Goldberg-Coxeter construction for 3- or 4-valent plane graphs

Michel Deza

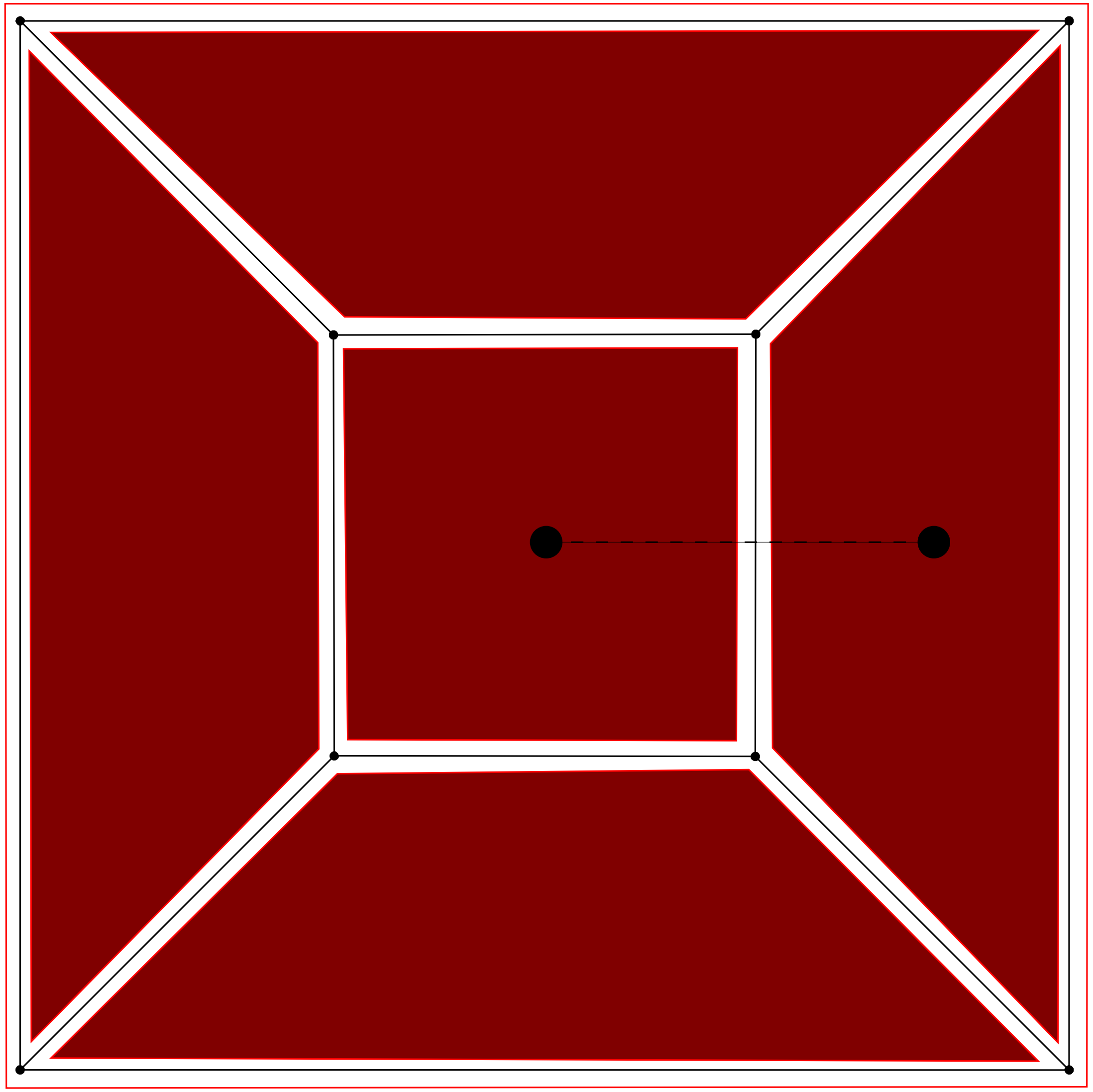
Ecole Normale Superieure, Paris

Mathieu Dutour Sikiric

Rudjer Boskovic Institute, Zagreb, and ISM, Tokyo

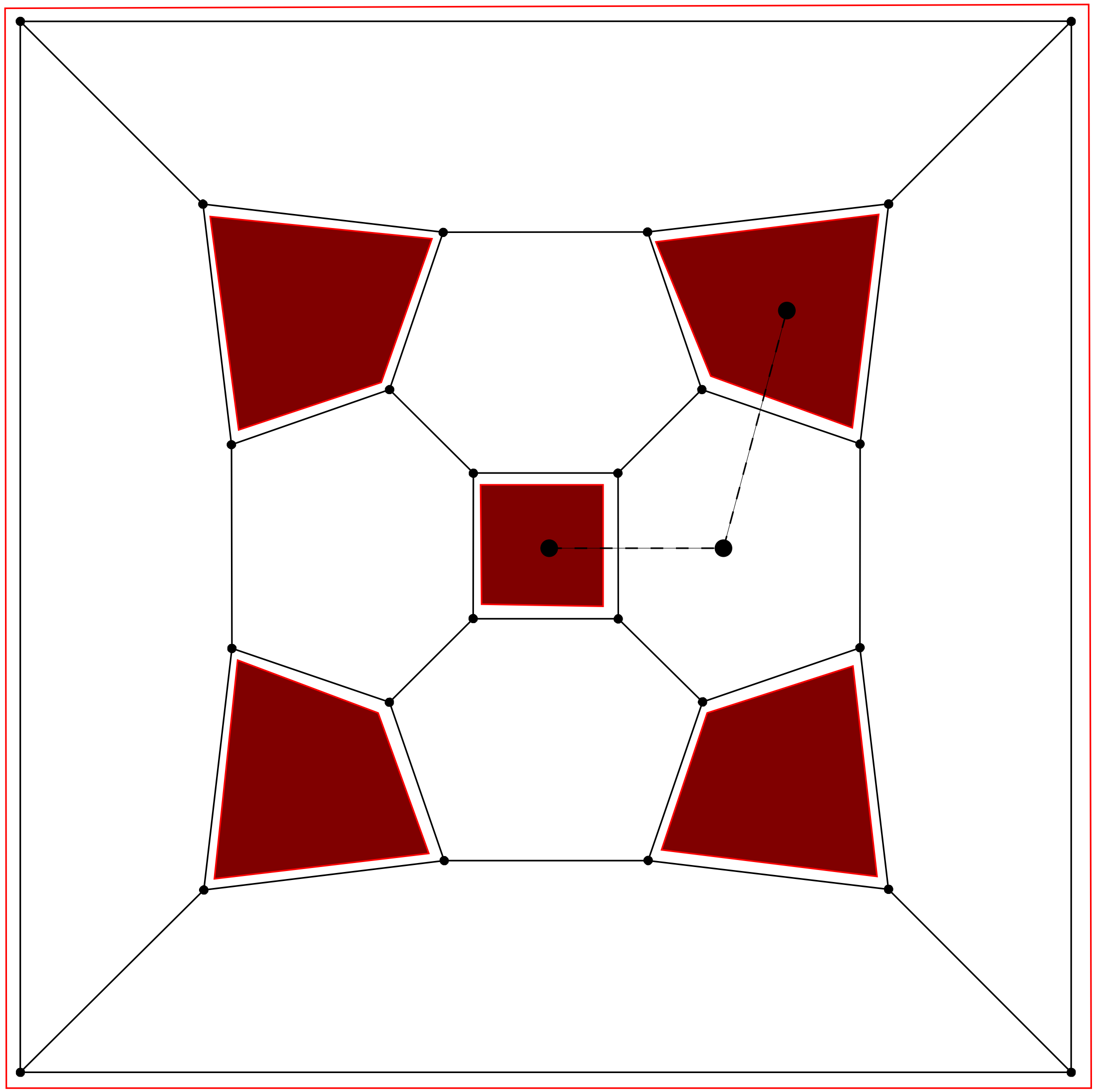
Goldberg-Coxeter for the Cube

1,0



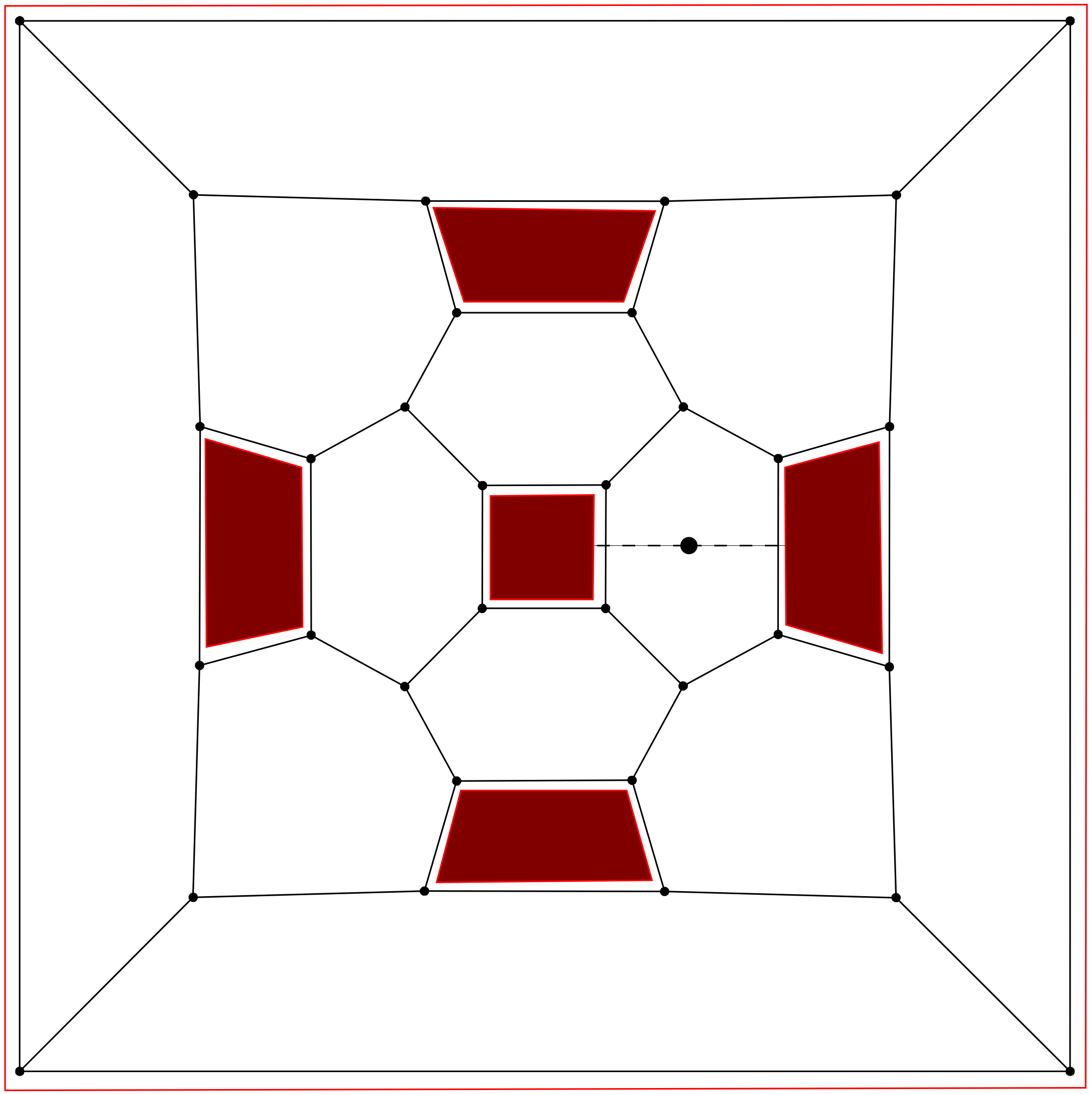
Goldberg-Coxeter for the Cube

1,1



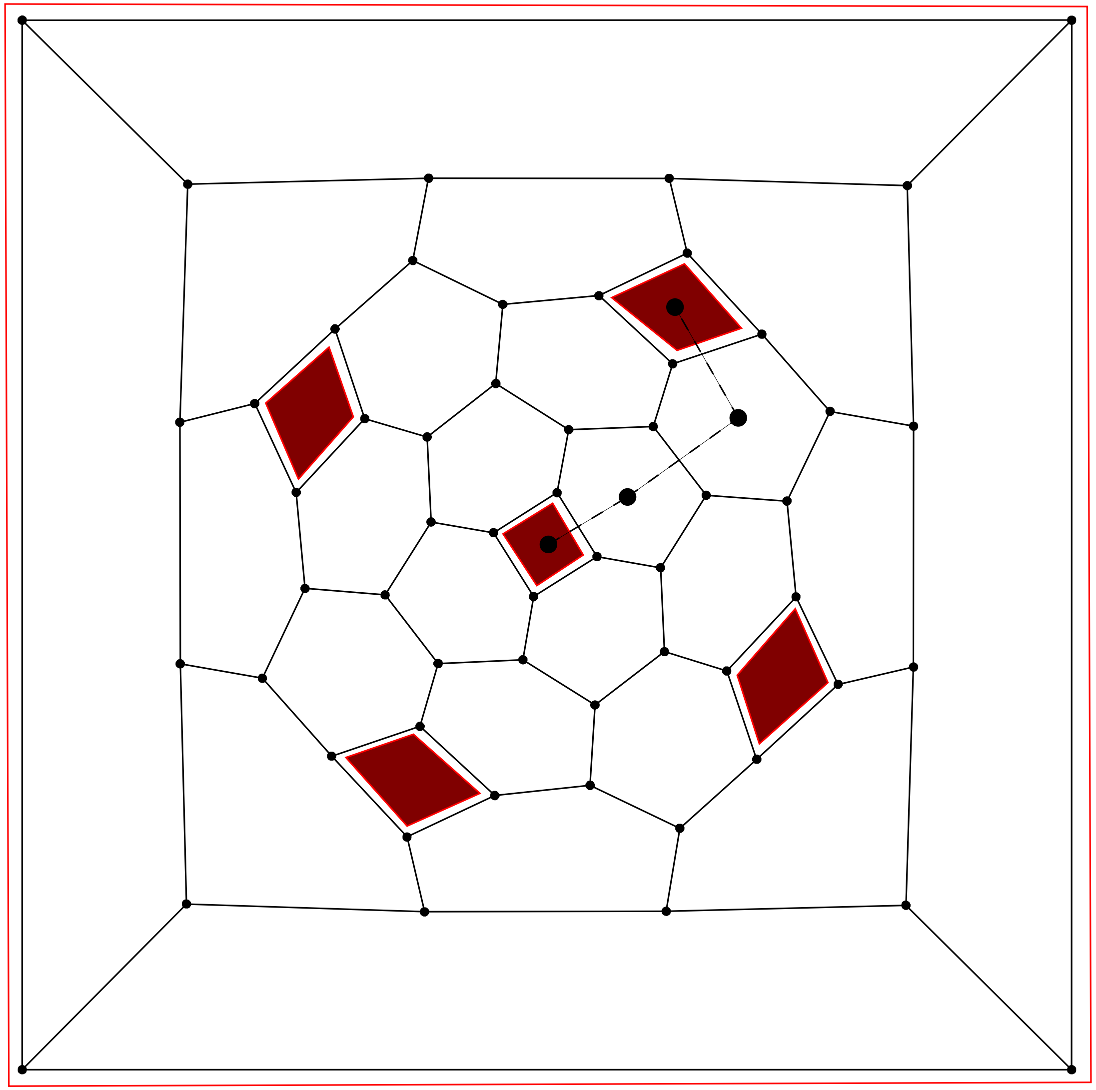
Goldberg-Coxeter for the Cube

2,0



Goldberg-Coxeter for the Cube

2,1



History

Mathematics: construction of planar graphs

M. Goldberg, *A class of multisymmetric polyhedra*, Tohoku Math. Journal, **43** (1937) 104–108.

Objective was to maximize the interior volume of the polytope, i.e. to find 3-dimensional analogs of regular polygons.

➠ search of equidistributed systems of points on the sphere for application to Numerical Analysis.

History

Biology: explanation of structure of icosahedral viruses

D.Caspar and A.Klug, *Physical Principles in the Construction of Regular Viruses*, Cold Spring Harbor Symp. Quant. Biol., **27** (1962) 1-24.

(k,l)	symmetry	capsid of virion
$(1,0)$	I_h	*gemini virus*
$(2,0)$	I_h	*hepathite B*
$(2,1)$	I, laevo	*HK97, rabbit papilloma virus*
$(3,1)$	I, laevo	*rotavirus*
$(4,0)$	I_h	*herpes virus, varicella*
$(5,0)$	I_h	*adenovirus*
$(6,3)$?	I, laevo	*HIV-1*

History

Architecture: construction of geodesic domes
Patent by Buckminster Fuller

EPCOT in Disneyland.

History

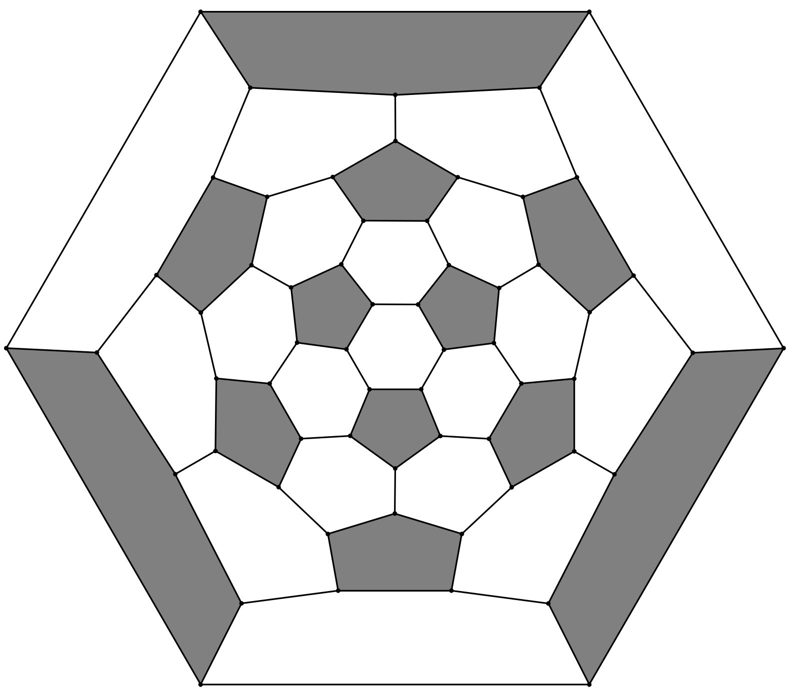
Mathematics:

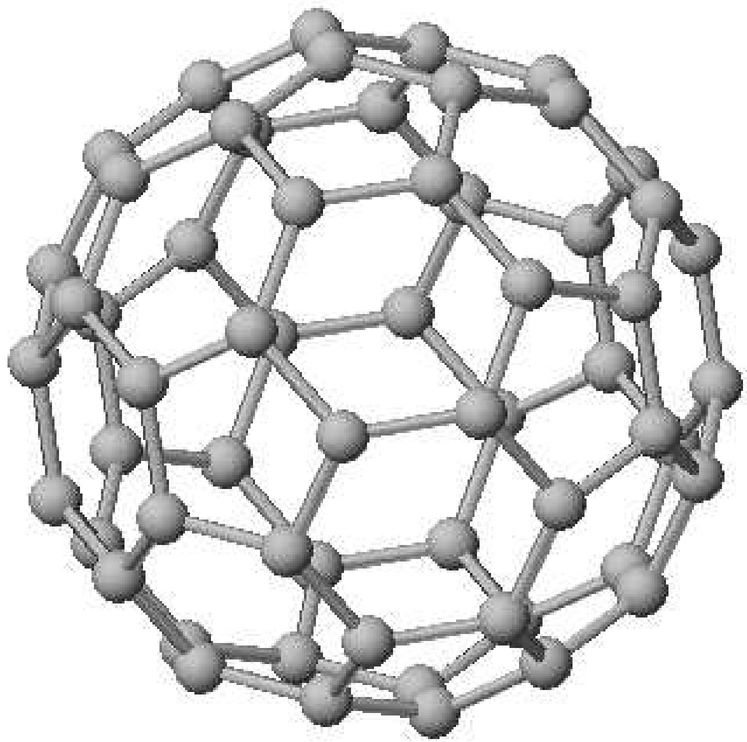
H.S.M. Coxeter, *Virus macromolecules and geodesic domes*, in *A spectrum of mathematics*; ed. by J.C.Butcher, Oxford University Press/Auckland University Press: Oxford, U.K./Auckland New-Zealand, (1971) 98–107.

History

Chemistry: Buckminsterfullerene C_{60}
(football, Truncated Icosahedron)

Kroto, Kurl, Smalley (Nobel prize 1996) synthetized in 1985 a new molecule, whose graph is $GC_{1,1}(Dodecahedron)$.
Osawa constructed theoretically C_{60} in 1984.





