Extended Family of Fullerenes and Lego-like Maps

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Overview

1. 8 families of parabolic $(\{a, b\}; k)$-spheres
2. Listing of $(\{a, b\}; k)$-spheres with small p_b
3. Symmetry groups of $(\{a, b\}; k)$-spheres
4. Goldberg–Coxeter construction and parameterizing
5. LEGO-LIKE $(\{a, b\}; k)$-SHERES AND TORI
6. Parabolic $(\{a, b\}; k)$-maps on surfaces $\mathbb{T}^2$, $\mathbb{K}^2$, $\mathbb{P}^2$
7. Other relatives: plane fullerenes, azulenoids, schwartzites
8. c-disk fullerenes

Definition of a fullerene

A (geometric) fullerene F_v is a simple (i.e., 3-valent) polyhedron (putative carbon molecule) whose v vertices (carbon atoms) are arranged in $p_5 = 12$ pentagons and $p_6 = (\frac{v}{2} - 10)$ hexagons.

- F_v exist for all even $v \geq 20$ except $v = 22$.
  $1, 0, 1, 1, 2, 3, 6 \dots, 1812, \dots 214127713, \dots$ isomers F_v for $v=$ $20, 22, 24, 26, 28, 30, 32 \dots, 60, \dots, 200, \dots$.
  Graphite lattice $\{6^3\}$ can be seen as *"largest fullerene"* F_∞.
- Thurston, 1998, implies: the number of F_v grows as v^9.

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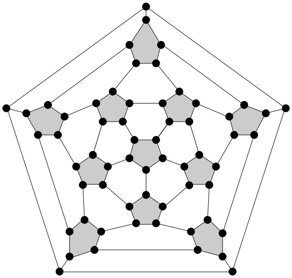
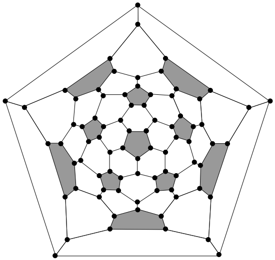
- Thurston, 1998, implies: the number of F_v grows as v^9.
- Only 4 Frank–Kasper fullerenes (having isolated hexagons): unique ones F_{20}, F_{24}, F_{26} and $F_{28}(T_d)$, one of two F_{28}. ∞ of IP fullerenes (isolated pentagons; denote such by C_v); the smallest is the truncated Icosahedron $C_{60}(I_h)$.
- Curl–Kroto–Smalley, 1985, synthesised it as carbon allotrope backminsterfullerene (Nobel Prize, 1996, in Chemistry). But Goldberg (1935, 1937) and rev. Kirkman, 1882: 80 of 89 F_{44}.

Original Goldberg–Coxeter construction

Any icosahedral fullerene (i.e., of symmetry I_h or I), has $v{=}20(p^2{+}pq{+}q^2)$ with $0{\leq}q{\leq}\ p$; I_h for $p=q\neq 0$ and for $q=0$. Below are cases of $C_{60}(I_h)$; $(p,q){=}(1,1)$, truncated Icosahedron, and $C_{80}(I_h)$; $(p,q){=}(2,0)$, chamfered Dodecahedron. Besides Dodecahedron, they are only icosahedral fullerenes with $v\leq 80$.





This construction: parameterization by Eisenstein integer $p{+}q\omega$.

Extended family of fullerenes; main considered ones are:

- $(\{a, b\}; k)$ on $\mathbb{S}^2$, $\mathbb{P}^2$, $\mathbb{T}^2$ or $\mathbb{K}^2$, i.e., k-valent maps with only a- and b-gonal faces, of curvature $1 + \frac{i}{k} - \frac{i}{2} \geq 0$ for $i = a, b$.
- b-icosahedrites, i.e., $(\{3, b\}, 5)$-$\mathbb{S}^2$ with $b \geq 4$.
- G-fulleroids, i.e., $(\{5, b\}, 3)$-$\mathbb{S}^2$ with $b > 6$ and symmetry G.
- c-disk-fullerenes, i.e., $(\{5, 6, c\}, 3)$-$\mathbb{S}^2$ with $p_c = 1$.
- c-near-fullerenes $(\{5, 6, c\}, 3)$-$\mathbb{S}^2$, with all 5- and c-gons forming $\min(12, p_c)$ lego (isomorphic disjoint clusters of faces) especially, lego-like fullerenes $(\{5, 6\}, 3)$-$\mathbb{S}^2$, with all faces forming $\min(p_5, p_6) = \min(12, p_6)$ legos.
- Azulenoids, i.e., $(\{5, 6, 7\}, 3)$-$\mathbb{T}^2$; such tori have $p_5 = p_7$.
- Schwartzits, i.e., $(\{6, 7\}, 3)$- and $(\{6, 8\}, 3)$-maps of genus $g \geq 2$ on minimal surfaces of constant negative curvature.
- Plane fullerenes, i.e., $(\{5, 6\}, 3)$-$\mathbb{E}^2$; such planes have $p_5 \leq 6$.
- Also, space fullerenes ($\mathbb{E}^3$-tilings by fullerenes) and fullerene manifolds (manifolds whose 2-faces are only 5- or 6-gonal).

Main considered properties of those maps

- Usual ones: symmetries, computer enumeration (when feasible), generation, connectivity and so on.
- Parameterization by complex numbers, esp. Goldberg–Coxeter construction (1-parameter case) using rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$.
- By analogy with v-, p-vectors enumerating map's vertices and faces, edges are represented by z-vector enumerating zigzags (left-right circuits doubly covering edge-set). Main interesting cases: knot (unique zigzag), pure (no zigzag self-intersects) and tight (no railroad, i.e. pair of "parallel" zigzags) maps. Similar theory is build for central circuits of even-valent maps.

This material, except lego-like and near-parabolic maps, to appear, is presented in our books: M.Deza and M.Dutour Sikirić, *Geometry of Chemical Graphs*, Cambridge University Press, 2008, and M.Deza, M.Dutour Sikirić and M.Shtogrin, *Geometric Structure of Chemistry-relevant Graphs*, Springer, 2015.

Fullerenes and other 7 families of parabolic $(\{a, b\}; k)$-spheres

(R,k)-spheres: curvature $\kappa_i = 1 + \frac{i}{k} - \frac{i}{2}$ of i-gons

- Fix $R \subset \mathbb{N}$. An (R,k)-sphere is a k-regular, $k \geq 3$, map on $\mathbb{S}^2$ whose faces are i-gons, $i \in R$. Let $m = \min$ and $M = \max_{i \in R} i$.
- Let v, e and $f = \sum_i p_i$ be the map's numbers of vertices, edges and faces, where p_i is the number of i-gonal faces. So, $kv = 2e = \sum_i ip_i$ and *Euler formula* $v - e + f = 2$ become $2 = \sum_i p_i \kappa_i$, where $\kappa_i = 1 + \frac{i}{k} - \frac{i}{2}$ is the curvature of i-gons.
- $\kappa_m \geq 0$ implies $m < \frac{2k}{k-2}$; so, $m \geq 3$, implies $3 \leq m, k \leq 5$, i.e. 5 Platonic parameters $(m,k) = (3,3), (4,3), (3,4), (5,3), (3,5)$.

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- $\kappa_m{\geq}0$ implies $m{<}\frac{2k}{k-2}$; so, $m{\geq}3$, implies $3\leq m,k\leq 5$, i.e. 5 Platonic parameters $(m,k){=}(3,3)$, $(4,3)$, $(3,4)$, $(5,3)$, $(3,5)$.
- (R,k)-sphere is elliptic if $M{<}\frac{2k}{k-2}$, i.e., $\min_{i\in R}\kappa_i>0$; then either 1) $k=3$, $M\leq 5$, or 2) $k\in\{4,5\}$, $M\leq 3$.
  So, for $m\geq 3$, such are only Octahedron, Icosahedron and 10 $(\{3,4,5\},3)$-spheres: 8 dual deltahedra and the Cube's truncations on 1 or 2 opposite vertices (Dürer octahedron).
  In other words, five Platonic and seven $(\{3,4,5\},3)$-spheres.

