G is called convex if, for any two vertices v, v' of H, all shortest paths between v and v' are included in H.
- If Z is a not self-intersecting alternating zone, then $G - Z$ consists of two graphs G_i. If both G_i are convex, then we say that Z *defines convex cut*.

Examples

Two $(3,5)$-polycycles with an non-convex alternating zone:

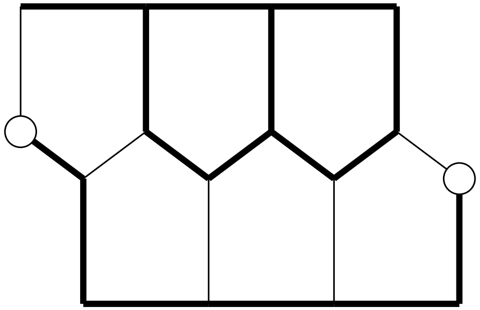


c_3

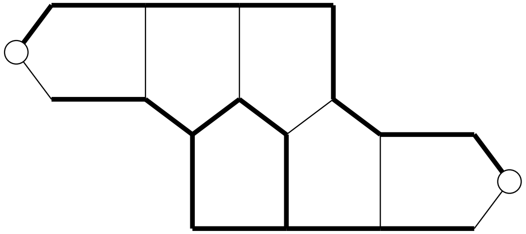


$d + e_2 + d$

Two $(5,3)$-polycycles with an alternating zone, which is not convex:



E_4



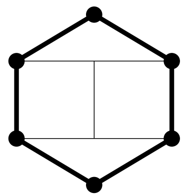
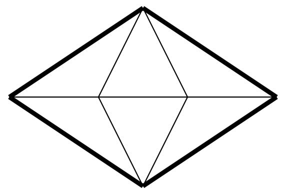
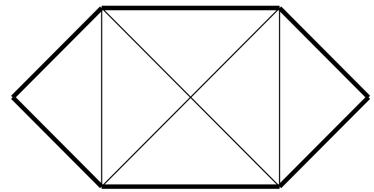
$D + E_2 + D$

Embedding of (r, q)-graph

- If the alternating zones of a plane graph G define convex cuts, then G is 2-embeddable.
- Above condition is not necessary.
- A (r, q)-graph is a plane graph such that all interior faces have at least r edges and all interior vertices have degree at least q.
- Chepoi et al.: $(4, 4)$-, $(3, 6)$- and $(6, 3)$-graphs are 2-embeddable.
- So, all parabolic and hyperbolic (r, q)-polycycle are 2-embeddable.

Elliptic 2-embeddable (r, q)-polycycles

- For elliptic $(r, q) \neq (5,3), (3,5)$ (i.e., $(3,3), (3,4), (4,3)$), only three polycycles are non-embeddable:

- A $(3,5)$-polycycle different from Icosahedron $\{3,5\}$ and $\{3,5\} - v$, is 2-embeddable if and only if it does not contain, as an induced subgraph, any of $(3,5)$-polycycles c_3 and $d + e_2 + d$.
- A $(5,3)$-polycycle different from Dodecahedron $\{5,3\}$ is 2-embeddable if and only if it does not contain, as an induced subgraph, any of $(5,3)$-polycycles E_4 and $D + E_2 + D$.

IX. Application to face-regular spheres

Euler formula

- Take a 3-valent plane map and denote by p_k the number of faces having k edges.
- Then one has the equality

$$12 = \sum_{k=3}^{\infty} (6-k) p_k$$

- So, every 3-valent plane map has at least one face of size less than 6.
- So, 3-valent plane graphs with faces of gonality at most 5
  - have at most 12 faces,
  - have at most 20 vertices.

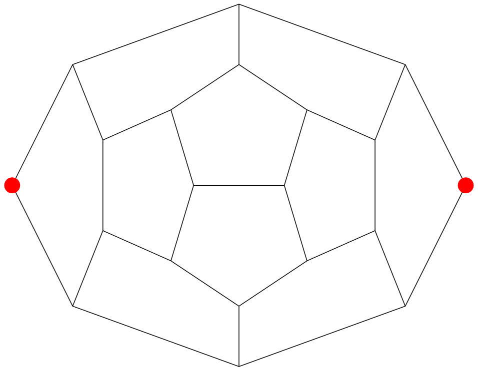
Face-regular maps

- A (p, q)-sphere is a 3-valent plane graphs, whose faces are p- or q-gonal.
- Take G a (p, q)-sphere. Then:
  - G is called pR_i if every p-gonal face is adjacent to exactly i p-gonal faces.
  - G is called qR_j if every q-gonal face is adjacent to exactly j q-gonal faces.
- The subject of enumerating them is very large. We intend to show non-trivial results obtained by using decomposition into elementary polycycles.
- $p \leq 5$. So, if one removes all q-gonal faces and all edges between any two q-gonal faces, then the result is a $(\{p\}, 3)$-polycycle.