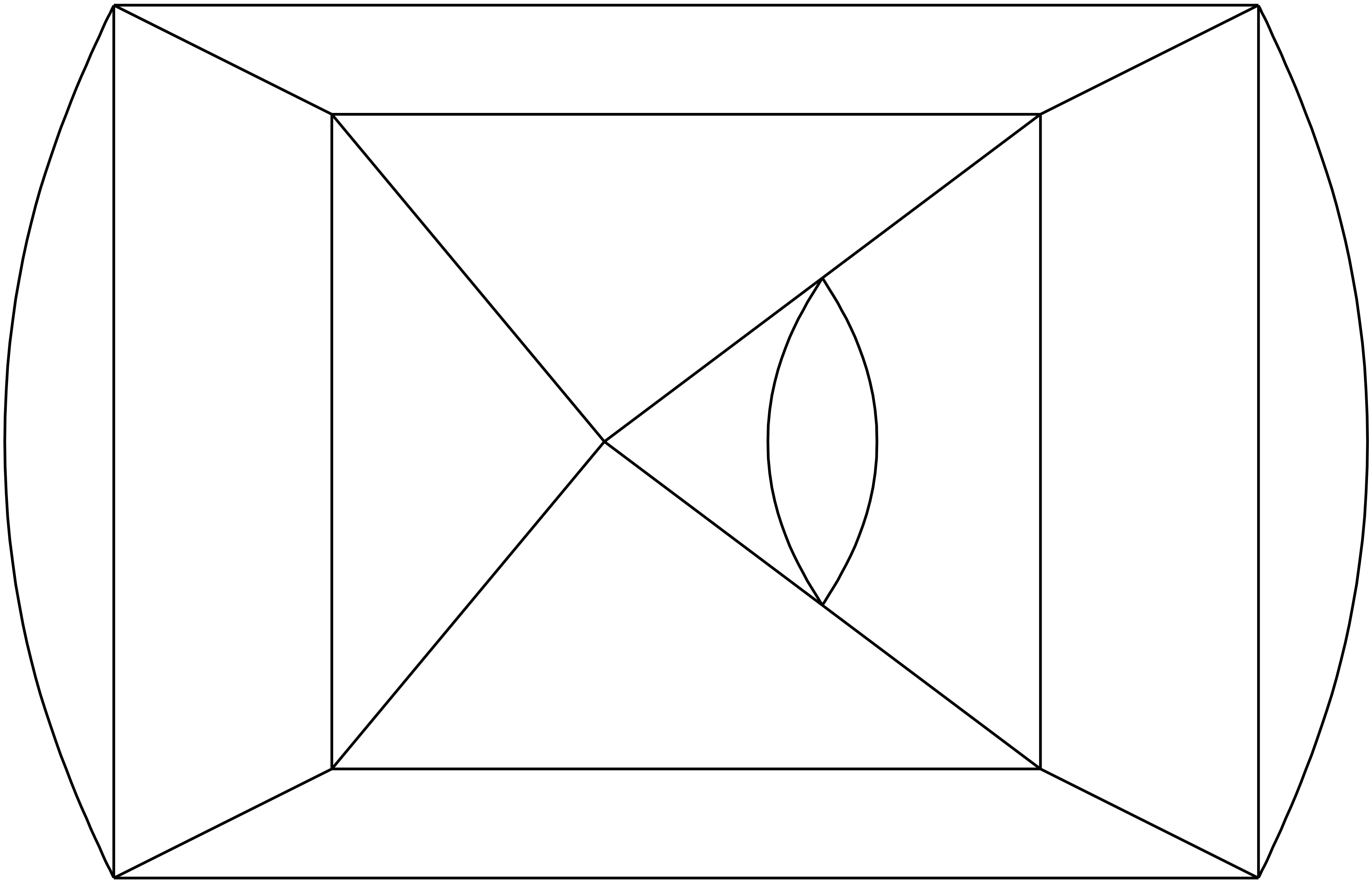
I. ZigZags

and

central circuits

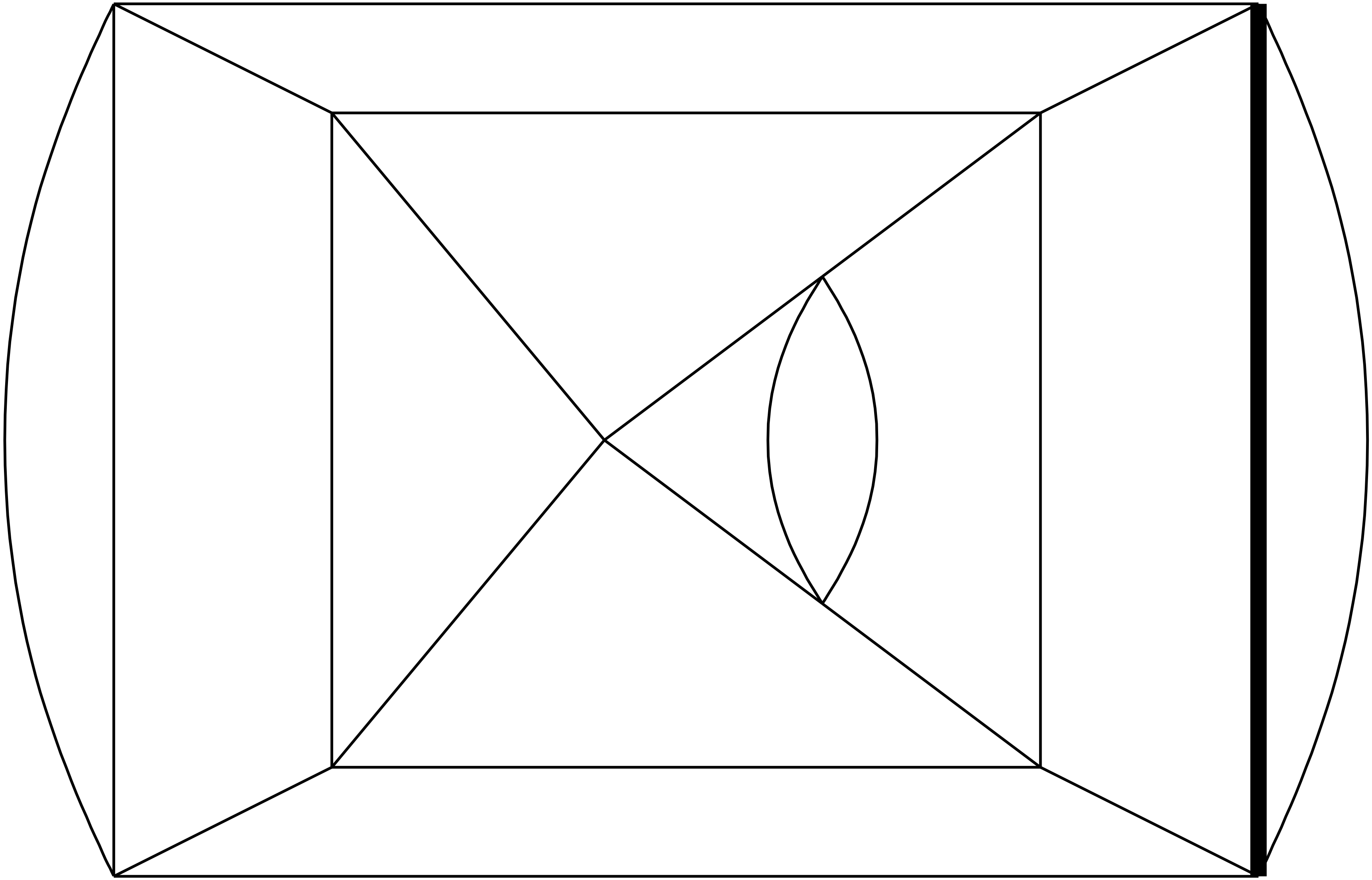
Central circuits

A 4−valent plane graph G



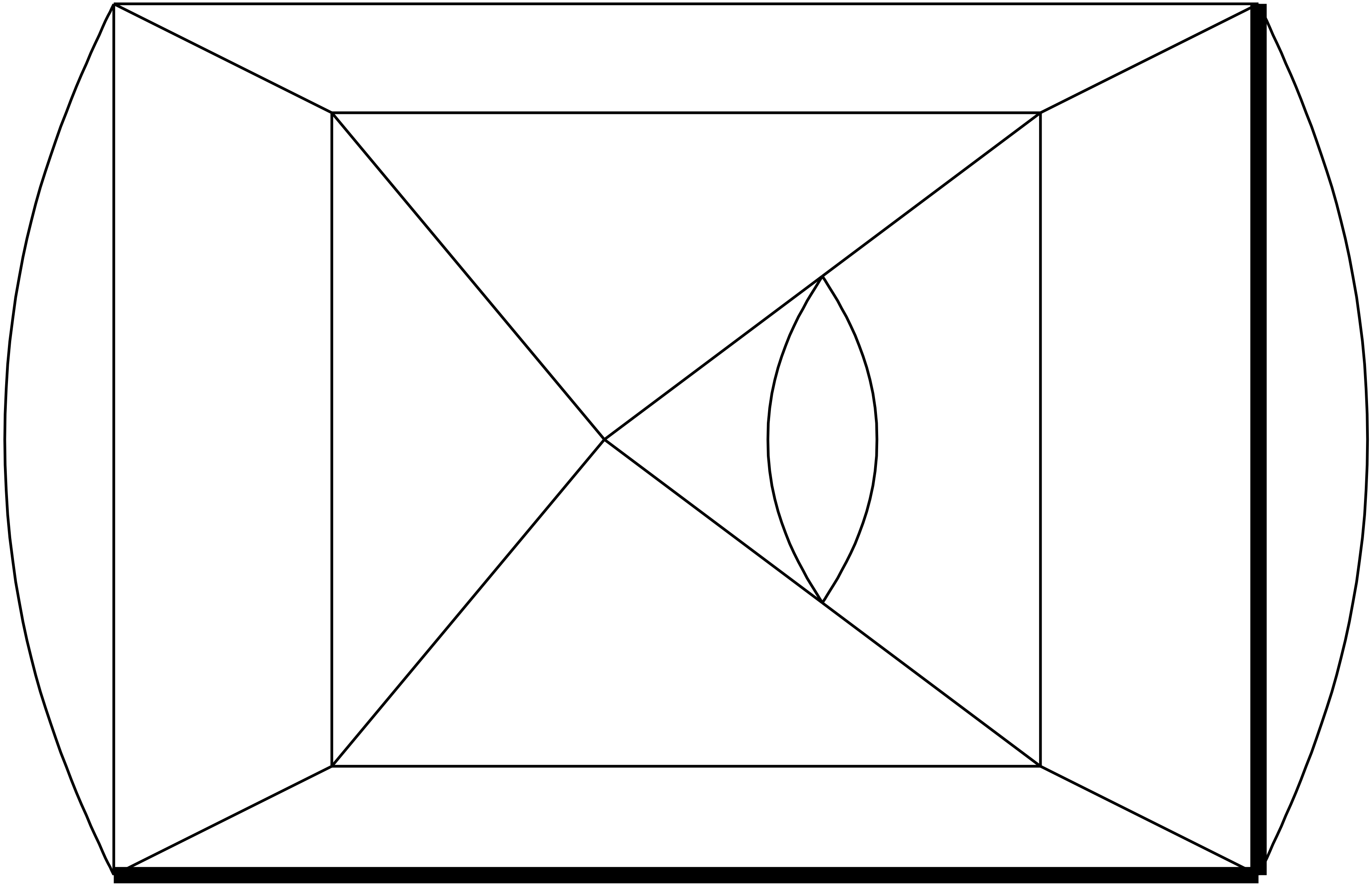
Central circuits

Take an edge of G



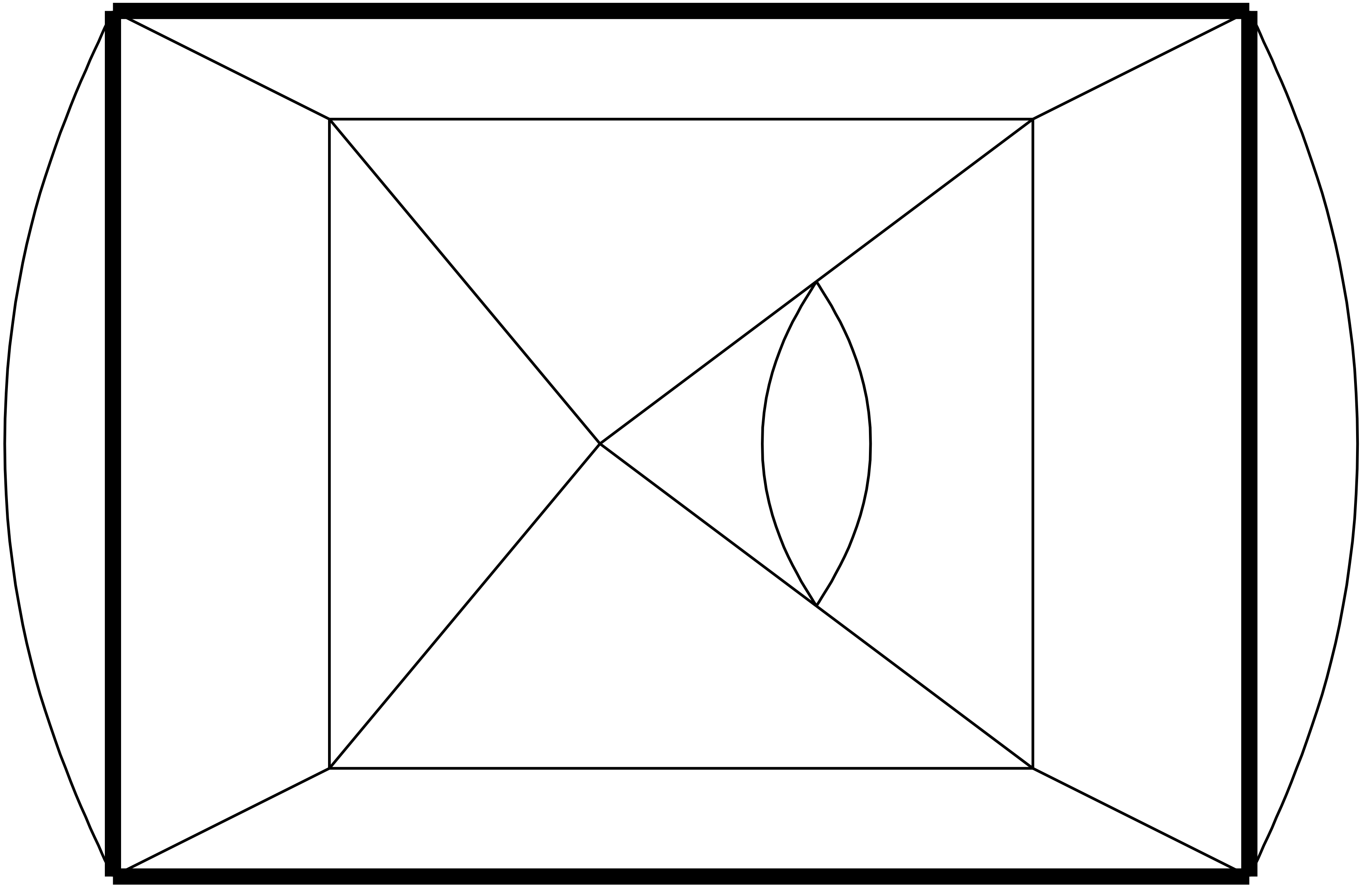
Central circuits

Continue it straight ahead ...