Parabolic (R,k)-spheres

- (R,k)-sphere is parabolic if $M=\frac{2k}{k-2}$, i.e. $\min_{i \in R} \kappa_i=0$.
  So, $(M,k)=(6,3)$, $(4,4)$, $(3,6)$ (Euclidean parameter pairs).
  Exclusion of i-faces with $\kappa_i<0$ simplifies enumeration, while number p_M of *flat* $(\kappa_M=0)$ M-gonal faces not being restricted, there is an infinity of such (R,k)-spheres.
- The number of such v-vertex (R,k)-spheres with $|R|=2$ increases polynomially with v.
  Such spheres admit parametrization and description in terms of rings of (*Gaussian* if $k=4$ and *Eisenstein* if $k=3,6$) *integers*.
- (R,k)-sphere is hyperbolic if $M>\frac{2k}{k-2}$, i.e. $\min_{i \in R} \kappa_i<0$; it do not admit above, in general. We considered only simplest cases, say: icosahedrites, i.e. $(\{3,4\},5)$-spheres, and special $(\{a,b,c\};k)$-spheres: those with $p_b=0$ or $p_c=0, p_b \le 3$ or $p_c=1$ or a- and c-gons forming disjoint isomorphic clusters).

(R,k)-maps on general surface $\mathbb{F}^2$

- Given $R \subset \mathbb{N}$ and a surface $\mathbb{F}^2$, an (R,k)-$\mathbb{F}^2$ is a k-regular map on surface $\mathbb{F}^2$ whose faces have gonalities $i \in R$.
- The Euler characteristic $\chi(\mathbb{F}^2)$ is v-e+f= $\sum_i p_i\kappa_i$, where κ_i=$1+\frac{i}{k}-\frac{i}{2}$ and p_i is the number of i-gons. So, elliptic and, with $|R|$>1, parabolic (R,k)-maps exist only on $\mathbb{S}^2$ and $\mathbb{P}^2$.
- In fact, all connected *closed* (compact and without boundary) irreducible surfaces $\mathbb{F}^2$ with $\chi(\mathbb{F}^2)\geq 0$ are (with $\chi = 2,0,1,0$, respectively): orientable: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ and non-orientable: real projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$.

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- In fact, all connected *closed* (compact and without boundary) irreducible surfaces $\mathbb{F}^2$ with $\chi(\mathbb{F}^2)\geq 0$ are (with $\chi=2,0,1,0,$ respectively): orientable: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ and non-orientable: real projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$.
- Again, let our (R,k)-maps be parabolic, i.e., $\min_{i\in R}\kappa_i=0$. Then M=:$\max\{i\in R\}=\frac{2k}{k-2}$, and $(M,k)=(6,3),(4,4),(3,6)$.
- Also, there are infinity of parabolic maps (R,k)-$\mathbb{F}^2$, since the number p_M of *flat* (κ_M=0) faces is not restricted.
- Also, if $\chi(\mathbb{F}^2)$=$\sum_i p_i\kappa_i$= 0, i.e. $\mathbb{F}^2$ is $\mathbb{T}^2$ or $\mathbb{K}^2$, then $R=\{M\}$