Polycycles of $(5,q)$-sphere qR_0

- The set of 5-gonal faces of $(5,q)$-sphere qR_0 is decomposed into elementary $(\{5\},3)$-polycycles.
- Let us see in the classification the elementary polycycles that could be ok
  - They should be finite (this eliminate $Barrel_\infty$ and α)
  - They should have some vertices of degree 2 (this eliminates Dodecahedron and $Barrel_k$)
  - It should be possible to fill open edges so as to have no pending vertices of degree 2.

Polycycles of $(5,q)$-sphere qR_0



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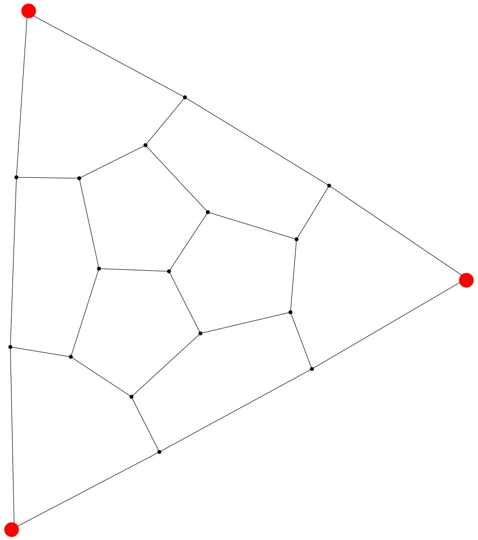

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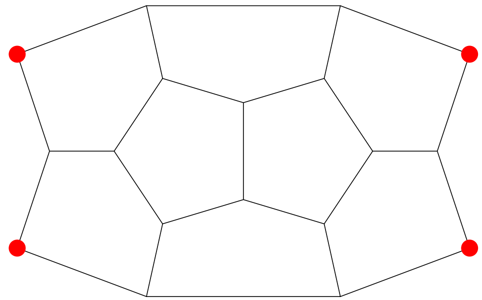

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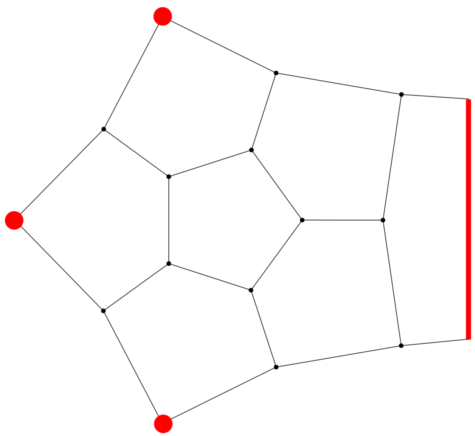

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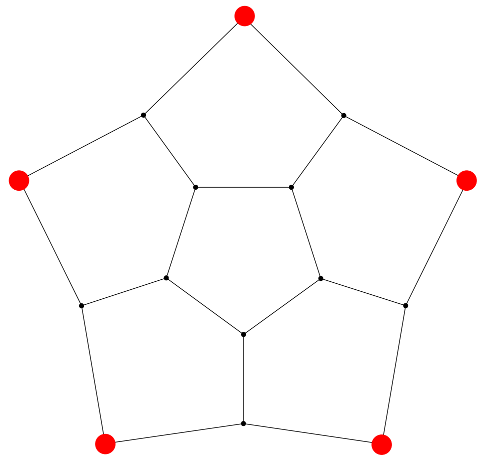
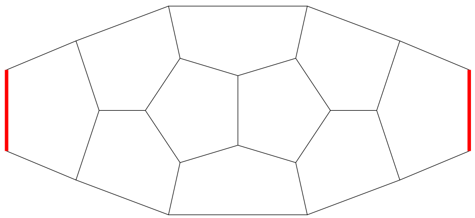