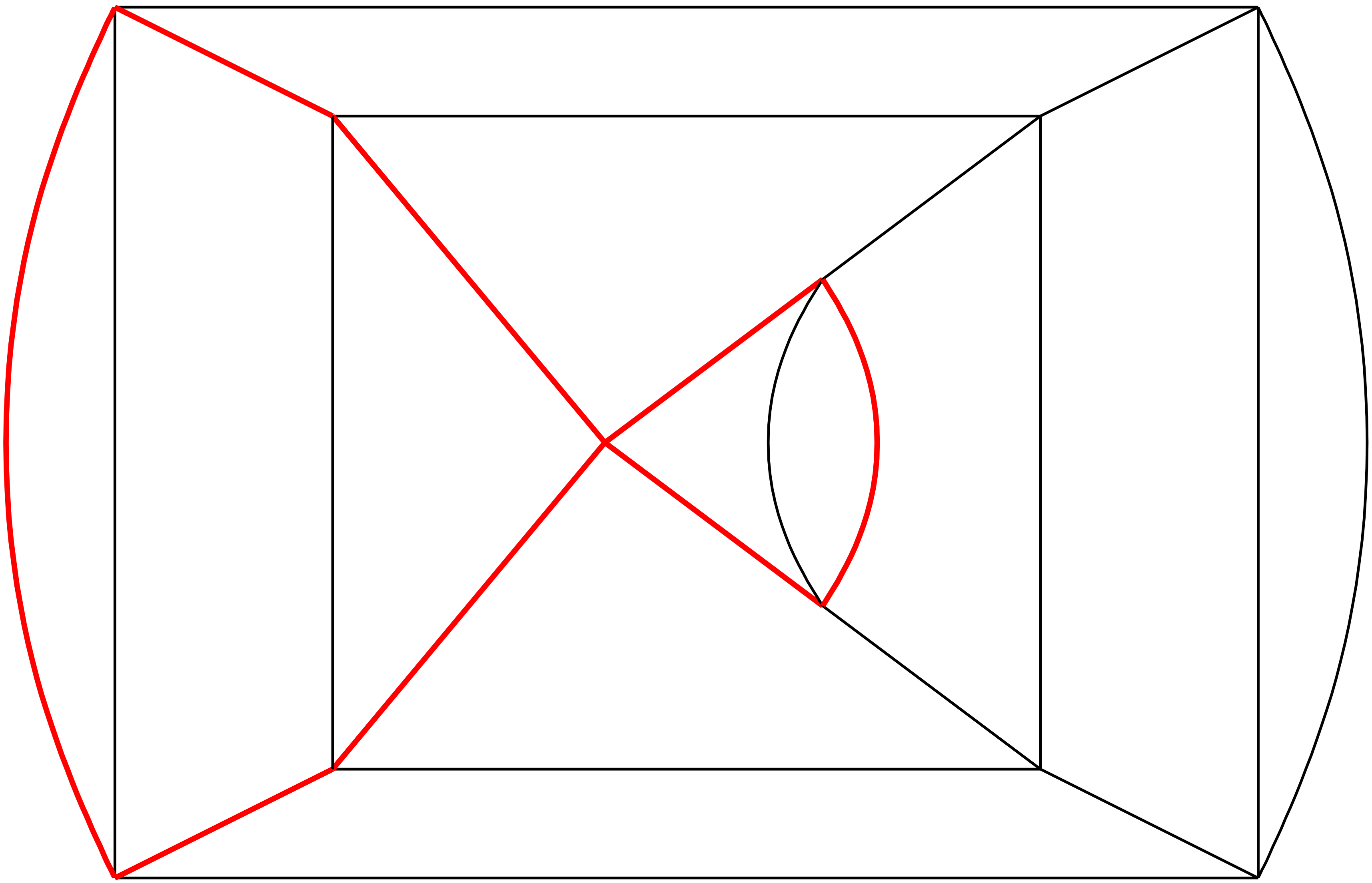
Central circuits

... until the end



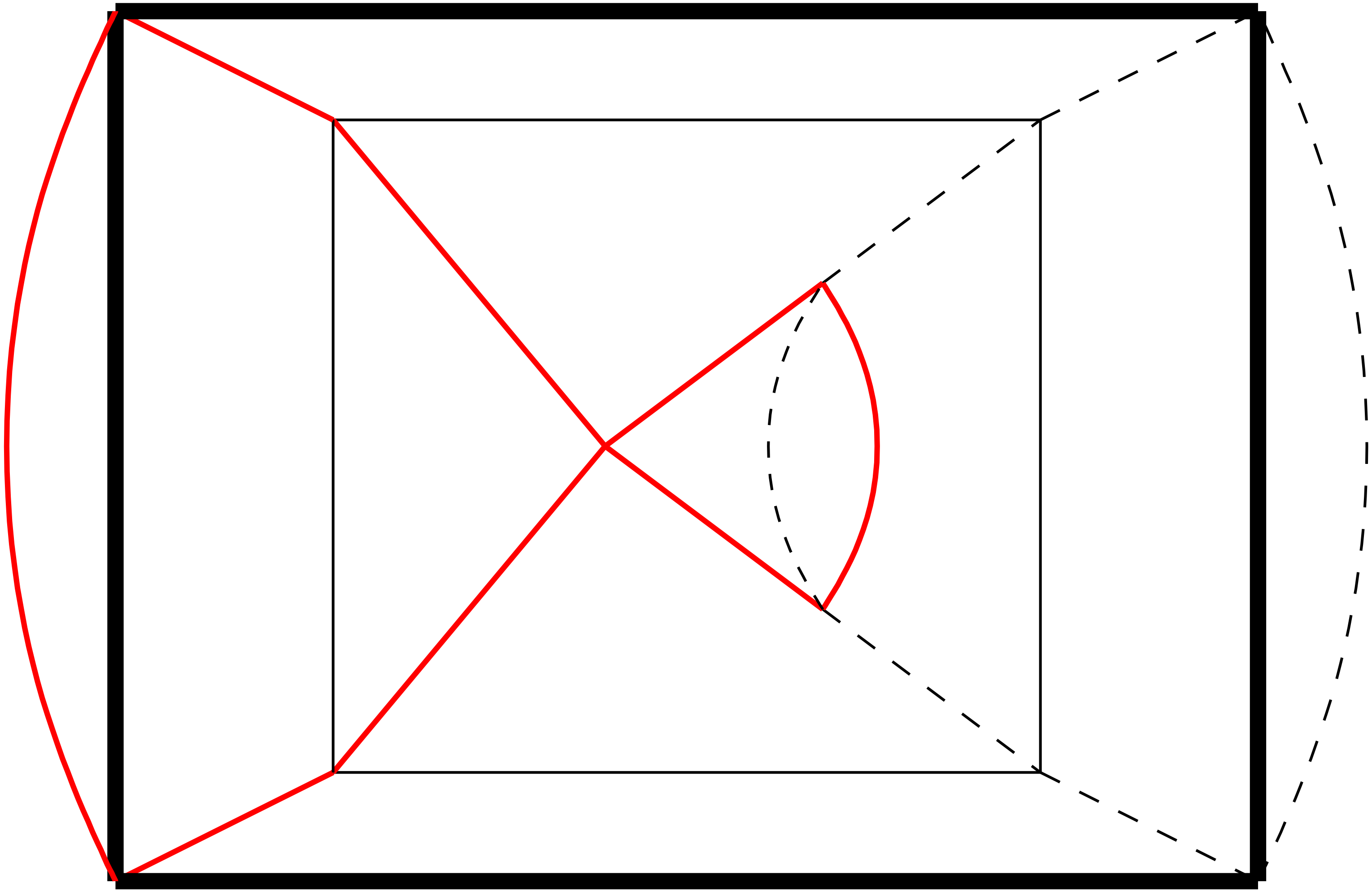
Central circuits

A self−intersecting central circuit



Central circuits

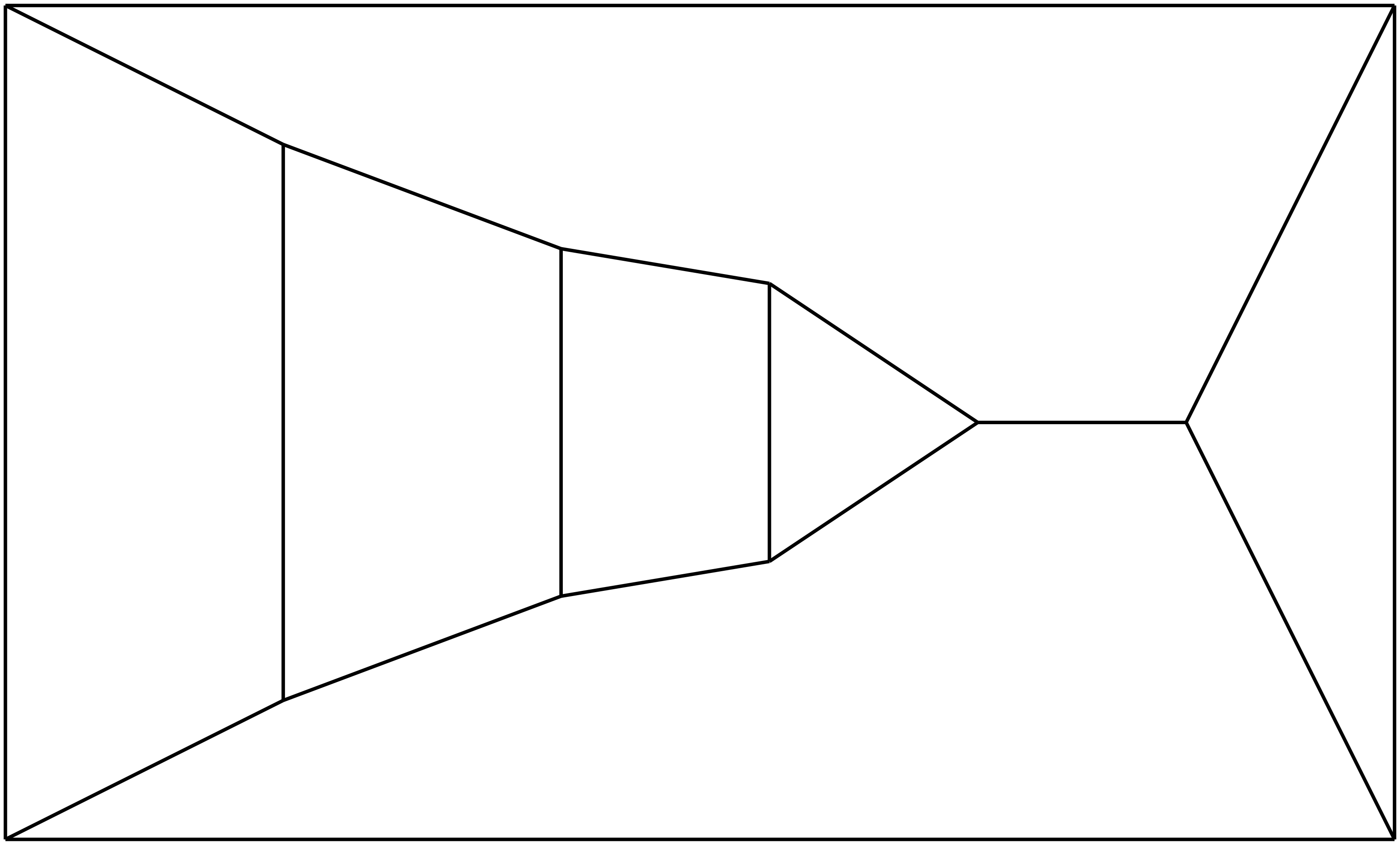
A partition of edges of G



CC=4^2, 6, 8

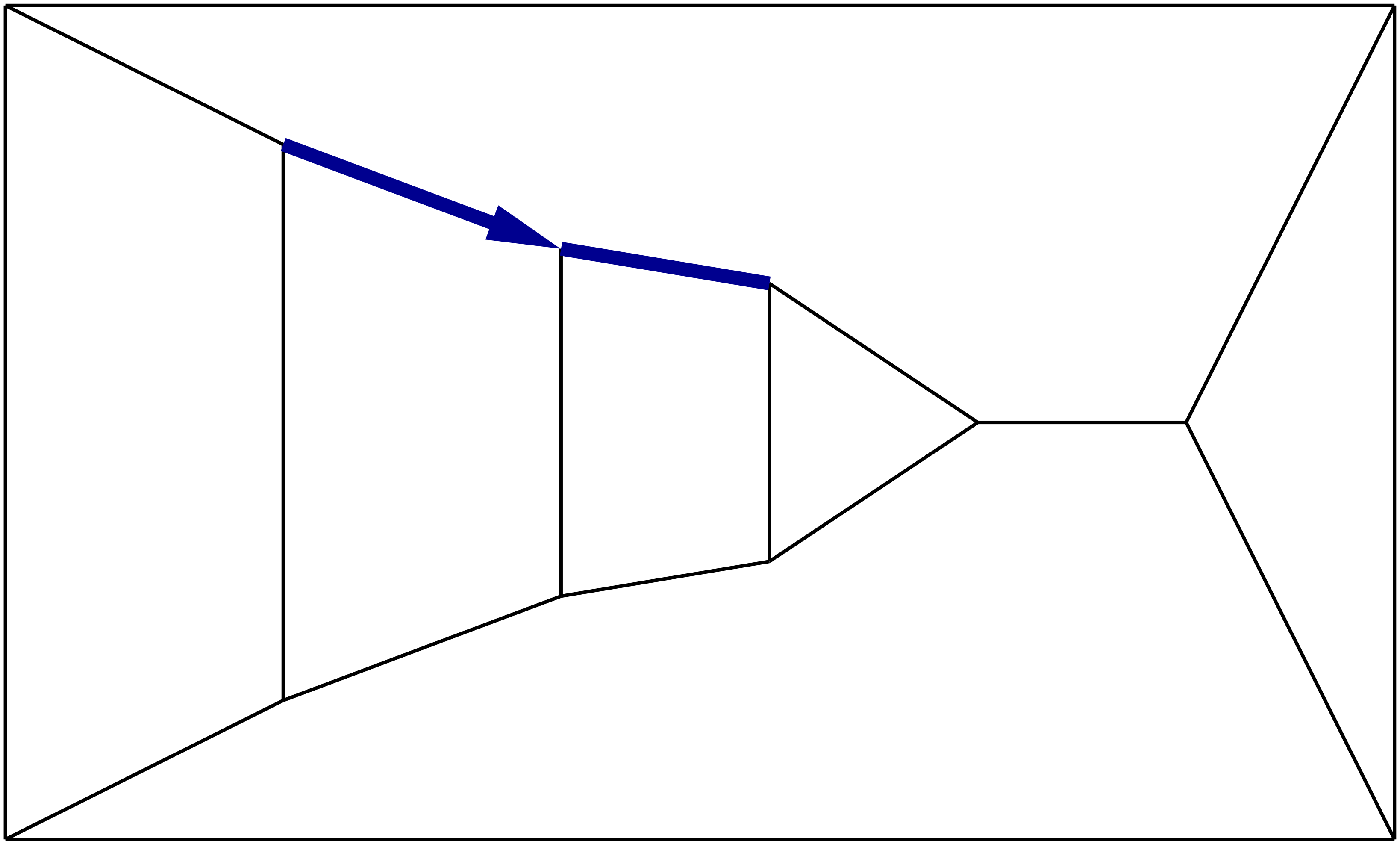
Zig Zags

A plane graph G



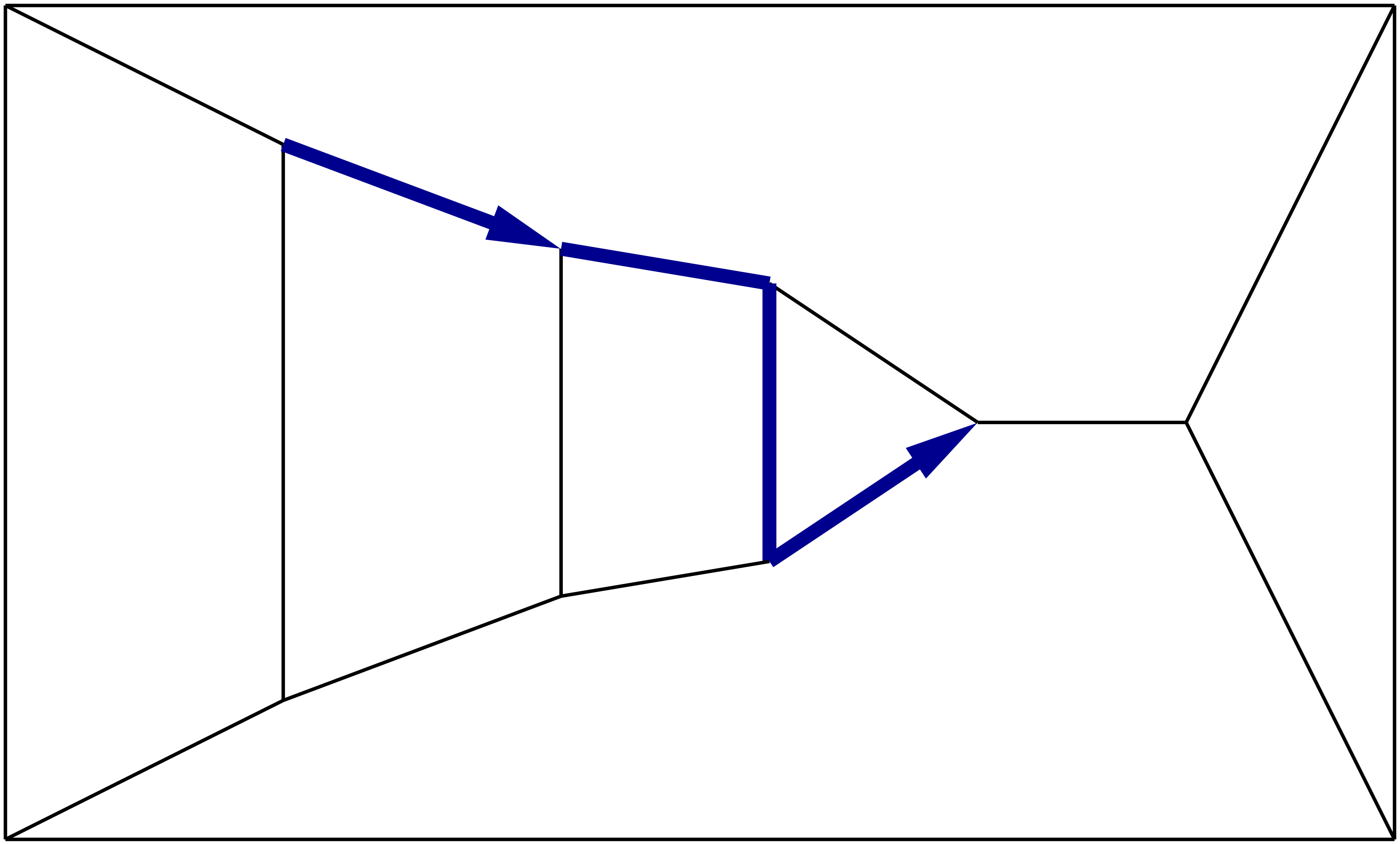
Zig Zags

take two edges



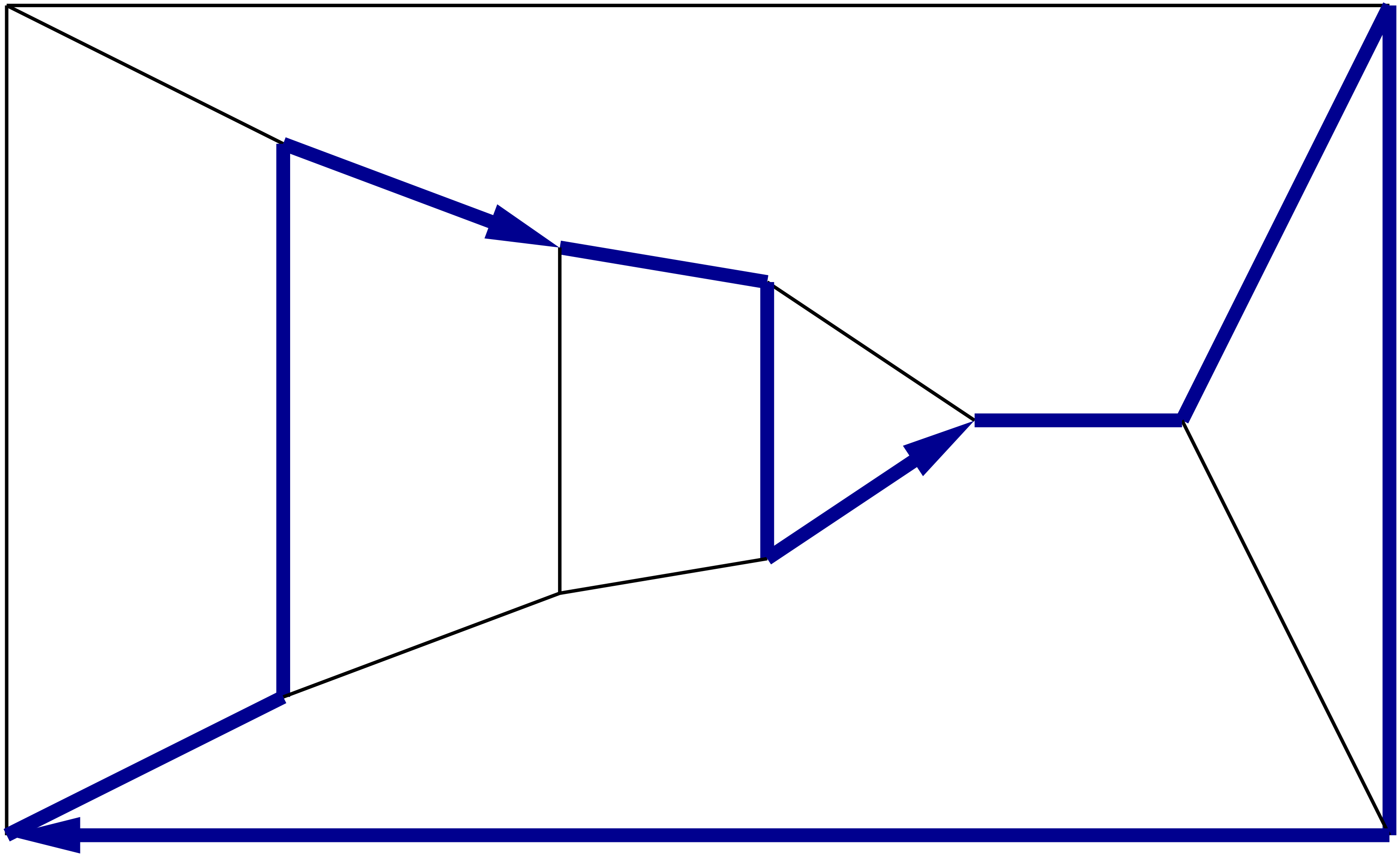
Zig Zags

Continue it left−right alternatively ....



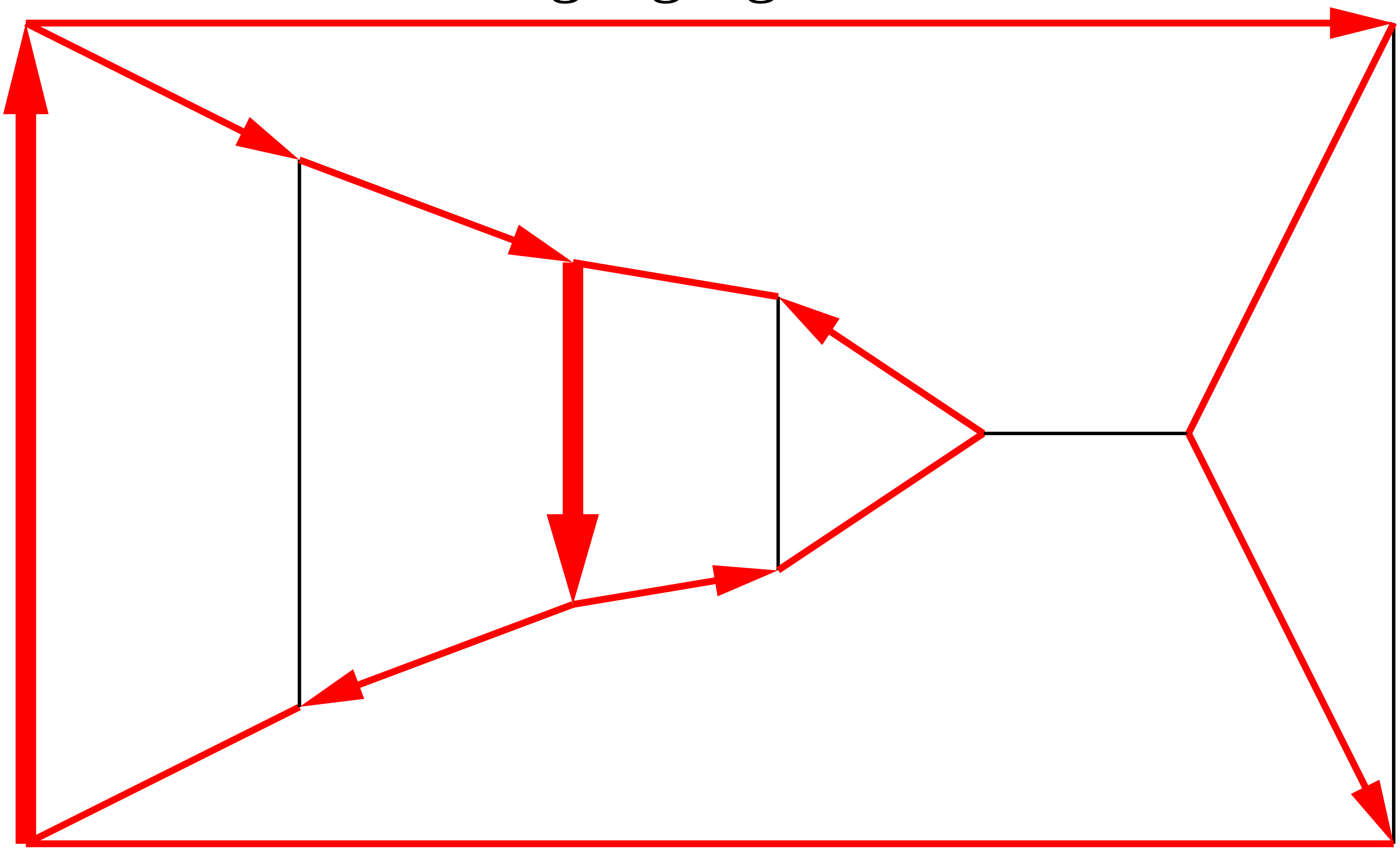
Zig Zags

... until we come back.



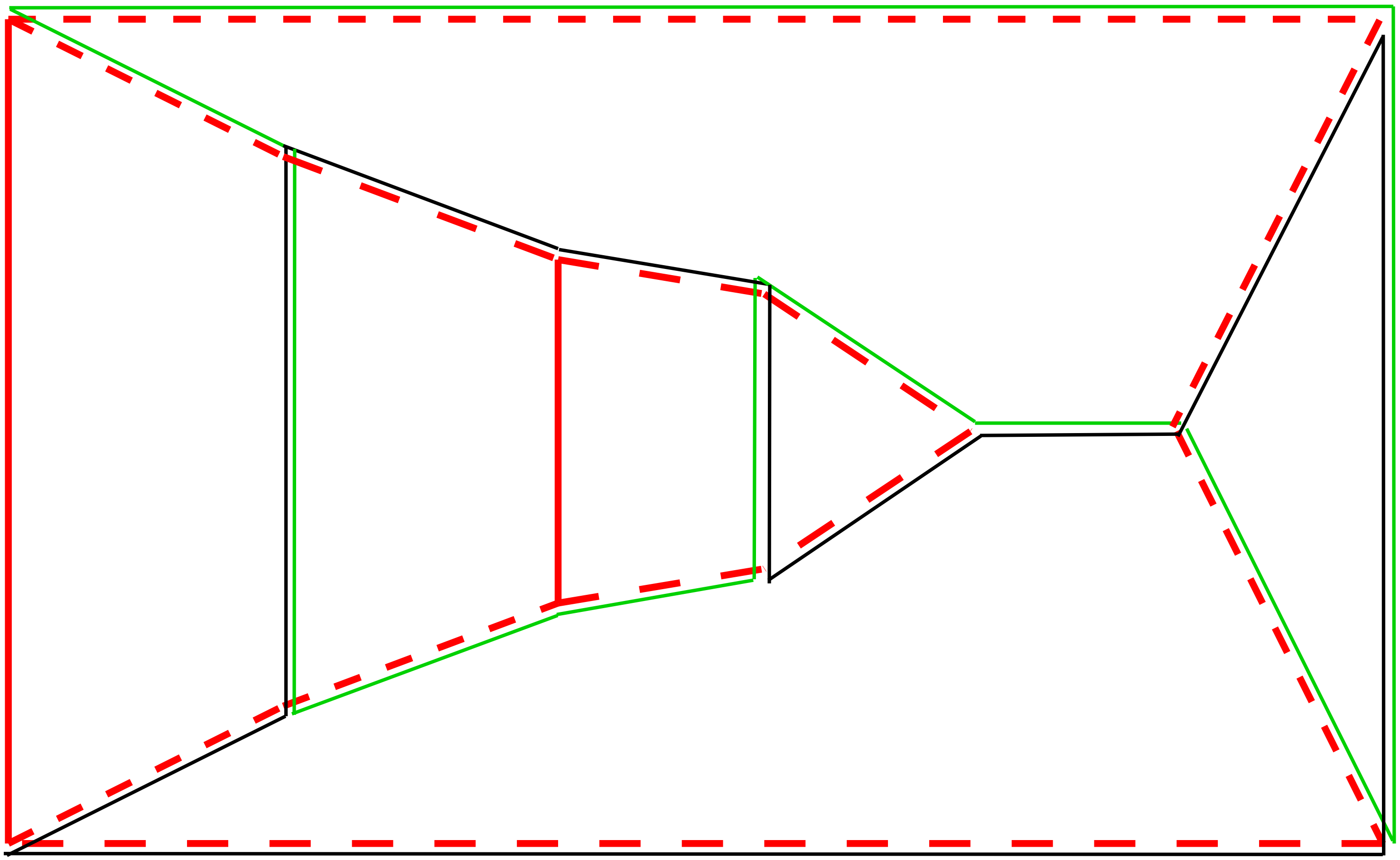
Zig Zags

A self−intersecting zigzag



Zig Zags

A double covering of 18 edges: 10+10+16



z−vector $z = 10^2, \; 16_{2,0}$

Notations

- ZC-circuit stands for “zigzag or central circuit” in 3- or 4-valent plane graphs.

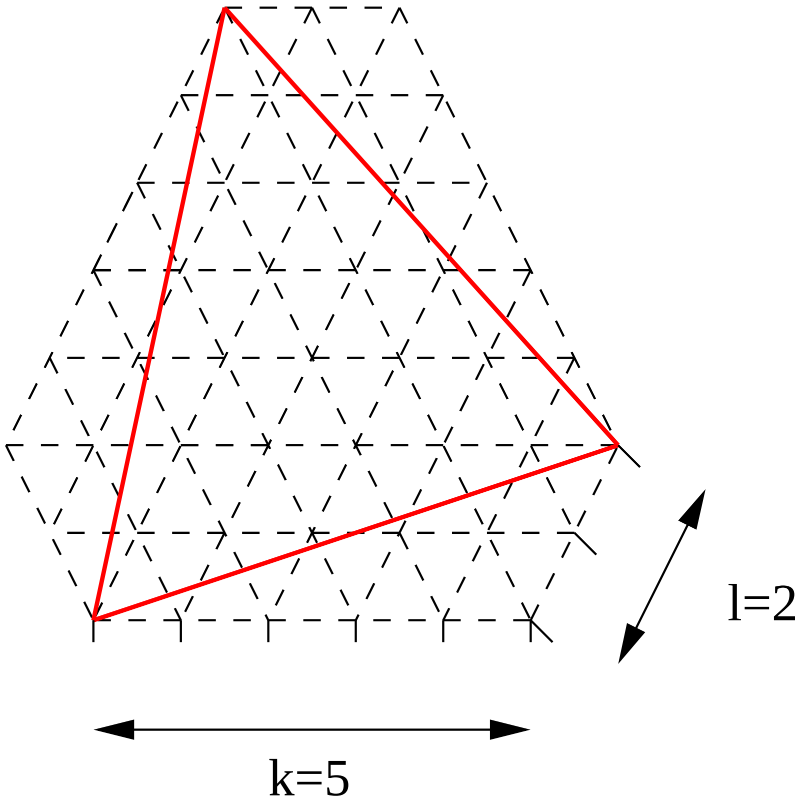
- The length of a ZC-circuit is the number of its edges.

- The ZC-vector of a 3- or 4-valent plane graph G_0 is the vector $\dots, c_k^{m_k}, \dots$ where m_k is the number of ZC-circuits of length c_k.