8 families of parabolic $(\{a, b\}; k)$-spheres

- An $(\{a, b\}; k)$-sphere is an (R, k)-sphere with $R = \{a, b\}$, $1 \le a < b$. It has $v = \frac{1}{k}(ap_a + bp_b)$ vertices.
- Such parabolic sphere has $b = \frac{2k}{k-2}$; so, $(b, k) = (6, 3), (4, 4), (3, 6)$ and Euler formula become $2 = \kappa_a p_a = (1 + \frac{a}{k} - \frac{a}{2})p_a = (1 - \frac{a}{b})p_a$.
- So, $p_a = \frac{2b}{b-a}$ and all possible (a, p_a) are:
  $(5, 12), (4, 6), (3, 4), (2, 3)$ for $(b, k) = (6, 3)$;
  $(3, 8), (2, 4)$ for $(b, k) = (4, 4)$;
  $(2, 6), (1, 3)$ for $(b, k) = (3, 6)$.
- Those 8 families can be seen as spheric analogs of the regular plane partitions $\{6^3\}$, $\{4^4\}$, $\{3^6\}$ with p_a *disclinations* ("defects") κ_a added to get the curvature 2 of the sphere.

8 parabolic families: existence criterions

Grűnbaum–Motzkin, 1963: criterion for $k{=}3 \le a$;
Grűnbaum, 1967: for $(\{3,4\},4)$-spheres;
Grűnbaum–Zaks, 1974: for $a = 1, 2$.

k	(a,b)	smallest one	it exists if and only if	p_a	v	Ord	Gr
3	$(5,6)$	Dodecahedron	$p_6 \neq 1$	12	$20{+}2p_6$	v^9	28
3	$(4,6)$	Cube	$p_6 \neq 1$	6	$8{+}2p_6$	v^3	16
4	$(3,4)$	Octahedron	$p_4 \neq 1$	8	$6{+}p_4$	v^5	18
6	$(2,3)$	Bundle$_6$=$6{\times}K_2$	p_3 is even	6	$2{+}\frac{p_3}{2}$	v^4	22
3	$(3,6)$	Tetrahedron	p_6 is even	4	$4{+}2p_6$	v	5
4	$(2,4)$	Bundle$_4$= $4{\times}K_2$	p_4 is even	4	$2{+}p_4$	v	5
3	$(2,6)$	Bundle$_3$=$3{\times}K_2$	$p_6{=}(k^2{+}kl{+}l^2){-}1$	3	$2{+}2p_6$	v	2
6	$(1,3)$	Trifolium	$p_3{=}2(k^2{+}kl{+}l^2){-}1$	3	$\frac{1+p_3}{2}$	v	3
5	$(3,4)$	Icosahedron	$p_4 \neq 1$	$2p_4{+}20$	$2p_4{+}12$	exp	38

8 families of parabolic $(\{a, b\}; k)$-spheres

- Let us denote $(\{a, b\}; k)$-sphere with v vertices by $\{a, b\}_v$.
- $(\{5, 6\}, 3)$- and $(\{4, 6\}, 3)$-spheres are models of molecules of (chemical) fullerenes and boron nitrides., respectively.
- $(\{a, b\}, 4)$-spheres are minimal projections of alternating links, whose components are their *central circuits* (those going only ahead) and crossings are the vertices.
- Bundle$_m$ is $m \times K_2$. Trifolium $\{1, 3\}_1$ is the 3-*rose* $3 \times (aa)$.
- b-icosahedrites $((\{3, b\}, 5)$-spheres) are hypebolic if $b>3$, $p_b>0$, since $p_3=p_b(3b\text{-}10)+20$ and $\kappa_b = \frac{10-3b}{10b} < 0$.

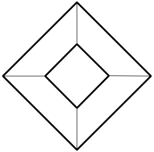
Generation of 4 simplest parabolic ($\{a, b\}; k$)-spheres

- ($\{3, 6\}, 3$)- (Grűnbaum–Motzkin, 1963) and ($\{2, 4\}, 4$)-spheres (Deza–Shtogrin, 2003) admit a 2-parametric description (by 2 complex numbers) and also a description by 3 integers.