NO

Polycycles of $(5, q)$-sphere qR_0



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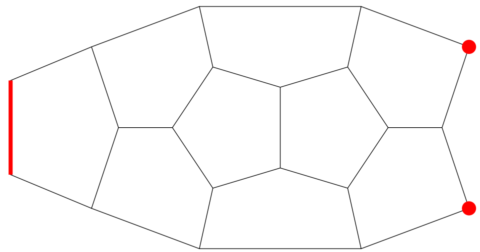

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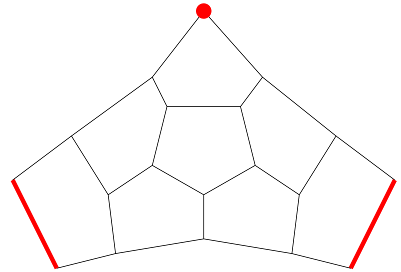

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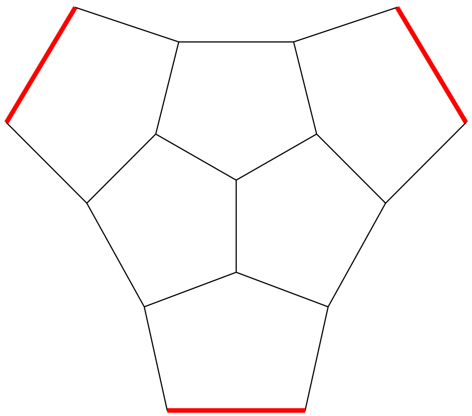

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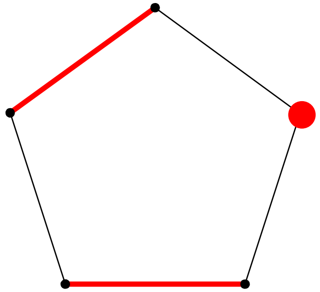
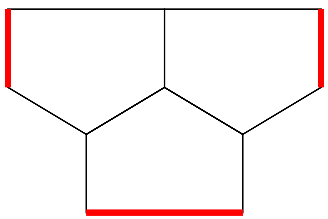


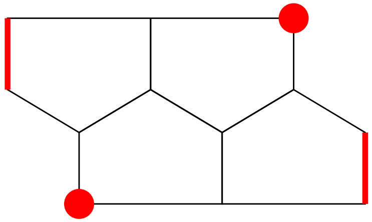
NO

Polycycles of $(5, q)$-sphere qR_0

The infinite series of elementary $(\{5\}, 3)$-polycycles $\alpha\alpha$:



YES



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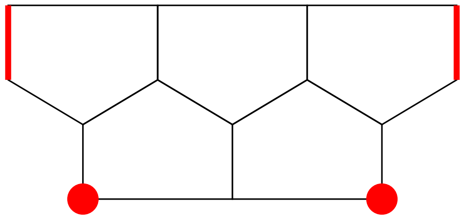
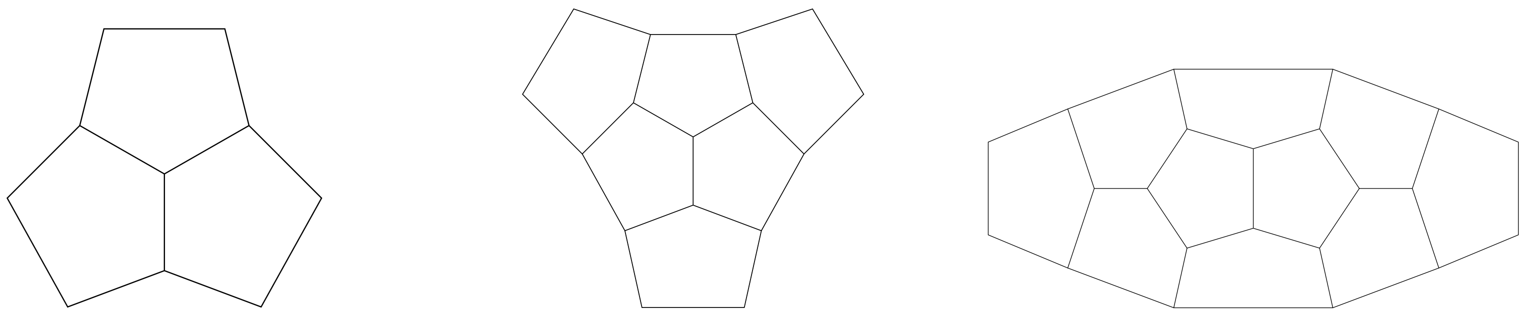