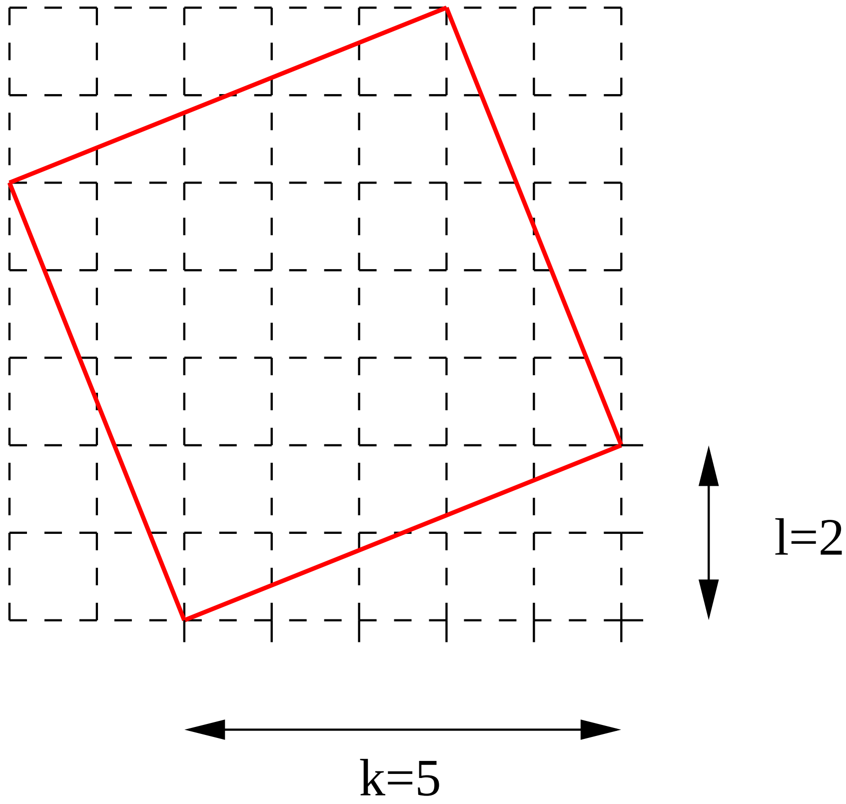
II. Goldberg-Coxeter construction

The construction

- Take a 3- or 4-valent plane graph G_0. The graph G_0^* is formed of triangles or squares.
- Break the triangles or squares into pieces:



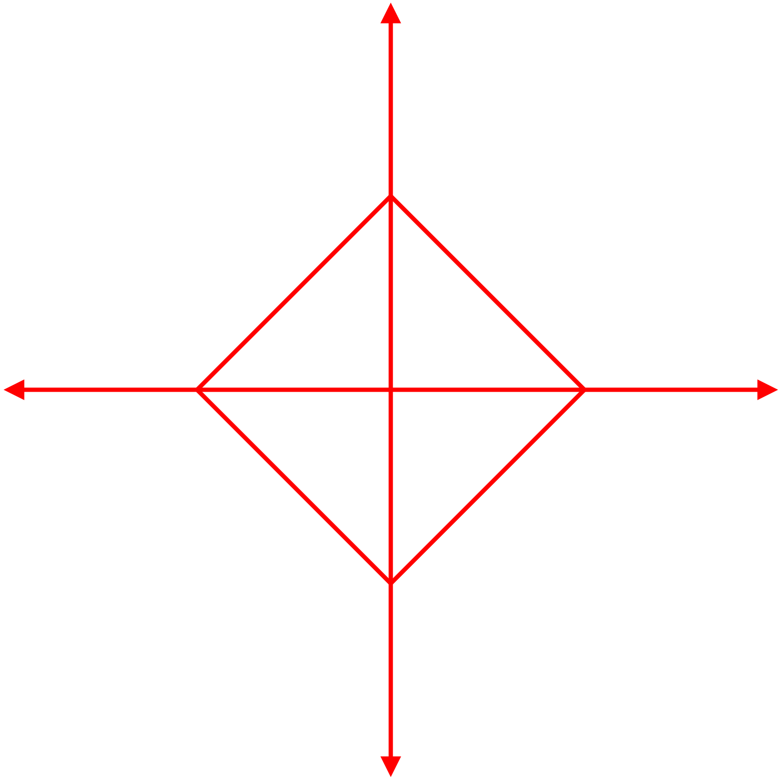
3−valent case



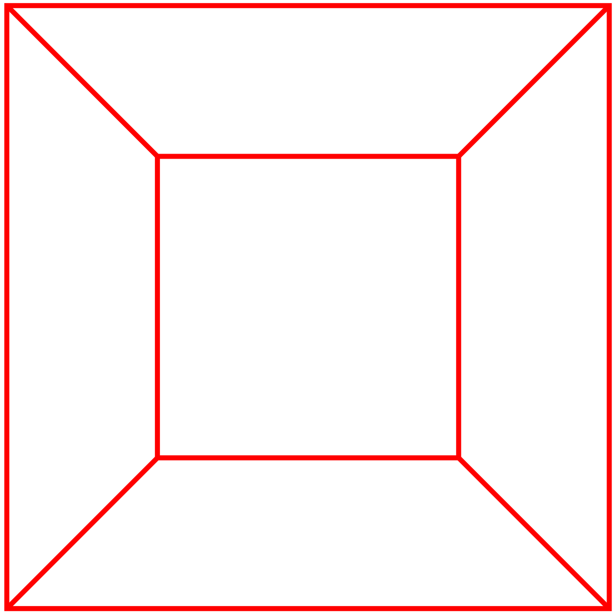
4−valent case

Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



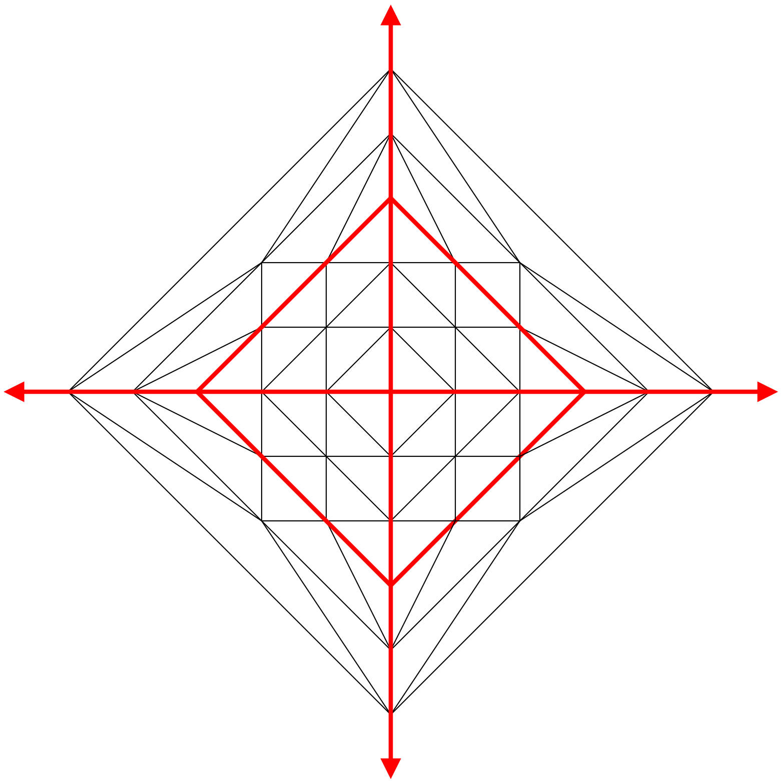
Case 3-valent



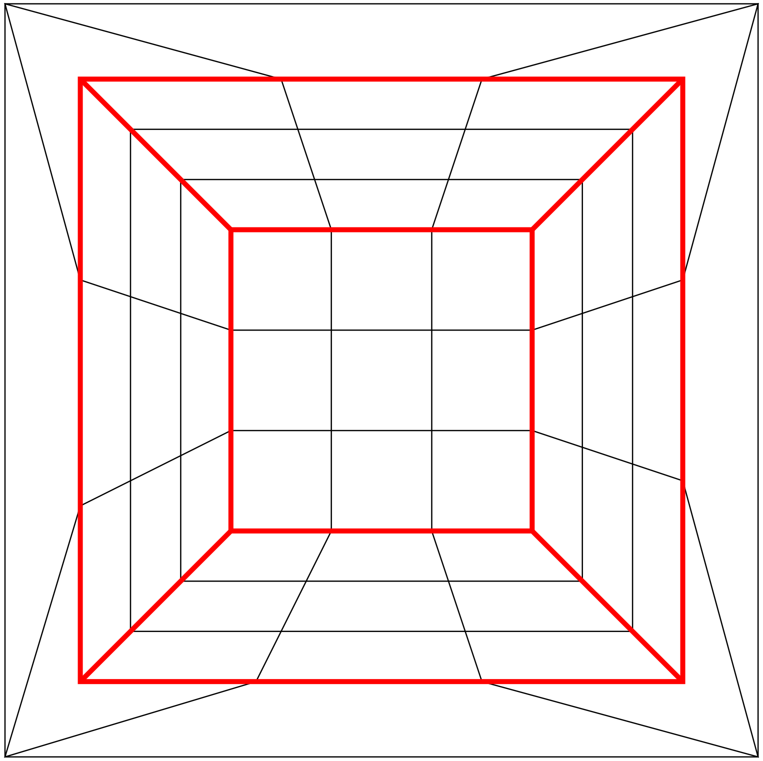
Case 4-valent

Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



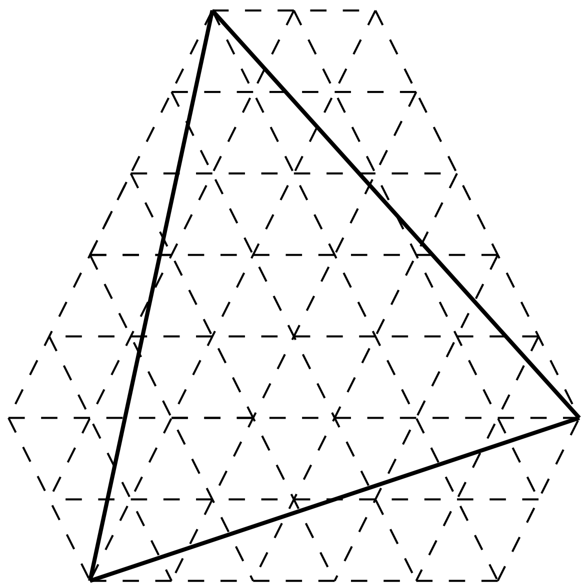
$(3,0)$: 3-valent



$(3,0)$: 4-valent

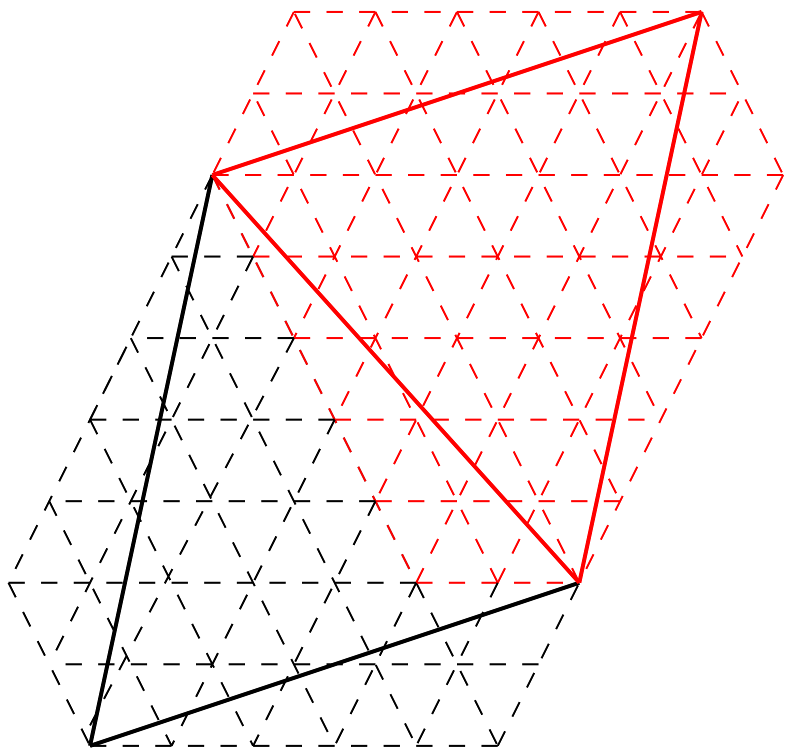
Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



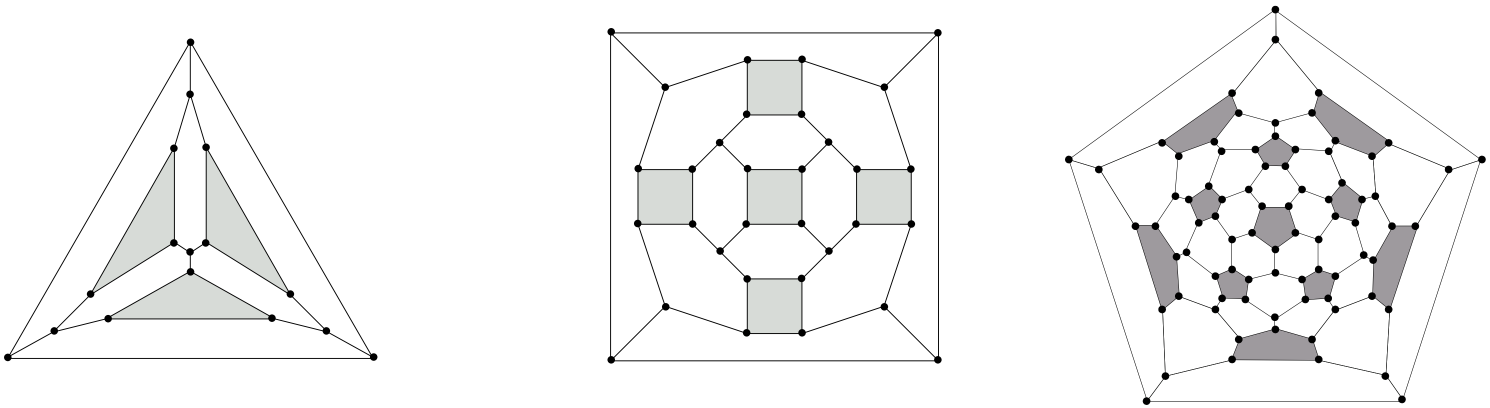
Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



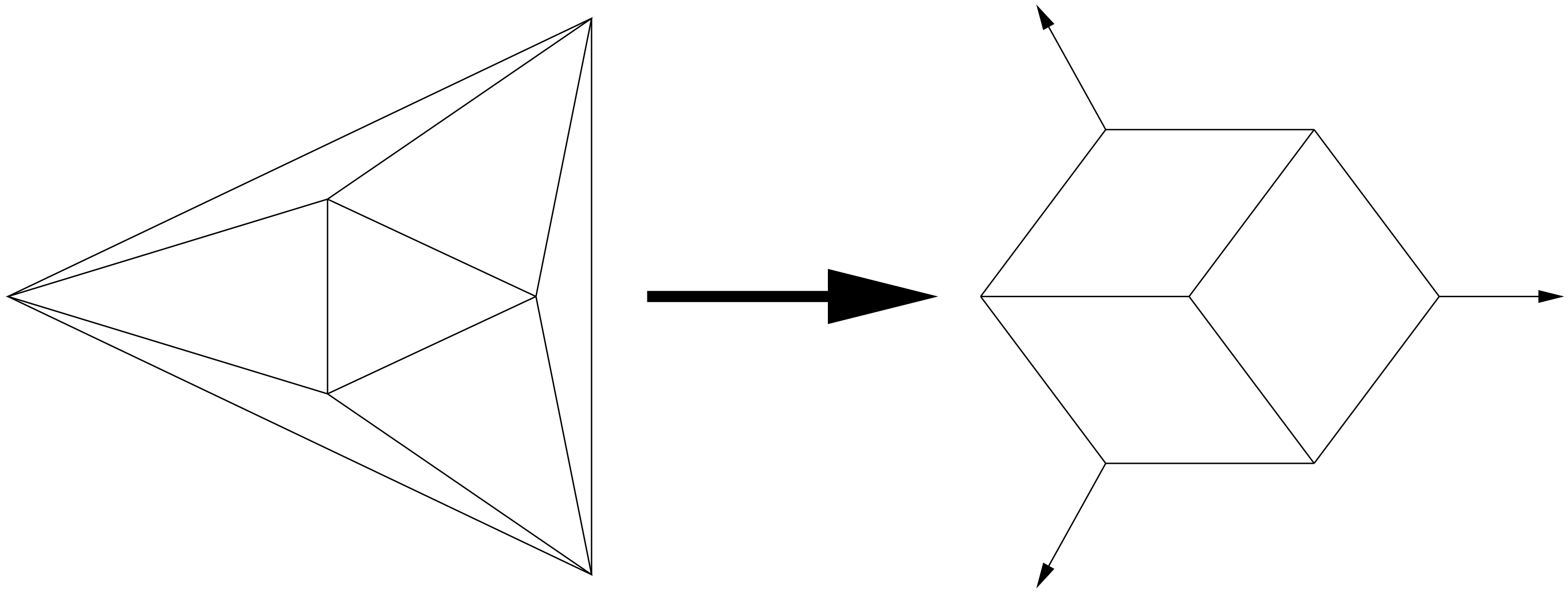
Final steps

- Go to the dual and obtain a 3- or 4-valent plane graph, which is denoted $GC_{k,l}(G_0)$ and called "Goldberg-Coxeter construction".
- The construction works for any 3- or 4-valent map on oriented surface.