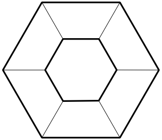
- 1-parametric description: ($\{2, 6\}, 3$)-spheres are given by the *Goldberg–Coxeter construction* from Bundle$_3$ $\{2, 6\}_2{=}3{\times}K_2$.
- ($\{1, 3\}, 6$)-spheres come by this construction (extended on 6-regular spheres) from Trifolium $\{1, 3\}_1{=}3{\times}(aa)$.

- ($\{2, 3\}, 6$)-spheres, except of $6 \times K_2$ and $3 \times K_3$, are the duals of ($\{3, 4, 5, 6\}, 3$)-spheres with six new vertices put on edges. Example: ($\{5, 6\}, 3$)-spheres with 5-gons organized in 6 pairs.
- ($\{1, 2, 3\}, 6$)-spheres with $v{>}3$, except of 5 infinite series, are the duals of ($\{3, 4, 5, 6\}, 3$)-$\mathbb{S}^2$ with splitting (into a 2-gon or into a 2-gon, enclosing a 1-gon) of some edges.

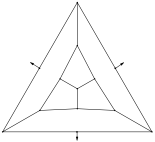
First four $(\{4,6\},3)$- and $(\{5,6\},3)$-spheres (fullerenes)



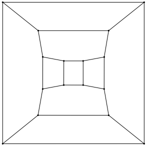
O_h (6^4)



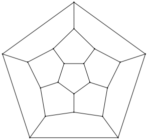
D_{6h} (18^2)



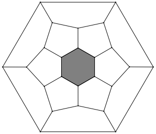
D_{3h} $(6^2; 30)$



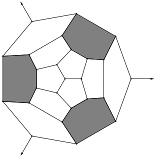
D_{2d} (24^2)



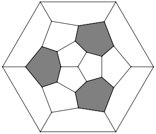
I_h (10^6)



D_{6d} $(12; 60)$



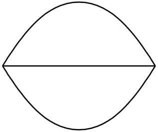
D_{3h} $(12^3; 42)$



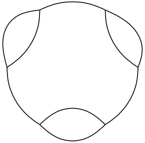
T_d (12^7)

First four $(\{2, 6\}, 3)$- and $(\{3, 6\}, 3)$-spheres

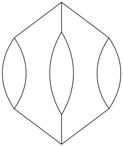
Number of $(\{2, 6\}_v$'s is nr. of representations $v{=}2(k^2 + kl + l^2)$, $0 \leq l \leq k$ $(GC_{k,l}(\{2, 6\}_2))$. It become 2 for $v{=}7^2{=}5^2{+}15{+}3^2$.



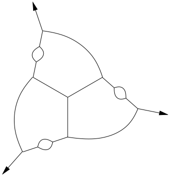
D_{3h} (6)



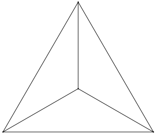
D_{3h} (6^3)



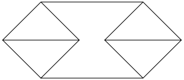
D_{3h} (12^2)



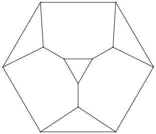
D_3 (42)



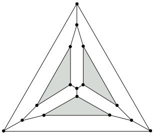
T_d (4^3)



D_{2h} $(8^2, 4^2)$

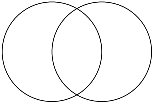


T_d (12^3)

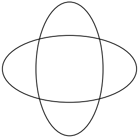


T_d (8^6)

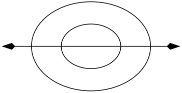
First four $(\{2, 4\}, 4)$- and $(\{3, 4\}, 4)$-spheres



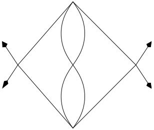
D_{4h} 2_1^2 (2^2)



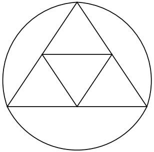
D_{4h} 4_1^2 (4^2)



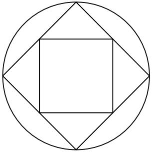
D_{2h} $2 \times 2_1^2$ $(2^2, 4)$



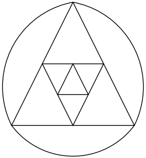
D_{2d} 6_2^2 (6^2)