NO

$(5, q)$-sphere qR_0

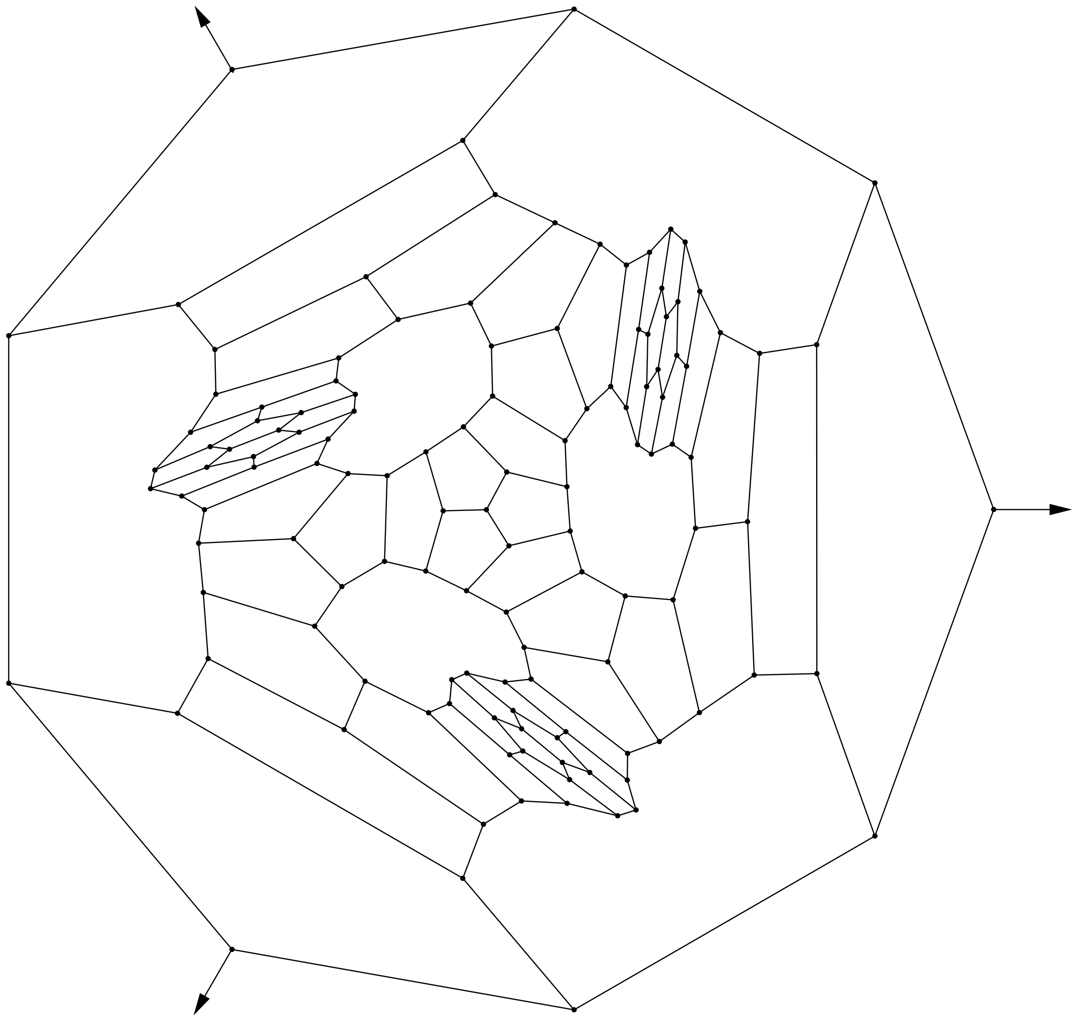
- The set of 5-gonal faces of $(5, q)$-sphere qR_0 is decomposed into the following elementary $(\{5\}, 3)$-polycycles:



E_1 $\qquad$ C_3 $\qquad$ C_1

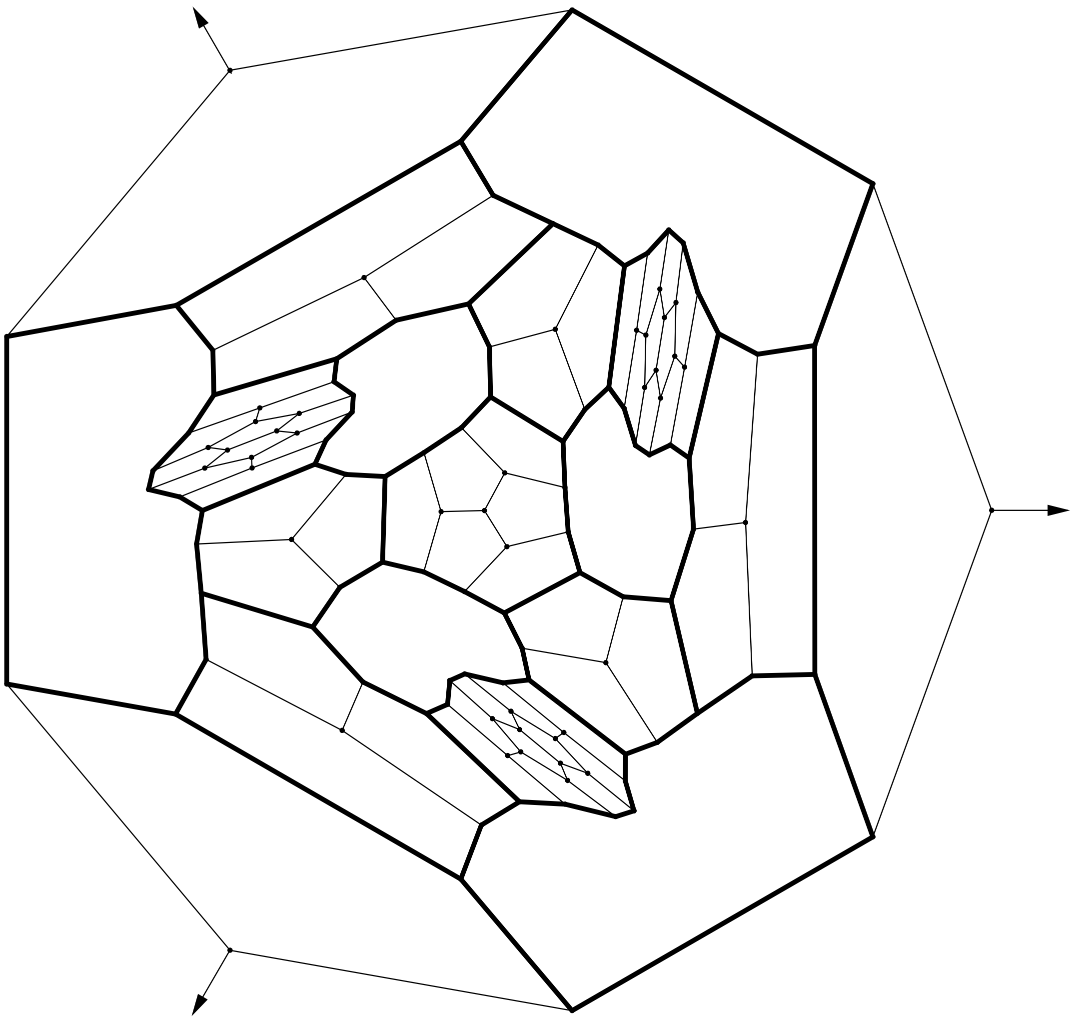
- The major skeleton $Maj(G)$ of a $(5, q)$-sphere qR_0 is a 3-valent map, whose vertex-set consists of polycycles E_1 and C_3.
- It consists of $el(G)$ with the vertices C_1 (of degree 2) being removed.