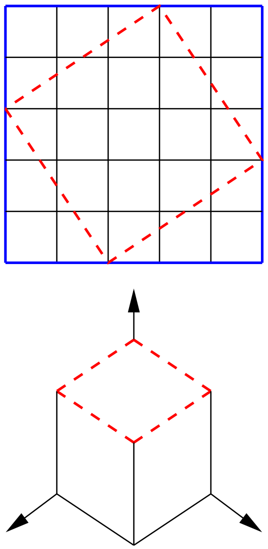


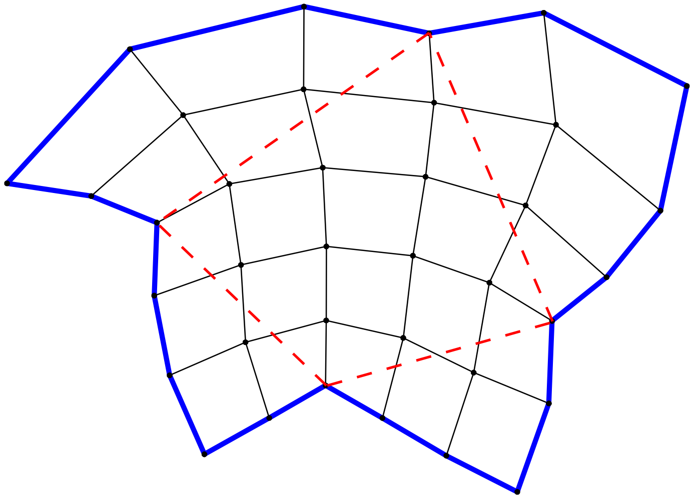
Operation $GC_{2,0}$

Example of $GC_{3,2}(Octahedron)$

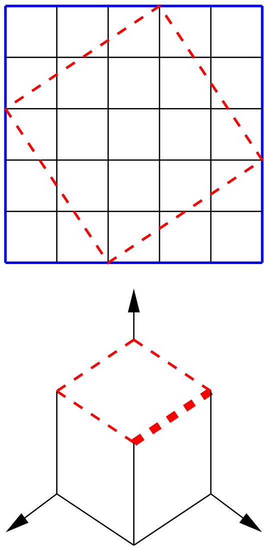


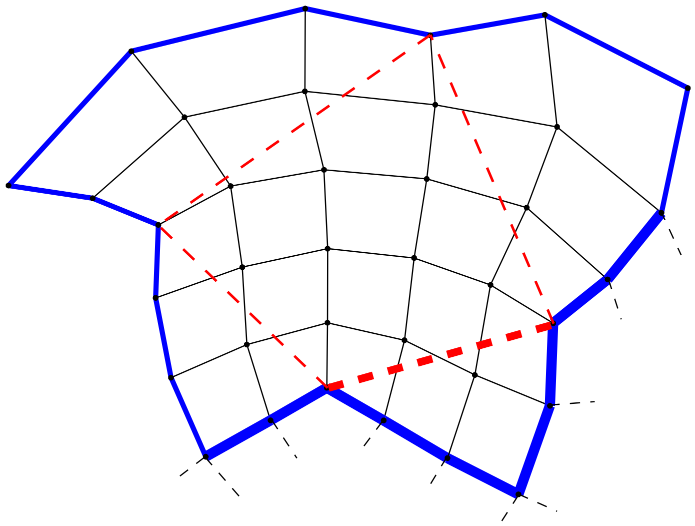
Example of $GC_{3,2}(Octahedron)$



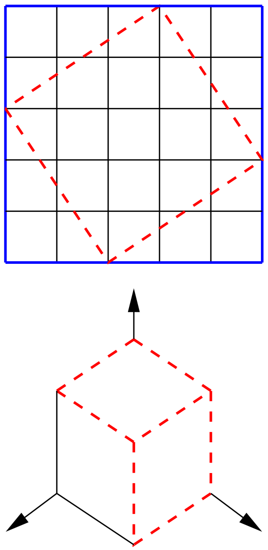


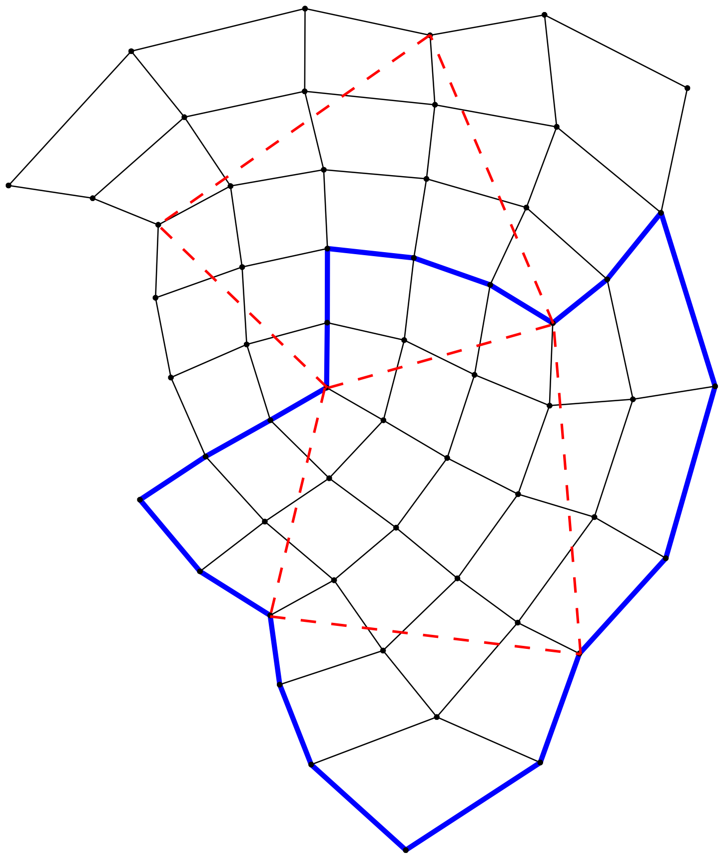
Example of $GC_{3,2}(Octahedron)$



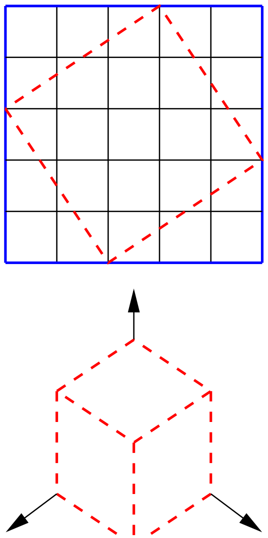


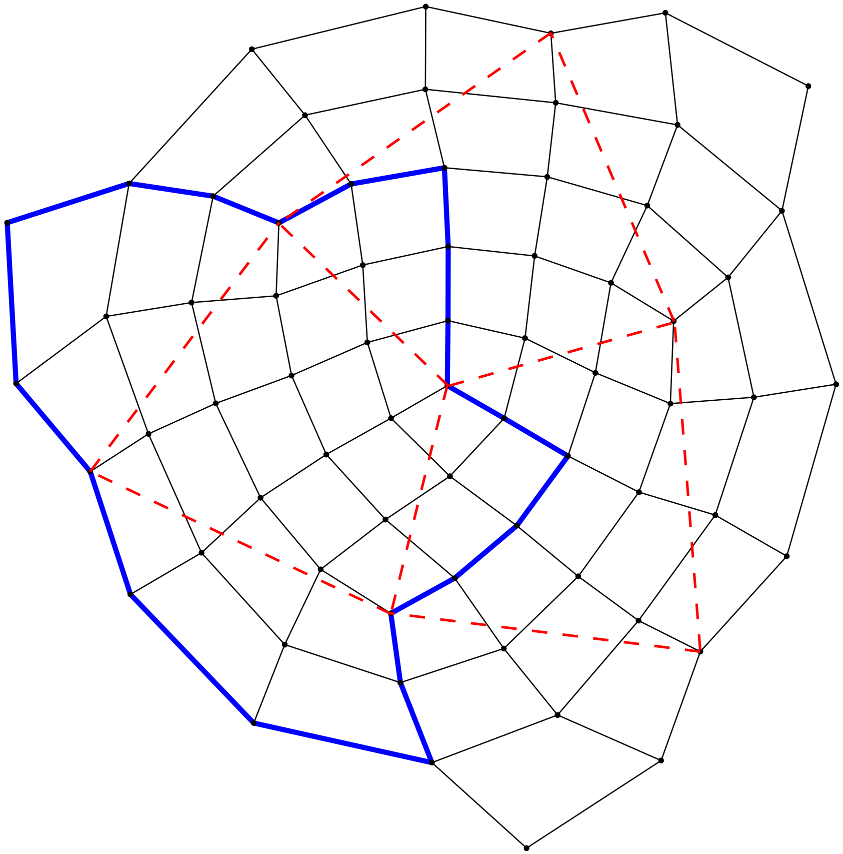
Example of $GC_{3,2}(Octahedron)$



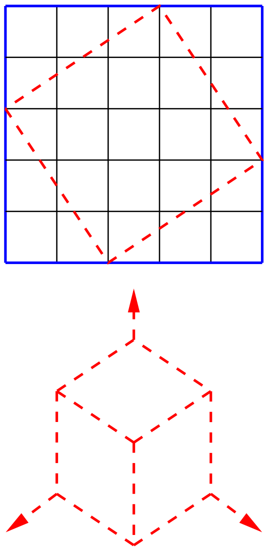


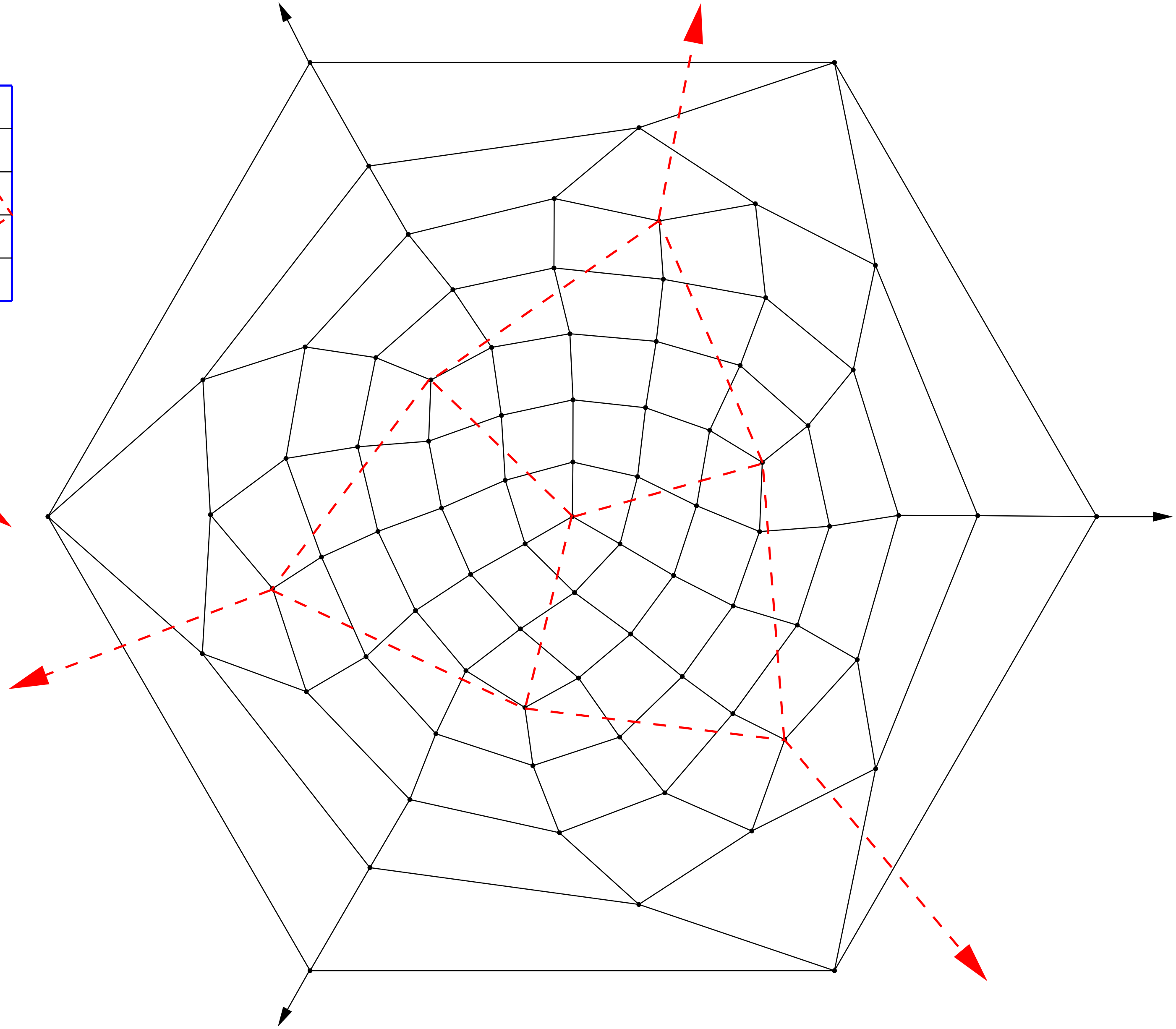
Example of $GC_{3,2}(Octahedron)$



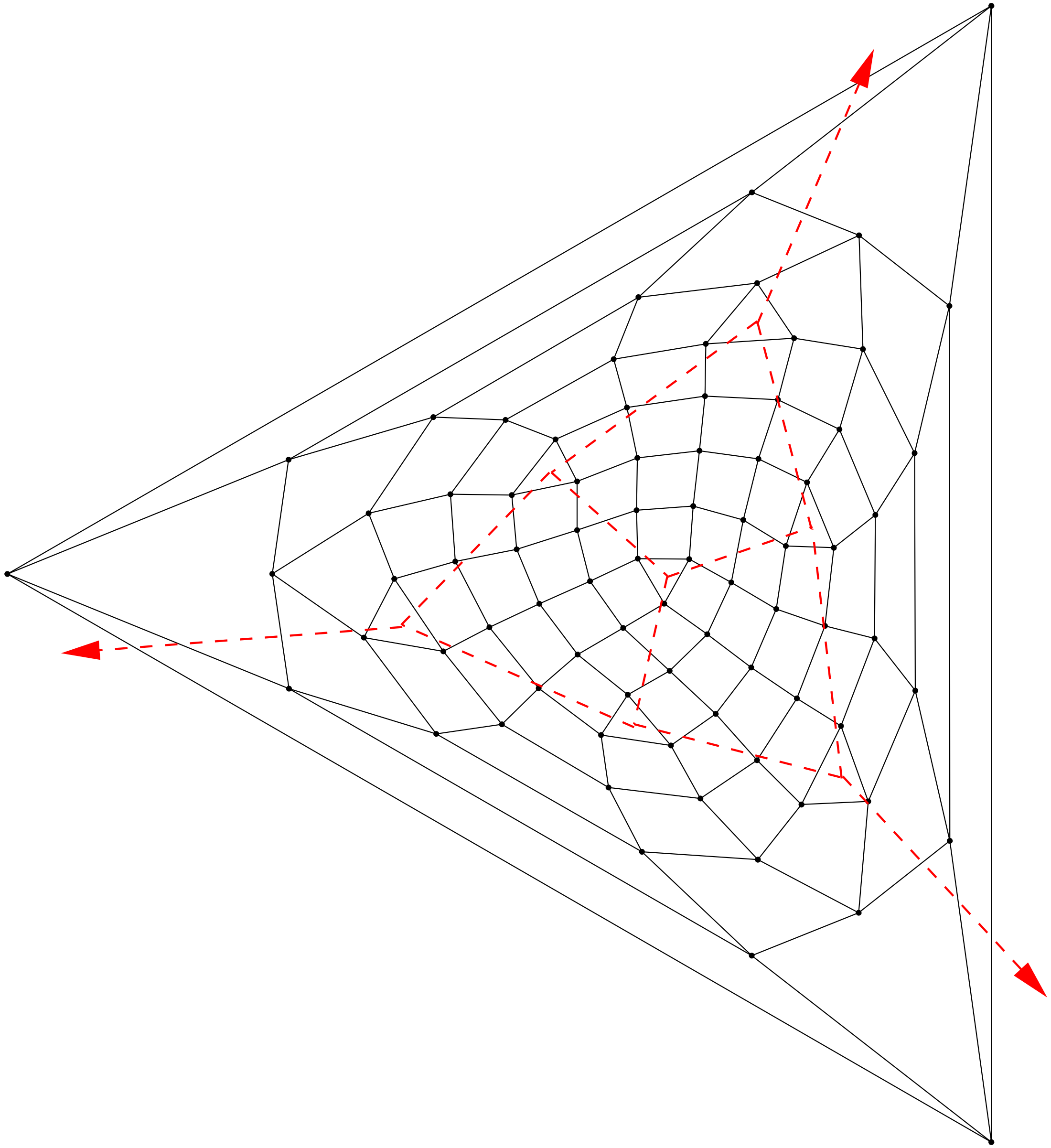


Example of $GC_{3,2}(Octahedron)$





Example of $GC_{3,2}(Octahedron)$



Properties

- One associates $z = k + le^{i\frac{\pi}{3}}$(Eisenstein integer) or $z = k + li$(Gaussian integer) to the pair (k, l) in 3- or 4-valent case.
- If one writes $GC_z(G_0)$ instead of $GC_{k,l}(G_0)$, then one has:

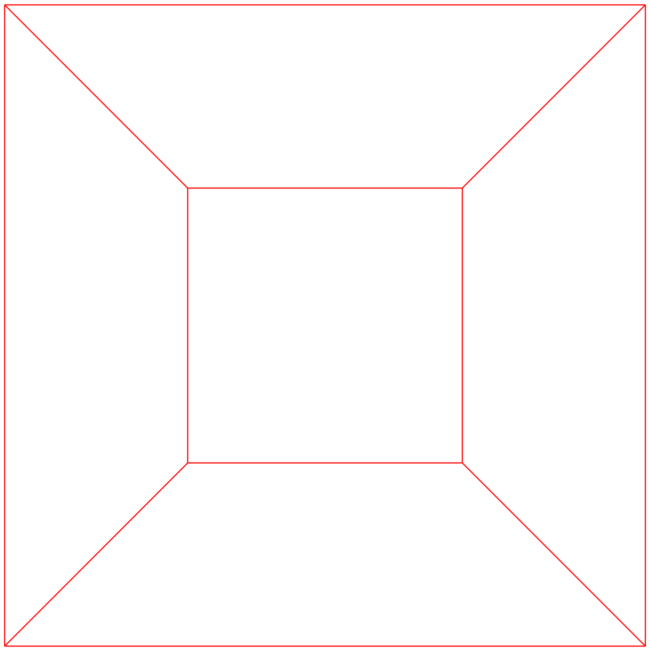
$$GC_z(GC_{z'}(G_0)) \quad = \quad GC_{zz'}(G_0)$$

- If G_0 has n vertices, then $GC_{k,l}(G_0)$ has

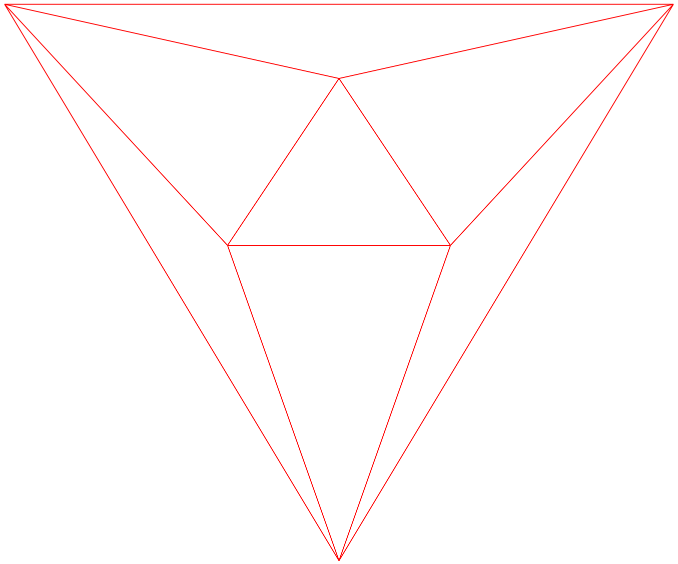
$$n(k^2 + kl + l^2) = n|z|^2 \text{ vertices} \quad \text{if} \quad G_0 \text{ is } 3\text{-valent,}$$
$$n(k^2 + l^2) = n|z|^2 \text{ vertices} \quad \text{if} \quad G_0 \text{ is } 4\text{-valent.}$$

- If G_0 has a plane of symmetry, we reduce to $0 \leq l \leq k$.
- $GC_{k,l}(G_0)$ has all rotational symmetries of G_0 and all symmetries if $l = 0$ or $l = k$.

The case $(k, l) = (1, 1)$

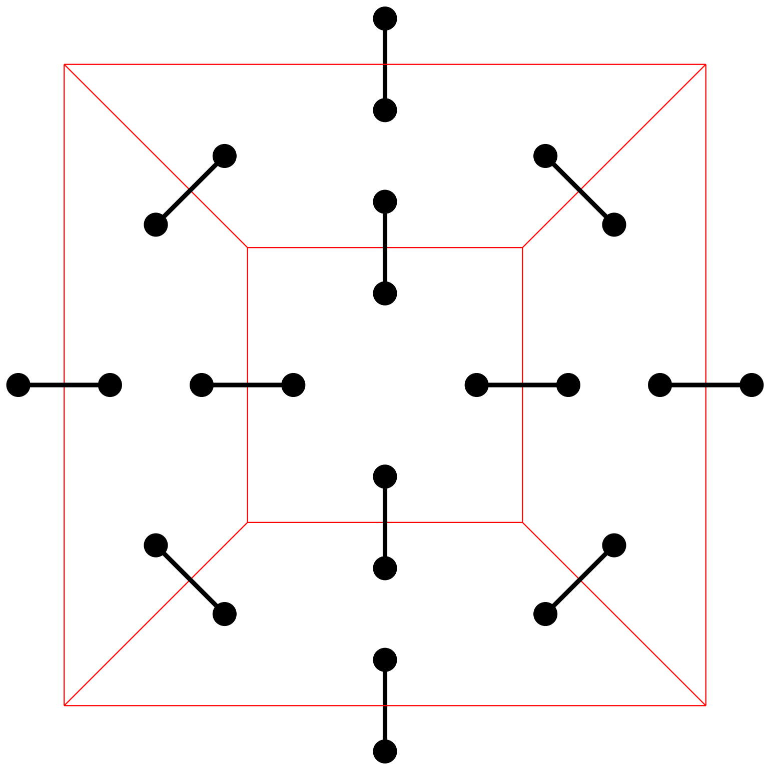


Case 3-valent

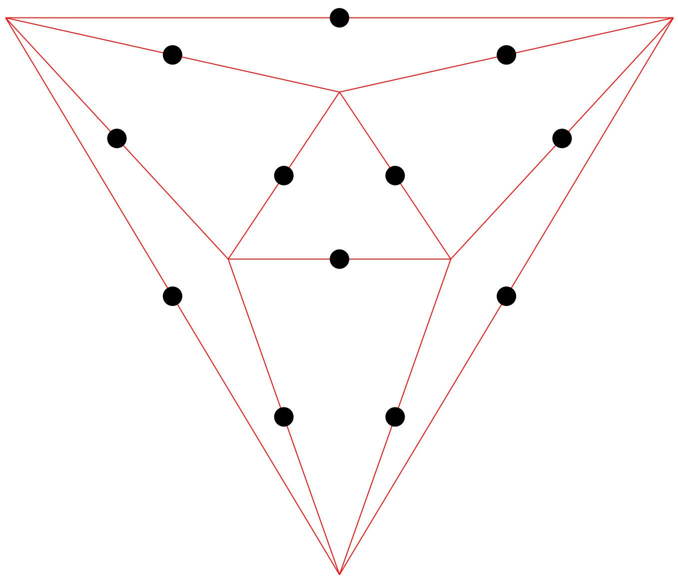


Case 4-valent

The case $(k, l) = (1, 1)$

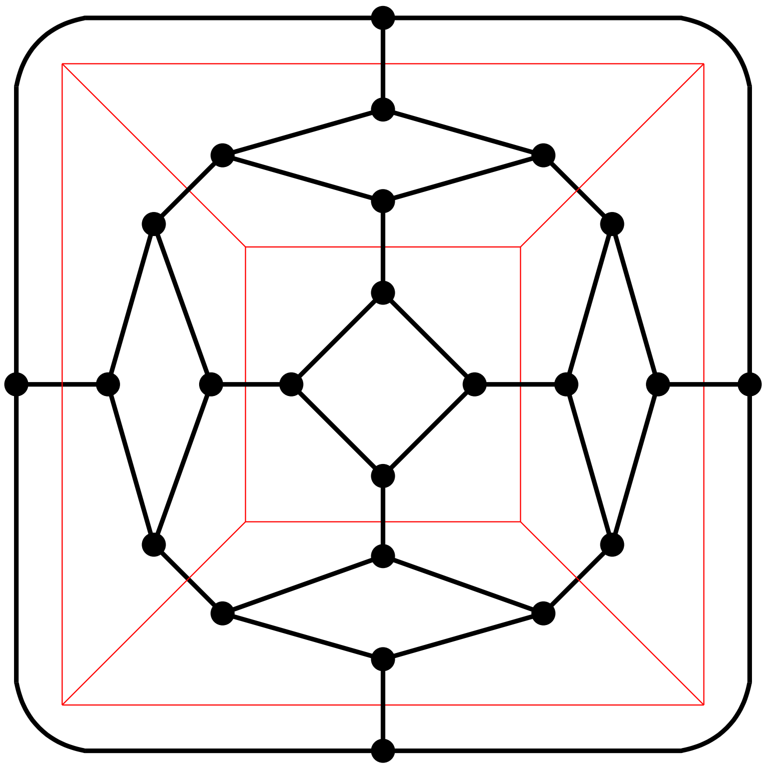


Case 3-valent

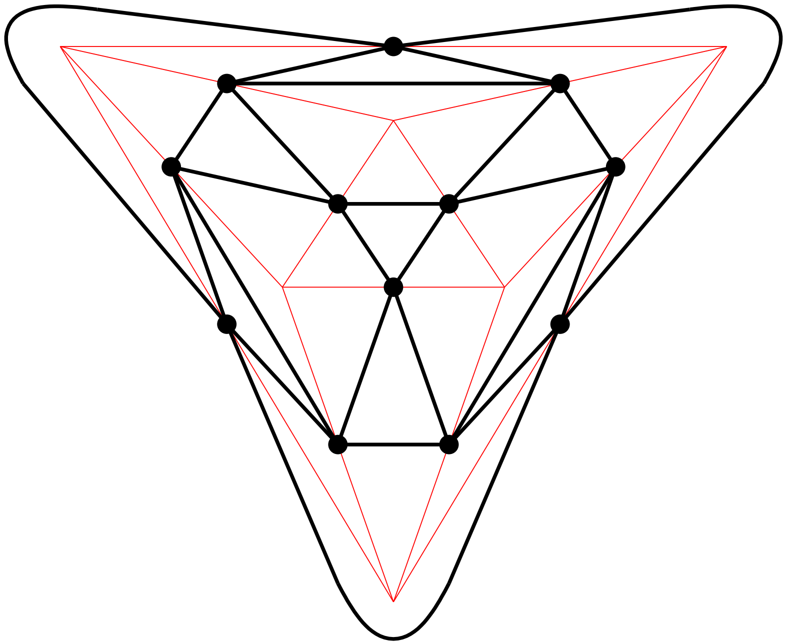


Case 4-valent

The case $(k, l) = (1, 1)$



Case 3-valent
$GC_{1,1}$ is called leapfrog
(=Truncation of the dual)



Case 4-valent
$GC_{1,1}$ is called medial

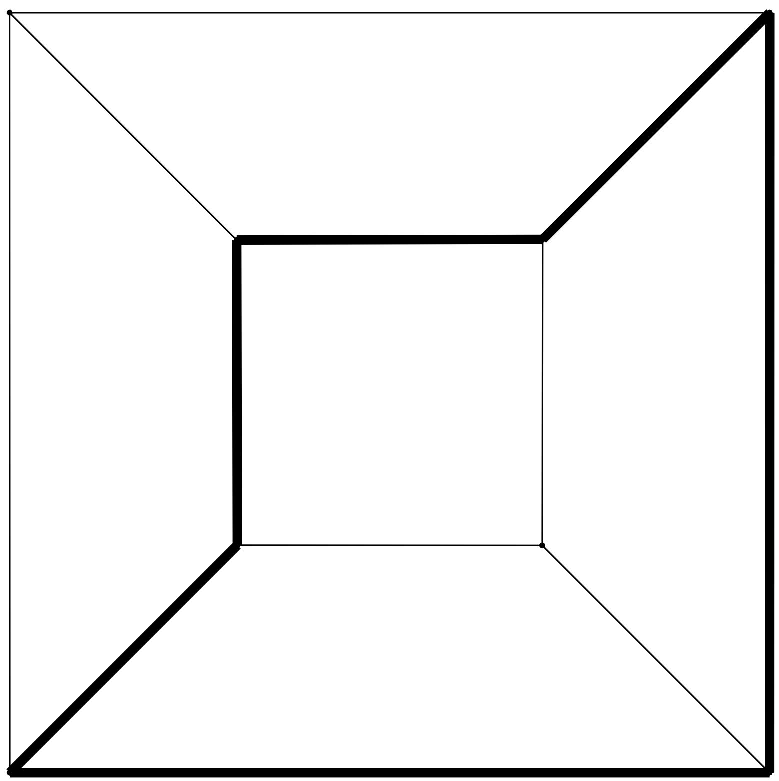
Goldberg Theorem

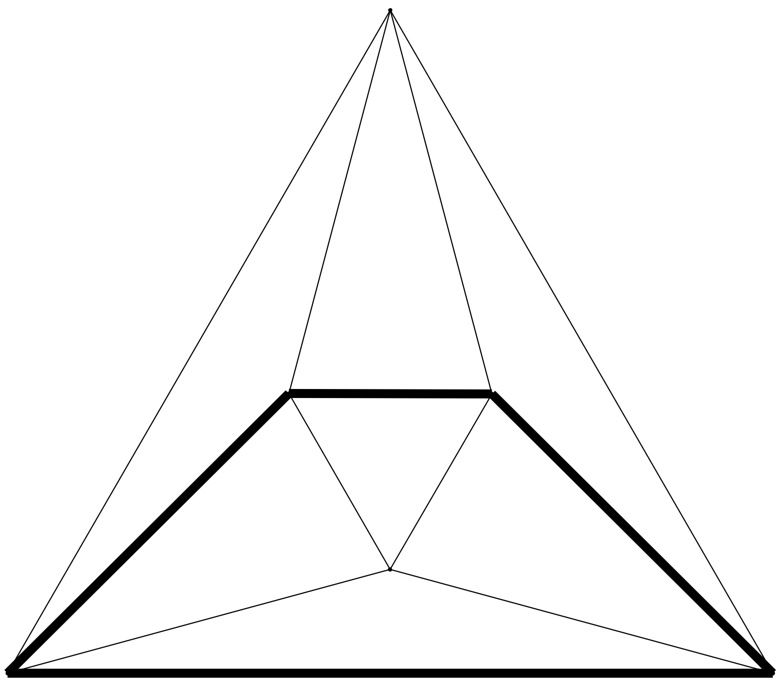
- q_n is the class of 3-valent plane graphs having only q- and 6-gonal faces.
- The class of 4-valent plane graphs having only 3- and 4-gonal faces is called Octahedrites.