$(5, q)$-sphere qR_0



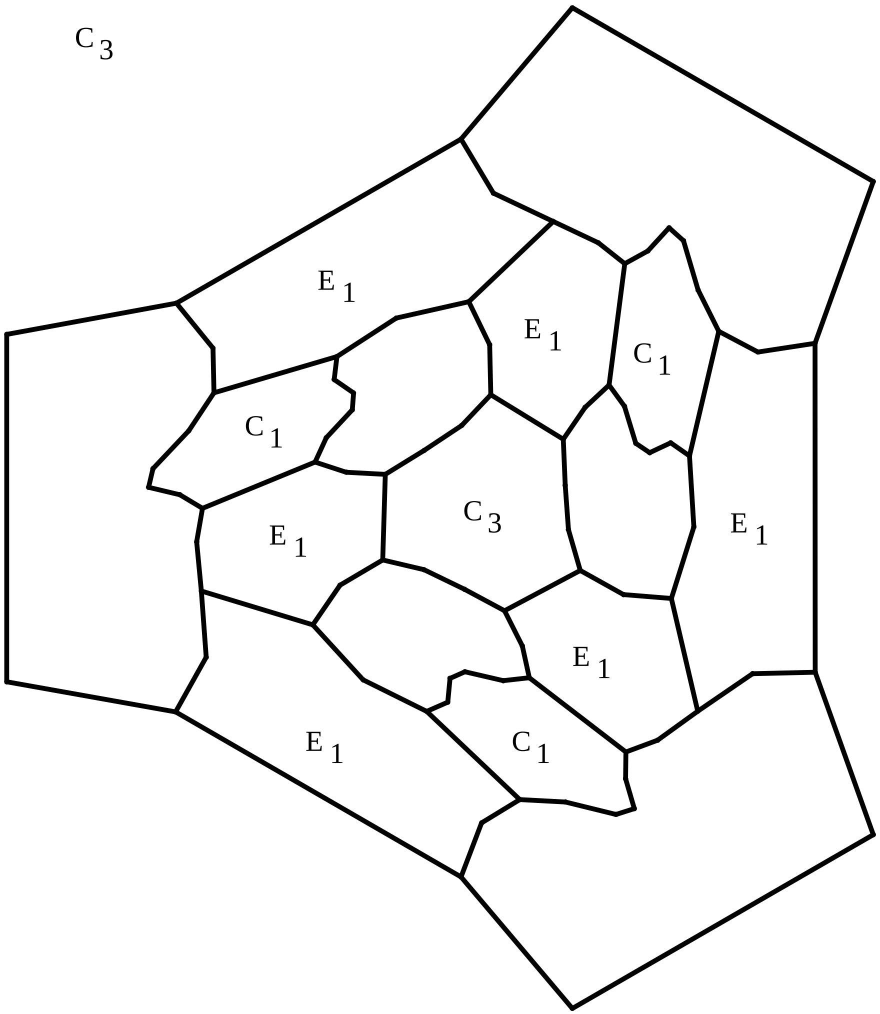
A $(5, 14)$-sphere $14R_0$

$(5, q)$-sphere qR_0



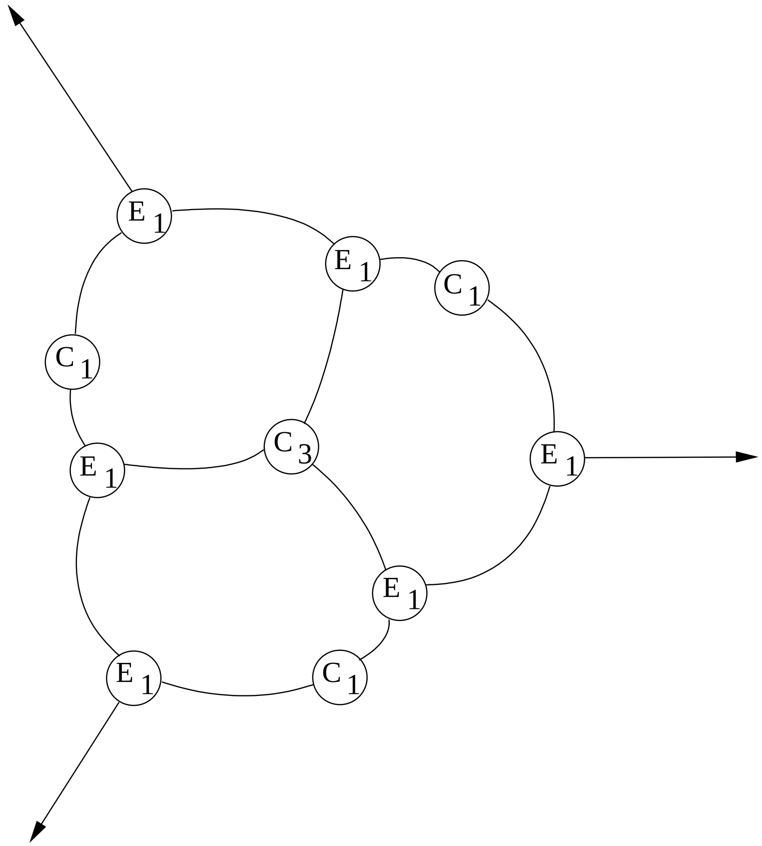
The decomposition into elementary polycycles.

$(5, q)$-sphere qR_0



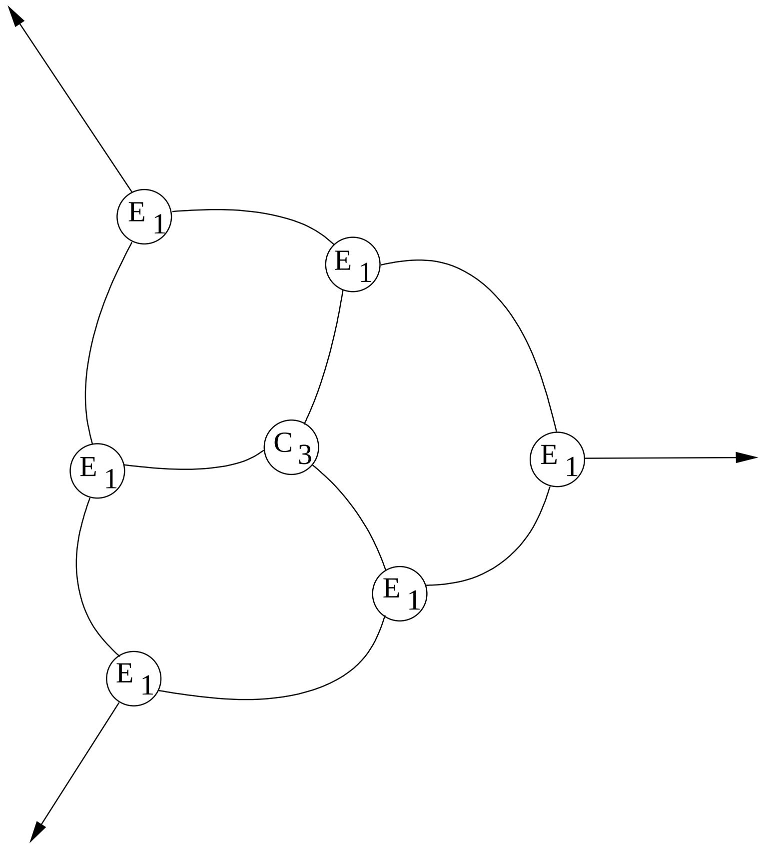
Their names in the classification of $(\{5\}, 3)$-polycycles.

$(5, q)$-sphere qR_0



The graph $el(G)$

$(5,q)$-sphere qR_0



$Maj(G)$: eliminate C_1, so as to get a 3-valent map

Results

For a $(5, q)$-sphere qR_0, the gonality of faces of the 3-valent map $Maj(G)$ is at most $\lfloor \frac{q}{2} \rfloor$.

- Proof: Take a q-gonal face F. Denote by x_{E_1}, x_{C_3} and x_{C_1} the number of $(\{5\}, 3)$-polycycles E_1, C_3 and C_1 incident to F.
- Counting edges, one gets:

$$q = 2x_{E_1} + 3x_{C_3} + 5x_{C_3}$$

which implies $q \geq 2(x_{E_1} + x_{C_3})$.

- But $x_{E_1} + x_{C_3}$ is the gonality of the face corresponding to F in $Maj(G)$.

Results

For $q < 12$, we have a finite number of $(5, q)$-spheres qR_0.

- Proof: Take such a plane graph G.
- The associated map $Maj(G)$ is 3-valent with faces of gonality at most 5.
- So, the number of $(\{5\}, 3)$-polycycles E_1 and C_3 is at most 20.
- The number of polycycles C_1 is bounded as well.
- This implies that the number of vertices of G is bounded and so, we have a finite number of spheres.