Class		Groups	Construction
3_n	$p_3 = 4$	T, T_d	$GC_{k,l}$(Tetrahedron)
4_n	$p_4 = 6$	O, O_h	$GC_{k,l}$(Cube)
4_n	$p_4 = 6$	D_6, D_{6h}	$GC_{k,l}$(Prism$_6$)
5_n	$p_5 = 12$	I, I_h	$GC_{k,l}$(Dodecahedron)
Octahedrites	$p_3 = 8$	O, O_h	$GC_{k,l}$(Octahedron)

The special case $GC_{k,0}$

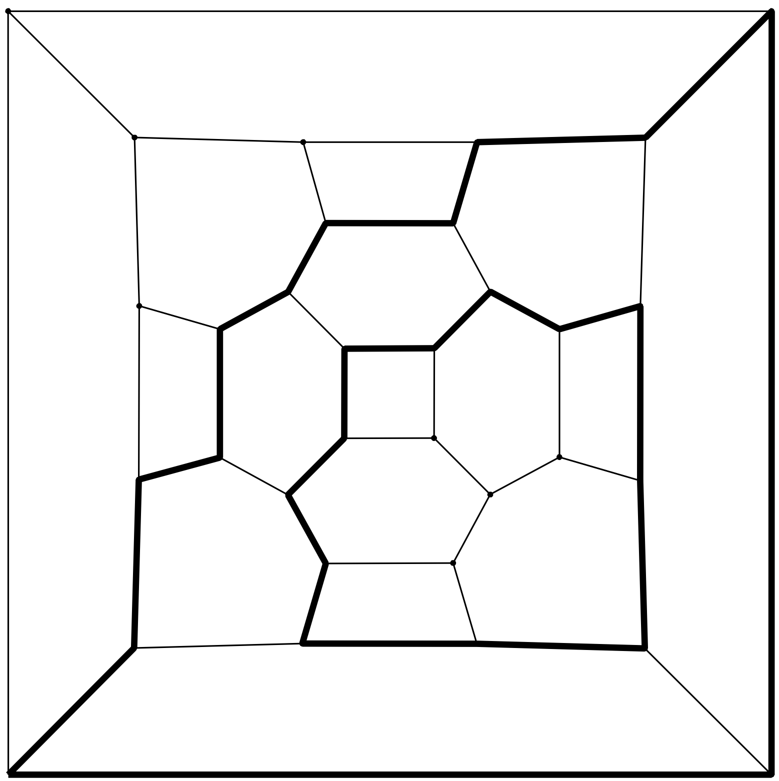
- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.

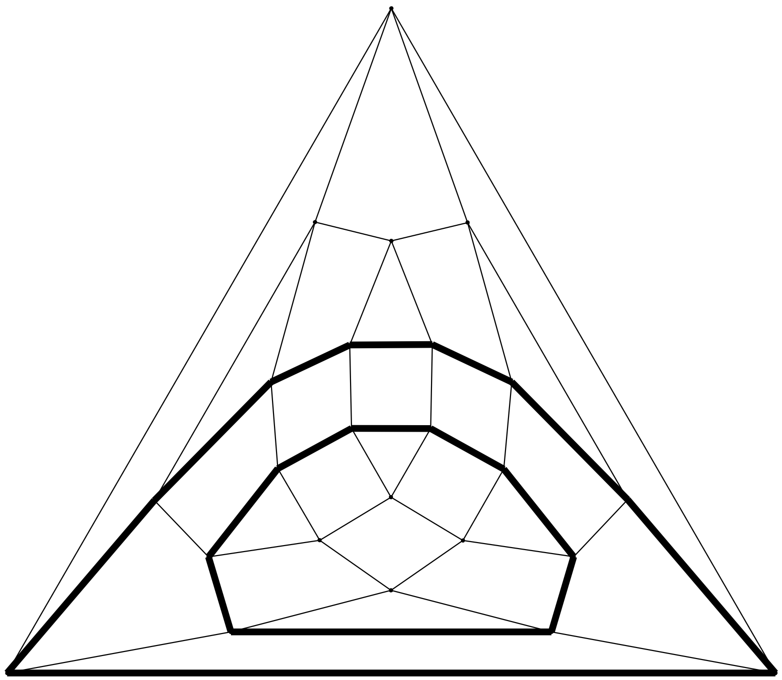




The special case $GC_{k,0}$

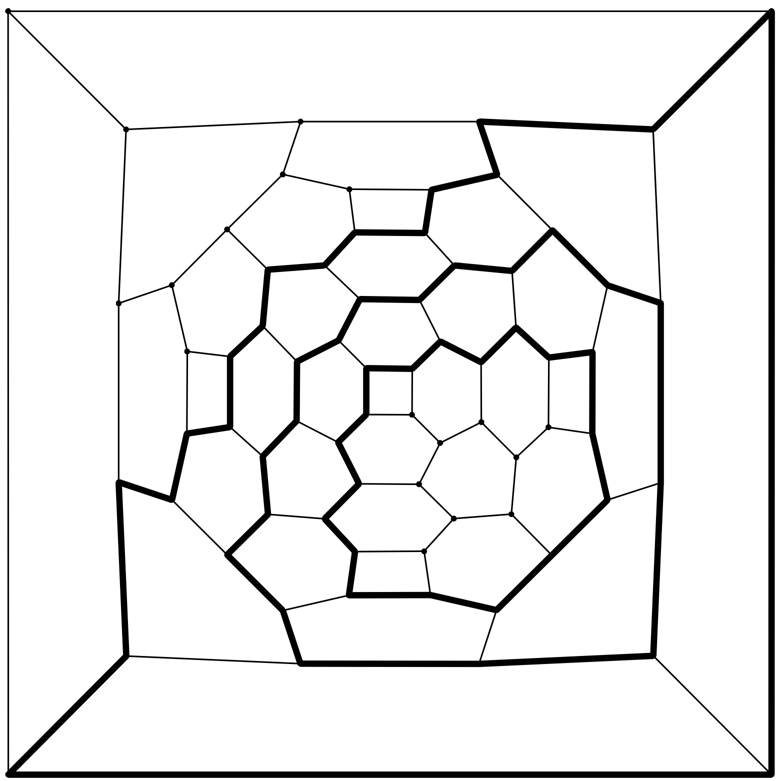
- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.

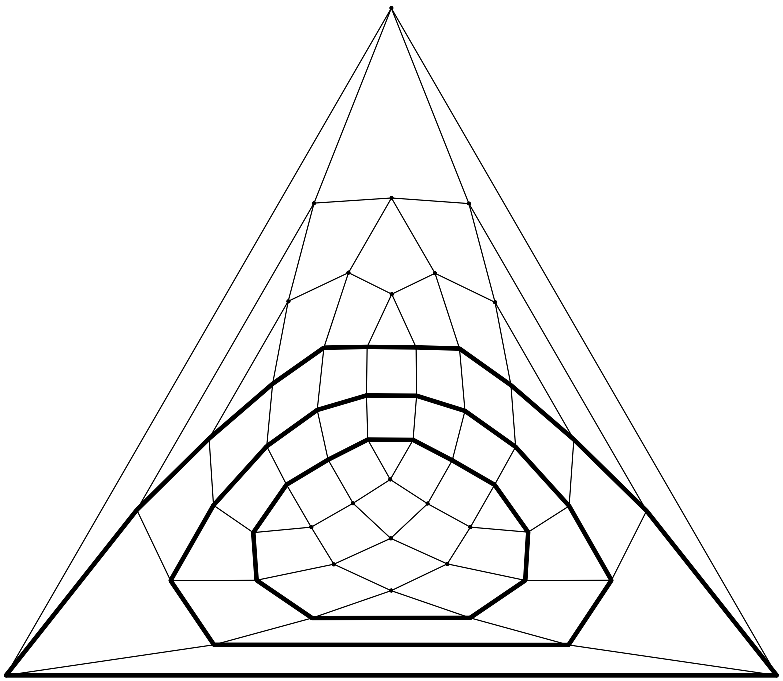




The special case $GC_{k,0}$

- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.





III. The (k, l)-product

The mapping $\phi_{k,l}$

We always assume $gcd(k,l) = 1$

$$\left\{ \begin{array}{rcl} \phi_{k,l} : \{1,\ldots,k+l\} & \longrightarrow & \{1,\ldots,k+l\} \\ u & \longmapsto & \left\{ \begin{array}{lll} u+l & \text{if} & u \in \{1,\ldots,k\} \\ u-k & \text{if} & u \in \{k+1,\ldots,k+l\} \end{array} \right. \end{array} \right.$$

is bijective and periodic with period $k+l$.

Example: Case $k = 5$, $l = 2$:

$\phi^{(s)}(1) = 1, 3, 5, 7, 2, 4, 6, 1, \ldots$

operations:$(+2), (+2), (+2), (-5), (+2), (+2), (-5)$

The (k,l)-product

- **Definition 1** *(The (k,l)-product)*
  If L and R are two elements of a group, $k,l \geq 0$ and $gcd(k,l) = 1$; we define $(p_0, \ldots, p_{k+l})$ by $p_0 = 1$ and $p_i = \phi_{k,l}(p_{i-1})$.
  Set $S_i = L$ if $p_i - p_{i-1} = l$ and $S_i = R$ if $p_i - p_{i-1} = -k$; then set
  $$L \odot_{k,l} R = S_{k+l} \ldots S_2 \cdot S_1.$$
  By convention, set $L \odot_{1,0} R = L$ and $L \odot_{0,1} R = R$.
  For $k = 5$, $l = 2$, one gets the expression
  $$L \odot_{5,2} R = RLLRLLL$$
- A similar notion is introduced by Norton (1987) in "Generalized Moonshine" for the Monster group.

Properties

- If L and R commute, $L \odot_{k,l} R = L^k R^l$
- Euclidean algorithm formula

$$\begin{cases} L \odot_{k,l} R = L \odot_{k-ql\,,\,l} RL^q & \text{if } k - ql \geq 0 \\ L \odot_{k,l} R = R^q L \odot_{k\,,\,l-qk} R & \text{if } l - qk \geq 0 \end{cases}$$

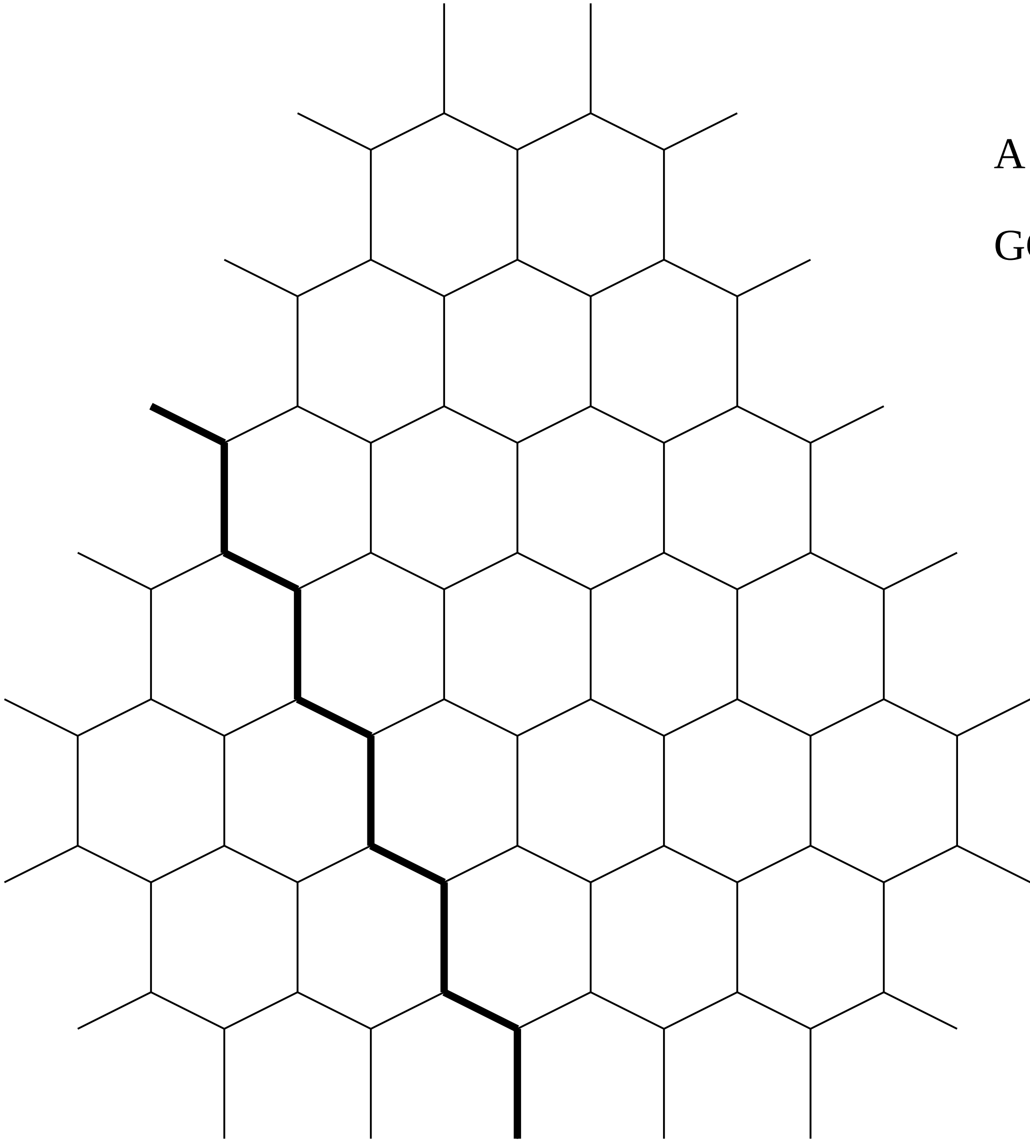
➠ If L and R do not commute, then $L \odot_{k,l} R \neq Id$.

IV. ZC-circuits

in

$GC_{k,l}(G_0)$

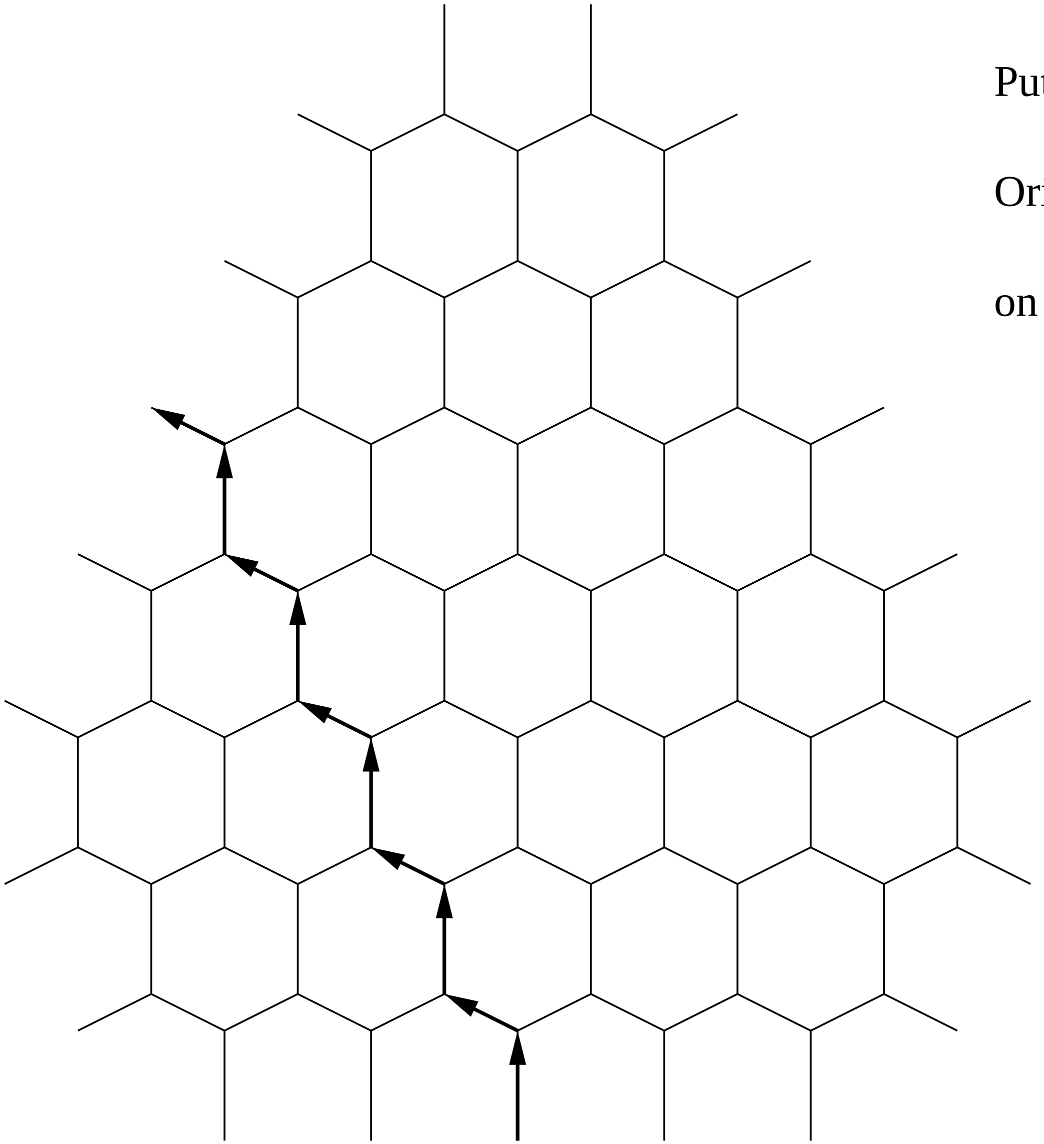
Position mapping, 3-valent case



A zigzag in

$GC_{5,2}$ (G_0)

Position mapping, 3-valent case

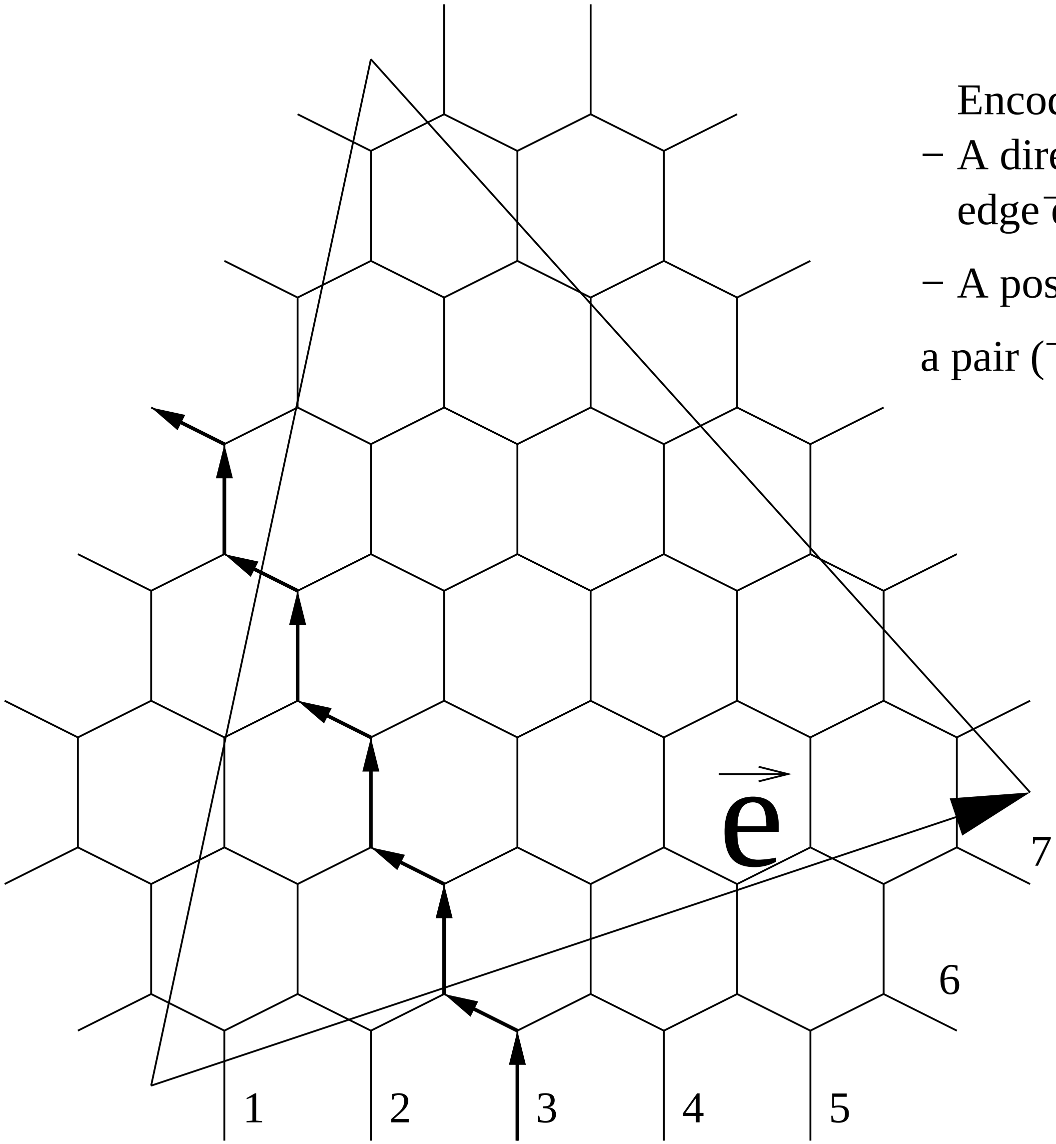


Put an

Orientation

on it

Position mapping, 3-valent case

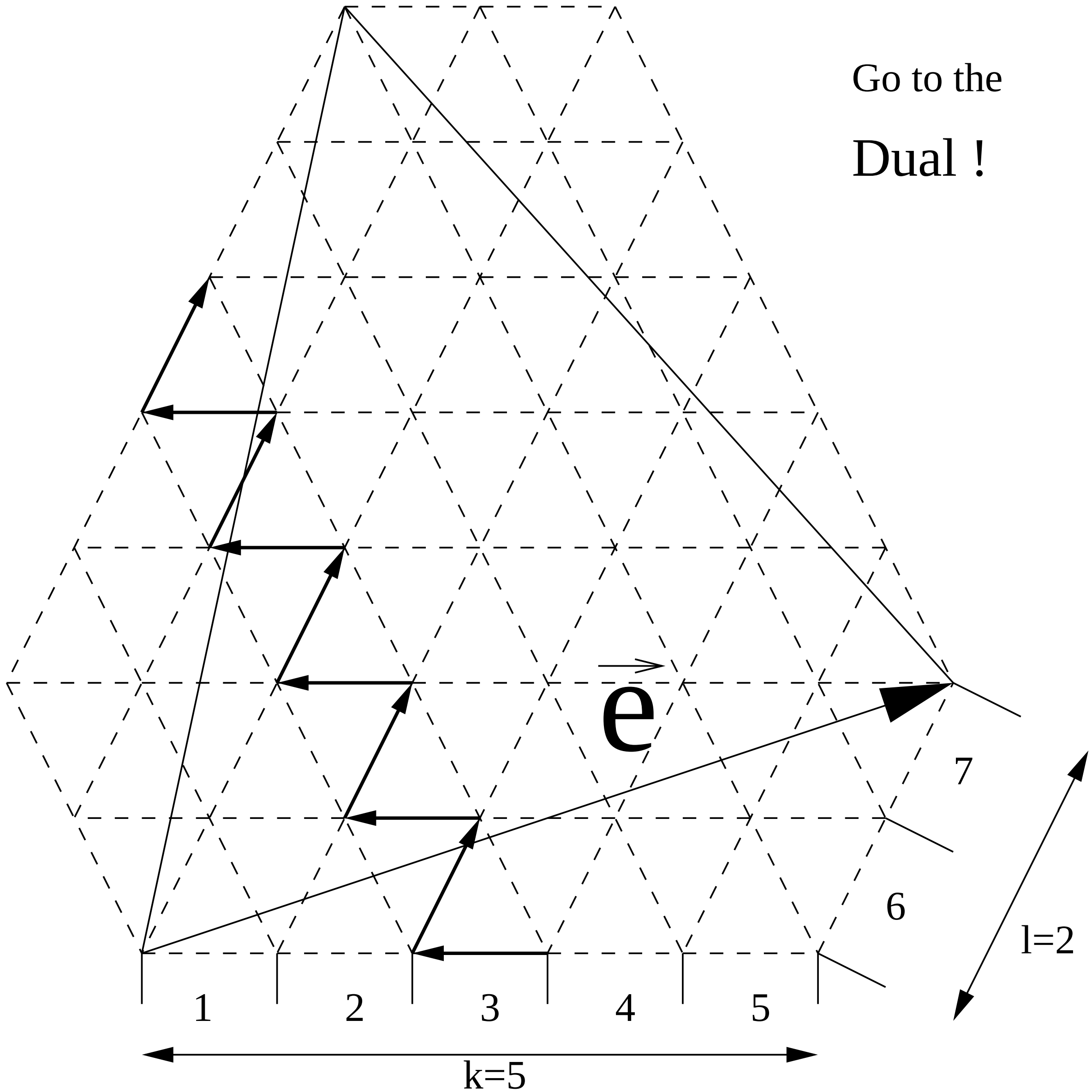


Encoding:

- A directed edge $\vec{e}$
- A position p=3

a pair ($\vec{e}$, 3)

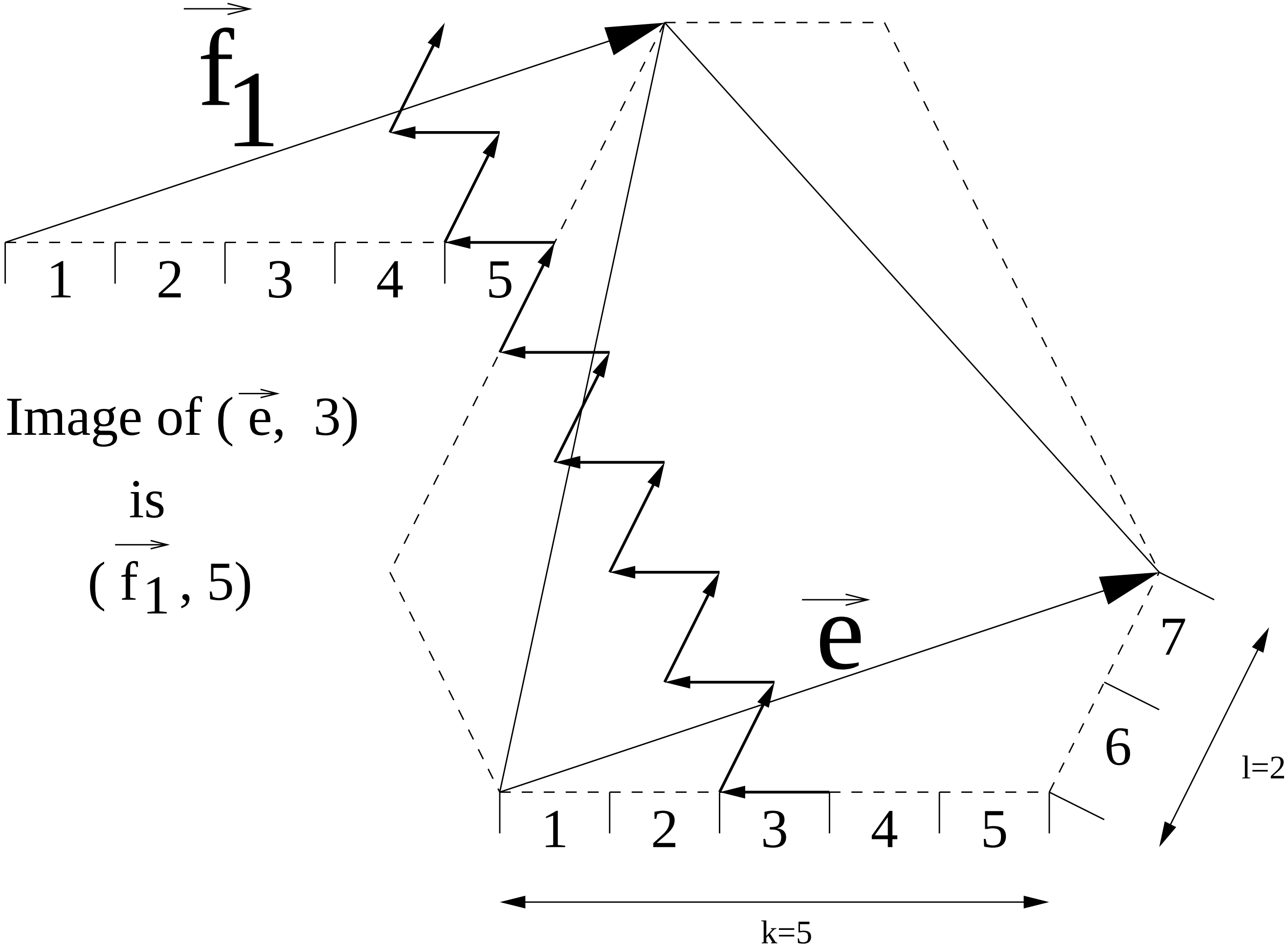
Position mapping, 3-valent case



Go to the

Dual !

Position mapping, 3-valent case

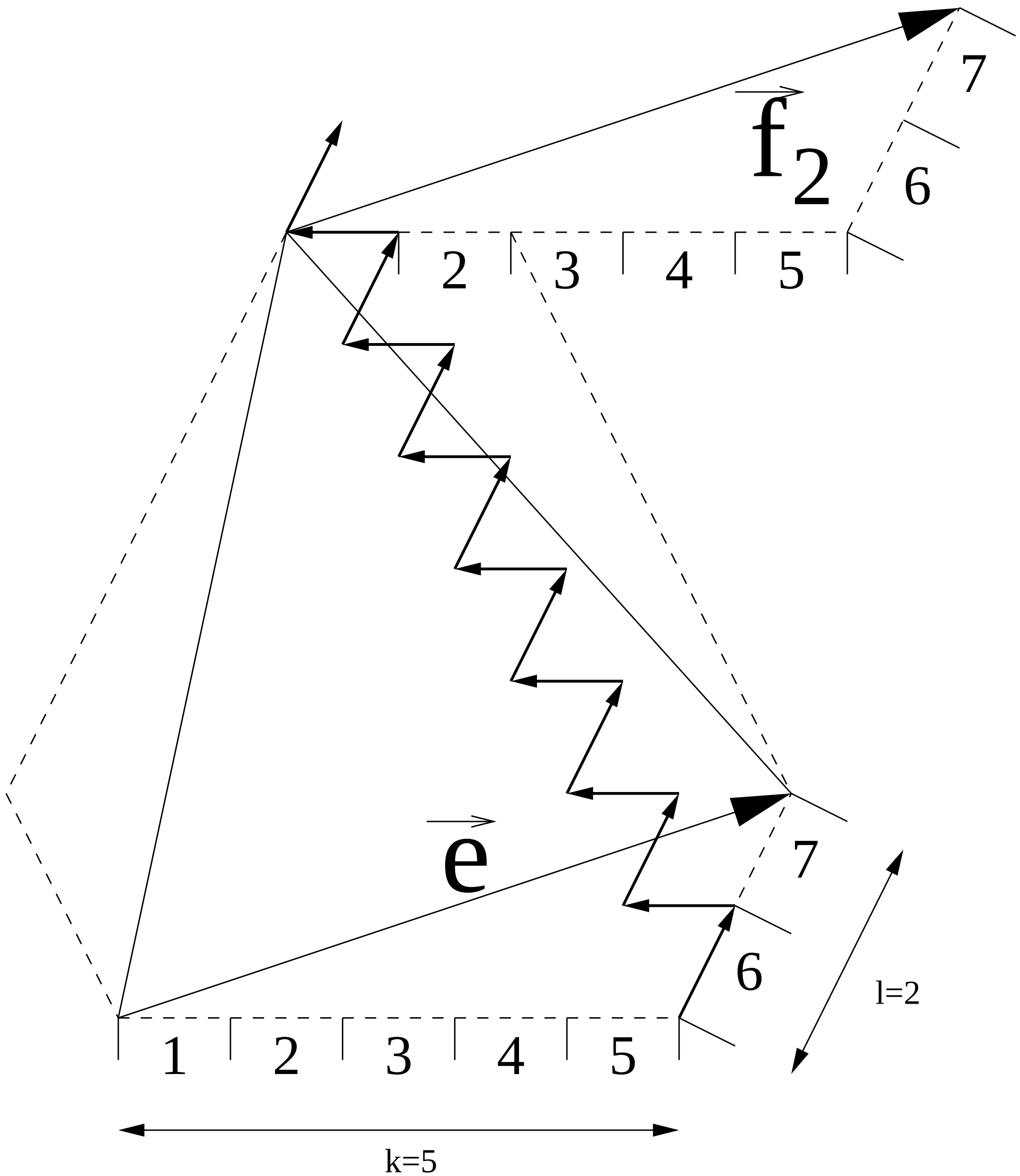


Position mapping, 3-valent case

Image of ($\vec{e}$, 6)

is

($\vec{f}_2$, 1)



Iteration

- "Position mapping" is denoted $PM(\overrightarrow{e}, p)$=$(\overrightarrow{f}_1, \phi_{k,l}(p))$ or $(\overrightarrow{f}_2, \phi_{k,l}(p))$
- $PM^{k+l}(\overrightarrow{e}, 1)$=$(\overrightarrow{e}', 1)$. So, one defines "Iterated position mapping" as $IPM(\overrightarrow{e}) = \overrightarrow{e}'$.
- $\mathcal{DE}$ is the set of directed edges of G_0^*. IPM is a permutation of $\mathcal{DE}$.
- For every ZC-circuit with pair $(\overrightarrow{e}, 1)$ denote $Ord(ZC)$ the smallest $s > 0$, such that $IPM^s(\overrightarrow{e}) = \overrightarrow{e}$.

➠ For any ZC-circuit of $GC_{k,l}(G_0)$ one has:

$\mathrm{length}(ZC)$=$2(k^2 + kl + l^2)Ord(ZC)$	3-valent case
$\mathrm{length}(ZC)$=$(k^2 + l^2)Ord(ZC)$	4-valent case

The [ZC]-vector of $GC_{k,l}(G_0)$ is the vector $\dots, c_k^{m_k}, \dots$ where m_k is the number of ZC-circuits with order c_k.

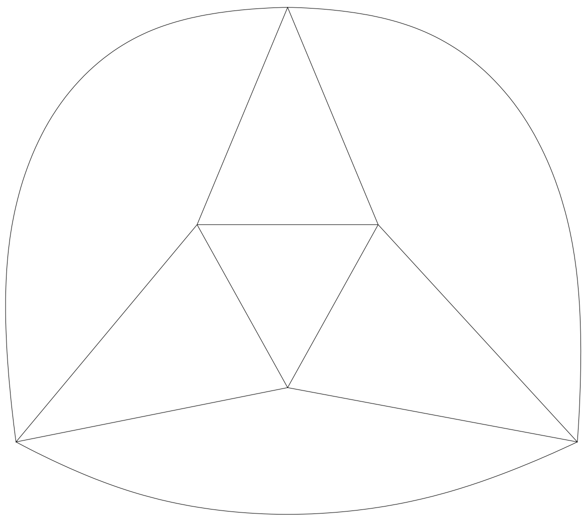
The mappings L and R

- L and R are the following permutation of $\mathcal{DE}$

$$L : \overrightarrow{e} \to \overrightarrow{f}_1 \qquad\qquad R : \overrightarrow{e} \to \overrightarrow{f}_2$$

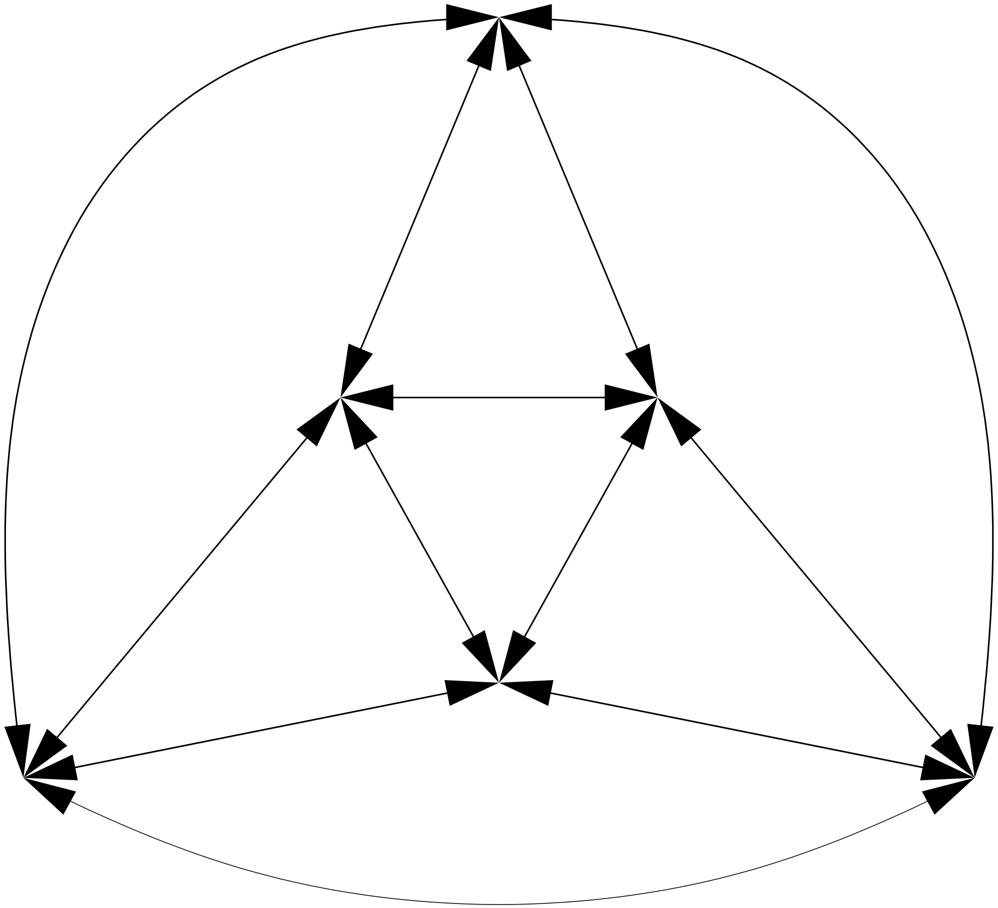
with $\overrightarrow{f}_1$ and $\overrightarrow{f}_2$ being the first and second choice.
Example of Cube

Dual Cube



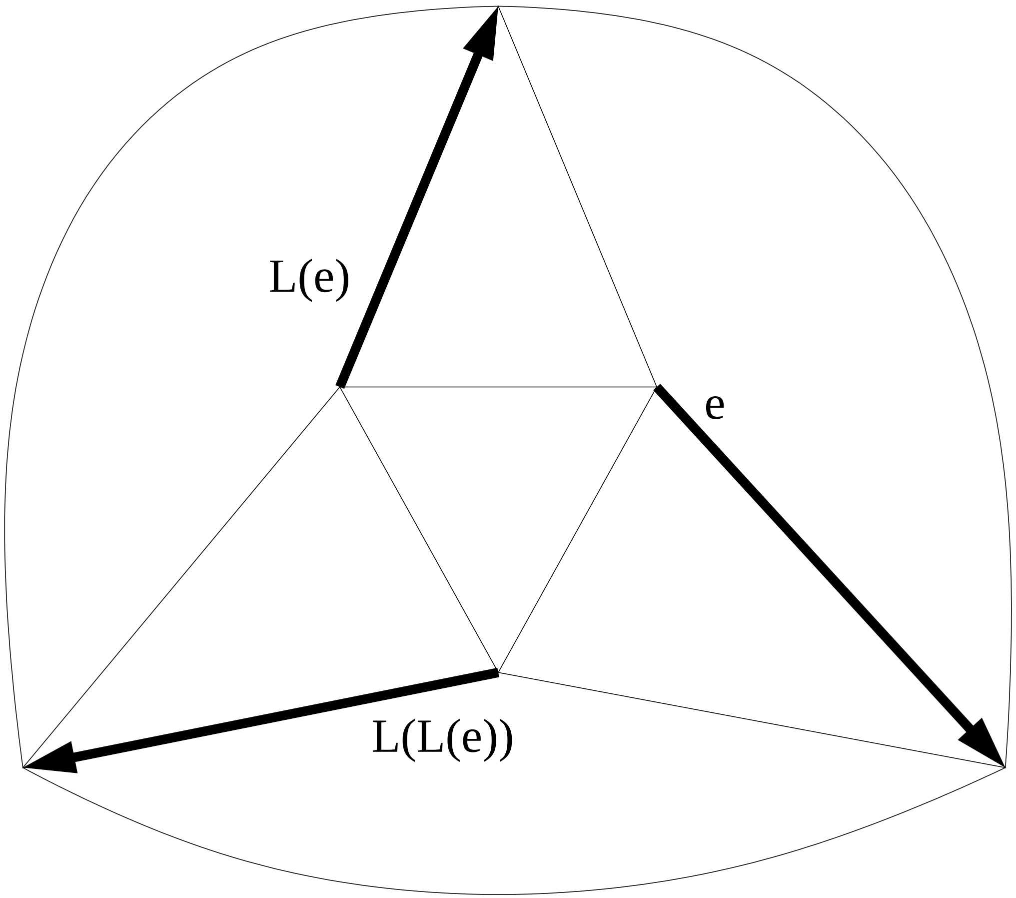
The mappings L and R

24 Directed Edges



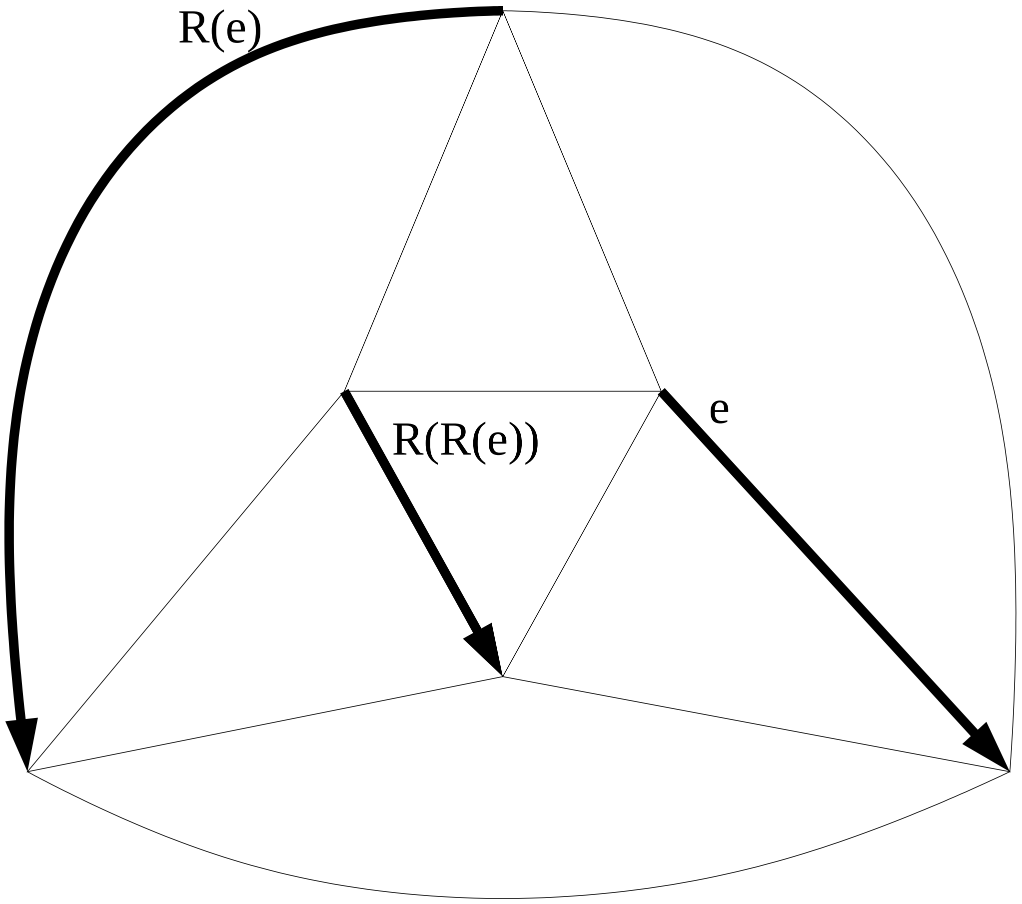
The mappings L and R

Successive images of e by L



The mappings L and R

Successive images of e by R



Moving group and Key Theorem

- $Mov(G_0) = \langle L, R \rangle$ is the moving group
  In Cube: a subgroup of $Sym(24)$.
- For $u \in Mov(G_0)$, denote $ZC(u)$ the vector $\dots, c_k^{m_k}, \dots$ with multiplicities m_k being the half of the number of cycles of length c_k in the permutation u acting on the set $\mathcal{DE}$.
  In Cube: $ZC(L) = ZC(R) = 3^4$
- Key Theorem One has for all 3- or 4-valent plane graphs G_0 and all $k, l \geq 0$

$$[ZC] - vector \text{ of } GC_{k,l}(G_0) = ZC(L \odot_{k,l} R)$$

Solution of the Cube case

- L and R do not commute ➠ $L \odot_{k,l} R \neq Id$.
- $Mov(Cube) = \langle L, R \rangle = Alt(4)$
- $K = \langle (1,2)(3,4), (1,3)(2,4) \rangle$ normal subgroup of index 3 of $Alt(4)$. $\overline{L}$ is of order 3.

$$\begin{cases} \overline{L \odot_{k,l} R} = \overline{L}^k \overline{R}^l = \overline{L}^{k-l} \\ L \odot_{k,l} R \in K \Leftrightarrow k - l \text{ divisible by } 3 \end{cases}$$

- Elements of $Alt(4) - K$ have order 3. Elements of $K - \{Id\}$ have order 2.

➠ $GC_{k,l}$(Cube) has [ZC]=2^6 if $k \equiv l \pmod 3$ and [ZC]=3^4, otherwise

Possible [ZC]-vectors

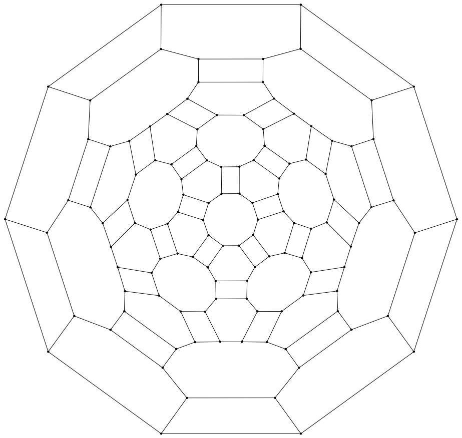
- Denote $\mathcal{P}(G_0)$ the set of all pairs (g_1, g_2) with $g_i \in Mov(G_0)$.
- Denote $U_{L,R}$ the smallest subset of $\mathcal{P}(G_0)$, which contains the pair (L, R) and is stable by the two operations

$$(x, y) \mapsto (x, yx) \quad \text{and} \quad (x, y) \mapsto (yx, y)$$

- Theorem: The set of possible [ZC]-vectors of $GC_{k,l}(G_0)$ is equal to the set of all vectors $ZC(v)$, $ZC(w)$ with $(v, w) \in U_{L,R}$.
- Computable in finite time for a given G_0.

Examples

- $Mov(\text{Dodecahedron}) = Alt(5)$ of order 60. Order of elements different from Id are 2, 3 or 5.
Possible [ZC] are 2^{15} or 3^{10} or 5^{6}.
- Mov(Klein Map)=$PSL_{F_7}(2)$ of order 168. Order of elements different from Id are 2, 3, 4 or 7.
Possible [ZC] are 3^{28} or 4^{21}.
- $Mov(\text{Truncated Icosidodecahedron})$ has size 139968000000



$2^{30}, 3^{40}$	$2^{30}, 5^{24}$	$3^{20}, 5^{24}$
$2^{60}, 3^{20}$	$2^{60}, 5^{12}$	$3^{40}, 5^{12}$
2^{90}	3^{60}	5^{36}
9^{20}	6^{30}	15^{12}

V. $SL_2(\mathbb{Z})$ action

$SL_2(\mathbb{Z})$ action?

$\mathcal{P}(G_0)$ is the set of pairs (g_1, g_2). One has

$$L \odot_{k,l} R = L \odot_{k-l,l} RL \quad \text{and} \quad L \odot_{k,l} R = RL \odot_{k,l-k} R$$

The matrices $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$.

We want to define ϕ, such that

(i) ϕ is a group action of $SL_2(\mathbb{Z})$ on $\mathcal{P}(G_0)$

(ii) If $M \in SL_2(\mathbb{Z})$, then the mapping $\phi(M) : \mathcal{P}(G_0) \to \mathcal{P}(G_0)$ satisfies

$$\phi(M)(g_1, g_2) = (h_1, h_2) \Rightarrow g_1 \odot_{(k,l)M} g_2 = h_1 \odot_{k,l} h_2$$

This is in fact not possible!

$SL_2(\mathbb{Z})$ **action**

- $SL_2(\mathbb{Z})$ is generated by matrices

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } U = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

all relations between T and U are generated by the relations

$$T^4 = I_2, \quad U^3 = I_2 \text{ and } T^2U = UT^2$$

- We write

$$\begin{aligned} \phi(T)(g_1, g_2) &= (g_2, g_2 g_1^{-1} g_2^{-1}) \\ \phi(U)(g_1, g_2) &= (g_2, g_2 g_1^{-1} g_2^{-2}) \end{aligned}$$

$SL_2(\mathbb{Z})$ action (continued)

- By computation

$$\begin{aligned} \phi(T)^4(g_1, g_2) = \phi(U)^3(g_1, g_2) &= Int_{g_1 g_2^{-1} g_1^{-1} g_2}(g_1, g_2), \\ \phi(T)^2\phi(U)(g_1, g_2) &= \phi(U)\phi(T)^2(g_1, g_2) \ . \end{aligned}$$

- Group action of $SL_2(\mathbb{Z})$ on $\mathcal{P}(G_0)/D(Mov(G_0))$.
- If M preserve the element $\overline{(L, R)}$ in $\mathcal{P}(G_0)/D(Mov(G_0))$, then for all pairs (k, l):

$$GC_{k,l}(G_0) \quad \text{and} \quad GC_{(k,l)M}(G_0)$$

have the same [ZC]-vector. This define a

finite index subgroup of $SL_2(\mathbb{Z})$

Conjectured generators

Graph G_0	Generators of $Stab(G_0)$
Dodecahedron	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -4 & -3 \\ 3 & 2 \end{pmatrix}$, $\begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}$
Cube	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$
Octahedron	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -4 & -3 \\ 3 & 2 \end{pmatrix}$, $\begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}$

VI. Remarks

$Rot(G_0)$ transitive

$\mathcal{DE}$ is the set of all directed edges of G_0.

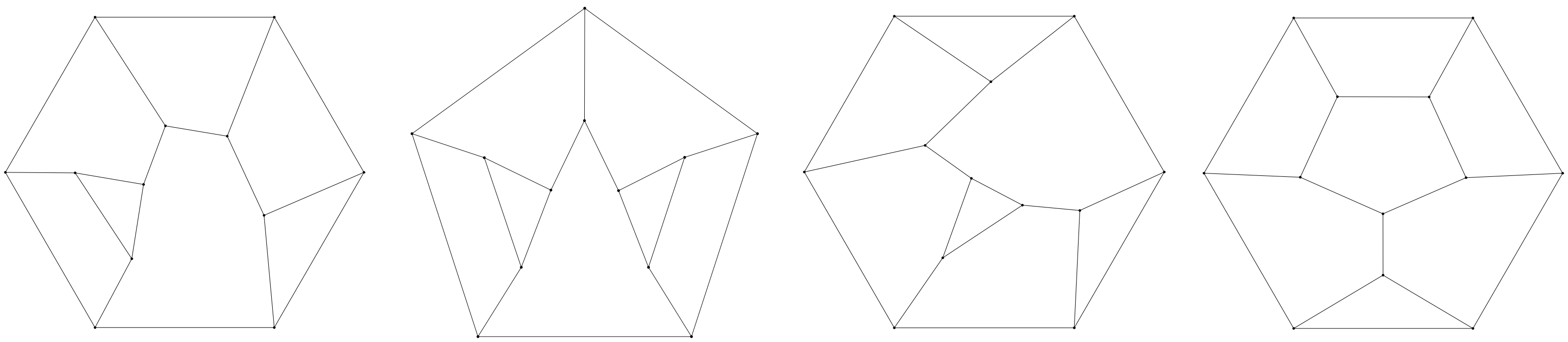
- $Rot(G_0)$: all rotations in automorphism group $Aut(G_0)$.
  - its action on $\mathcal{DE}$ is free.
  - action of $Rot(G_0)$ and $Mov(G_0)$ on $\mathcal{DE}$ commute.
- If $Rot(G_0)$ is transitive on $\mathcal{DE}$, then its action on ZC-circuit is transitive too and

$$\begin{cases} \phi_{\overrightarrow{e}} : Mov(G_0) & \longrightarrow & Rot(G_0) \\ u & \longmapsto & \phi_{\overrightarrow{e}}(u) \end{cases}$$

defined by $u^{-1}(\overrightarrow{e}) = \phi_{\overrightarrow{e}}(u)(\overrightarrow{e})$, is an injective group morphism. $\phi_{\overrightarrow{e}}(Mov(G_0))$ is normal in $Rot(G_0)$.

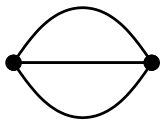
Extremal cases

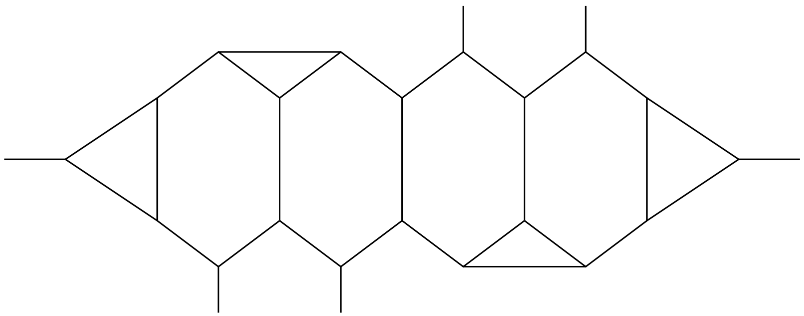
- $Rot(G_0)$ non-trivial $\Rightarrow$ restrictions on $Mov(G_0)$.
- $Rot(G_0)$ transitive on $\mathcal{DE} \Rightarrow |Mov(G_0)|$=$3n$ (3-valent case) or $= 4n$ (4-valent case).
- $Mov(G_0)$ is formed of even permutation on $3n$ or $4n$ directed edges.
- In some cases $Mov(G_0) = Alt(3n)$.



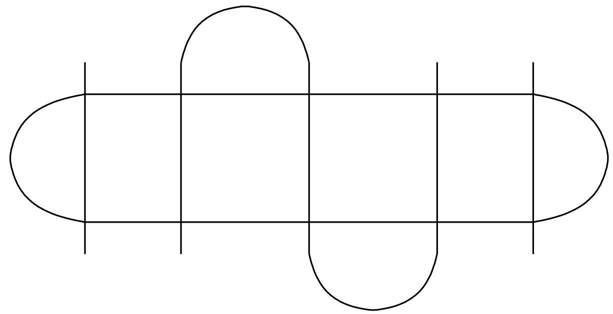
- We have no example of 4-valent plane graph G_0 with $Mov(G_0) = Alt(4n)$.

$Mov(G_0)$ commutative

- $Mov(G_0)$ commutative $\Leftrightarrow G_0$ is either a graph 2_n, a graph 3_n or a 4-hedrite.
- Class 2_n (Grunbaum-Zaks): Goldberg-Coxeter of the Bundle 



Class 3_n
(Grunbaum-Motzkin)



Class 4-hedrites
(Deza-Shtogrin)

- No other classes of graphs q_n or i-hedrites is known to admit such simple descriptions.

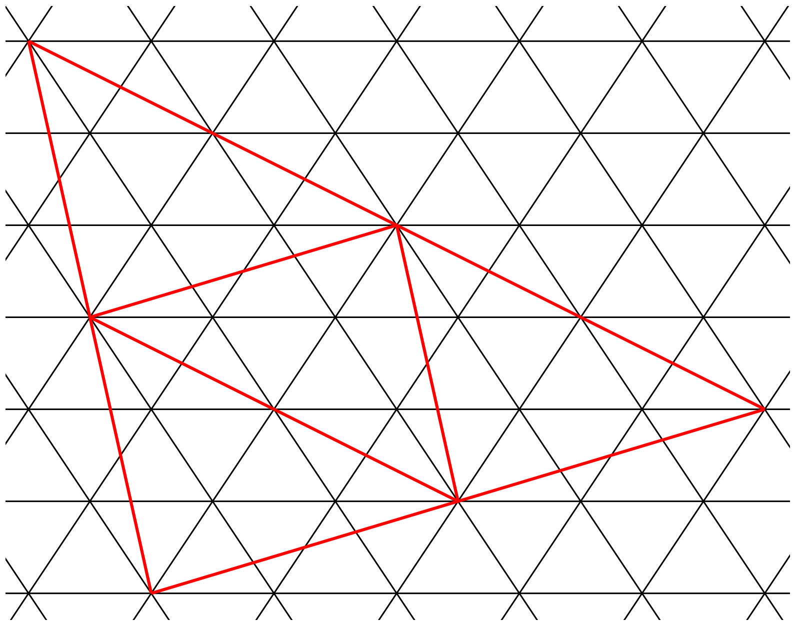
VII. Parametrizing graphs q_n

Parametrizing graphs q_n

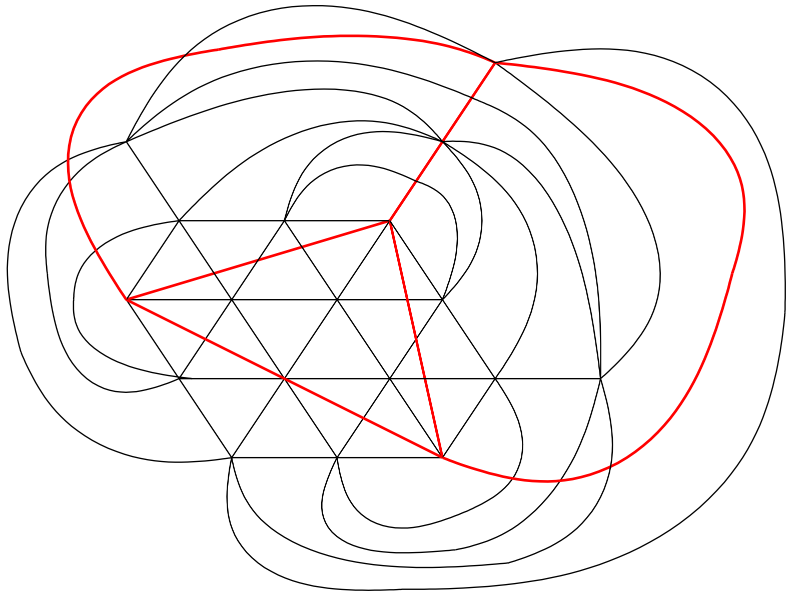
Idea: the hexagons are of zero curvature, it suffices to give relative positions of faces of non-zero curvature.

- Goldberg (1937): All 3_n, 4_n or 5_n of symmetry (T, T_d), (O, O_h) or (I, I_h) are given by Goldberg-Coxeter construction $GC_{k,l}$.
- Fowler and al. (1988) All 5_n of symmetry D_5, D_6 or T are described in terms of 4 parameters.
- Graver (1999) All 5_n can be encoded by 20 integer parameters.
- Thurston (1998) The 5_n are parametrized by 10 complex parameters.
- Sah (1994) Thurston's result implies that the Nrs of 3_n, 4_n, $5_n \sim n$, n^3, n^9.

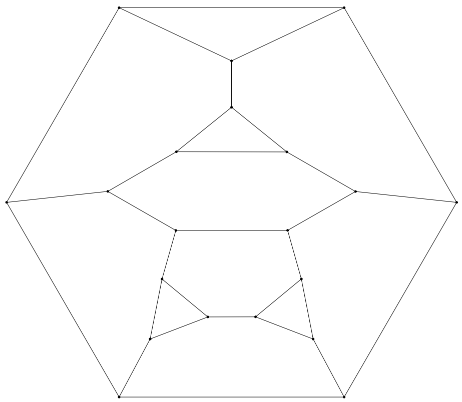
The structure of graphs 3_n



4 triangles in $Z[\omega]$



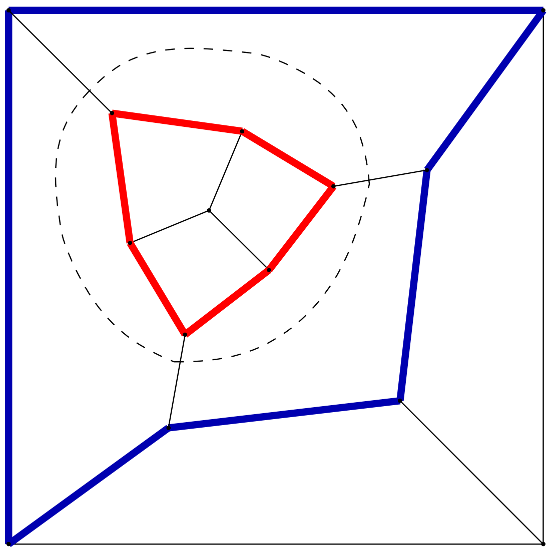
The corresponding triangulation

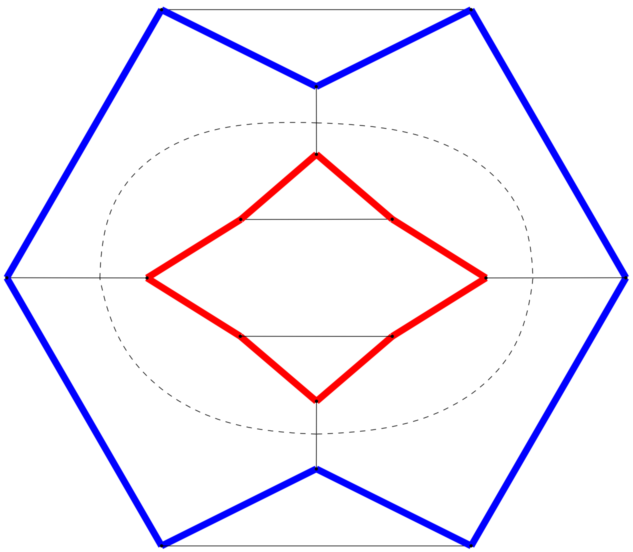


The graph $3_{20}(D_{2d})$

Tightness

- A railroad in a 3-valent plane graph G is a circuit of hexagons with any two of them adjacent on opposite edges.





They are bounded by two zigzags.

- A graph is called tight if and only if it has no railroads.
- If a 3- (or 4-)valent plane graph G_0 has no q-gonal faces with q=6 (or 4) and $gcd(k, l) = 1$ then $GC_{k,l}(G_0)$ is tight.

z- and railroad-structure of graphs 3_n

All zigzags are simple.

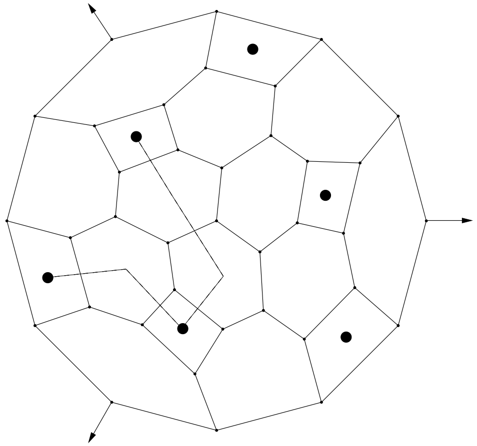
- The z-vector is of the form

$$(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3} \quad \text{with} \quad s_i m_i = \frac{n}{4};$$

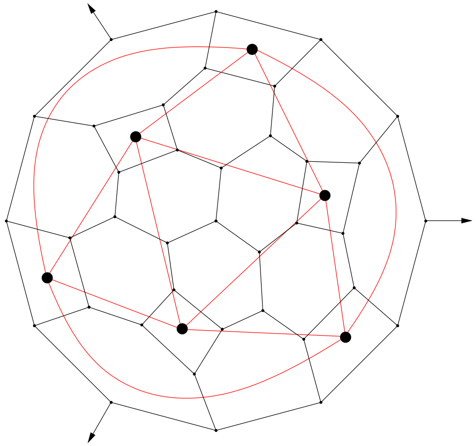
  the number of railroads is $m_1 + m_2 + m_3 - 3$.
- G has ≥ 3 zigzags with equality if and only if it is tight.
- If G is tight, then $z(G) = n^3$ (so, each zigzag is a Hamiltonian circuit).
- All 3_n are tight if and only if $\frac{n}{4}$ is prime.
- There exists a tight 3_n if and only if $\frac{n}{4}$ is odd.

Conjecture on $4_n(D_{3h}, D_{3d}\ or\ D_3)$

- $4_n(D_3 \subset D_{3h}, D_{3d}, D_6, D_{6h}, O, O_h)$ are described by two complex parameters. They exists if and only if $n \equiv 0, 2 \pmod 6$ and $n \geq 8$.



$4_n(D_3)$ with one zigzag



The defining triangles

- $4_n(D_{3d} \subset O_h, D_{6h})$ exists if and only if $n \equiv 0, 8 \pmod{12}$, $n \geq 8$.
- If n increases, then part of $4_n(D_3)$ amongst $4_n(D_{3h}, D_{3d}, D_3)$ goes to 100%

More conjectures

- All 4_n with only simple zigzags are:
  - $GC_{k,0}(Cube)$, $GC_{k,k}(Cube)$ and
  - the family of $4_n(D_3 \subset \dots)$ with parameters $(m, 0)$ and $(i, m-2i)$ with $n = 4m(2m-3i)$ and $z = (6m-6i)^{3m-3i}, (6m)^{m-2i}, (12m-18i)^i$

  They have symmetry D_{3d} or O_h or D_{6h}
- Any $4_n(D_3 \subset \dots)$ with one zigzag is a $4_n(D_3)$.
- For tight graphs $4_n(D_3 \subset \dots)$ the z-vector is of the form a^k with $k \in \{1, 2, 3, 6\}$ or a^k, b^l with $k, l \in \{1, 3\}$
- Tight $4_n(D_{3d})$ exist if and only if $n \equiv 0 \pmod{12}$, they are z-transitive with
  - $z = (n/2)^6_{n/36,0}$ iff $n \equiv 24 \pmod{36}$ and, otherwise,
  - $z = (3n/2)^2_{n/4,0}$ iff $n \equiv 0, 12 \pmod{36}$

The End