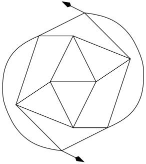
O_h 6_2^3 (4^3)
Borr. rings



D_{4d} 8_{18} (16)



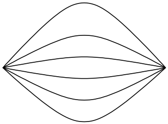
D_{3h} 9_{40} (18)
(Herschel)*



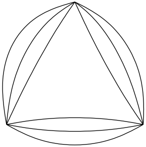
D_2 10_{56}^2
$(6; 14)$

Above links/knots are given in Rolfsen, 1976 and 1990, notation.
Herschel graph: smallest non-Hamiltonian polyhedral graph.

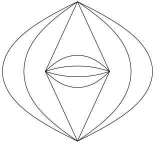
First four $(\{2,3\},6)$- and $(\{1,3\},6)$-spheres



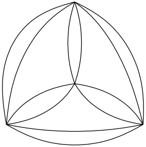
D_{6h} (2^3)



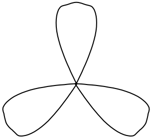
D_{3h} $(3;6)$



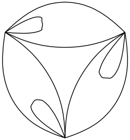
D_{2d} $(2^2;8)$



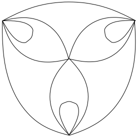
T_d (3^4)



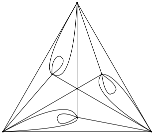
C_{3v} (3)



C_{3h} $(3;6)$



C_{3v} (6^2)



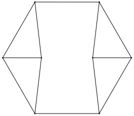
C_3 (21)

$(\{a, b\}; k)$-spheres with $p_b \leq 3$: listings

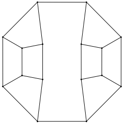
$(\{a,b\};k)$-spheres with $p_b \leq 2 < a < b$

- Remind: $(a,k)=(3,3),(4,3),(3,4),(5,3),(3,5)$ if $k,a \geq 3$.
- The only $(\{a,b\};k)$-spheres with $p_b \leq 1$ are 5 Platonic (a^k): Tetrahedron, Cube ($Prism_4$), Octahedron ($APrism_3$), Dodecahedron (snub $Prism_5$), Icosahedron (snub $APrism_3$).
- There exists unique trivial 3-connected $(\{a,b\};k)$-sphere with $p_b=2$ for $(\{4,b\},3)$-, $(\{3,b\},4)$-, $(\{5,b\},3)$-, $(\{3,b\},5)$-: D_{bh} $Prism_b$ and D_{bd} $APrism_b$, snub $Prism_b$, snub $APrism_b$: two b-gons separated by b-ring of 4-gons, $2b$-ring of 3-gons, two b-rings of 5-gons, two $3b$-rings of 3-gons.
- Also, for $t \geq 2$, 10 non-trivial $(\{a,at\};k)$-spheres with $p_{at}=2$: 5 $(\{a,ta\};k)$-spheres are (D_{th}) necklaces of polycycles $\{a^k\}$-e; 3 are (D_{th}) necklaces of t v-split $\{3^4\}$ and e-split $\{5^3\}$, $\{3^5\}$; $(\{3,3t\},5)$-spheres C_{th}, D_t are necklaces of t v-, f-split $\{3^5\}$.

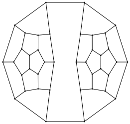
$(\{a, b{=}ta\}; k)$-spheres with $p_b{=}2 < a$, $k{=}3, 4, 5$; case $t{=}2$



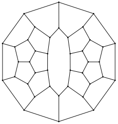
D_{2h}: $a{=}3$



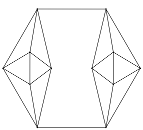
$a{=}4$



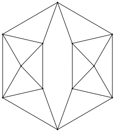
$a{=}5$



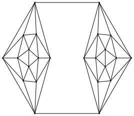
$a{=}5$



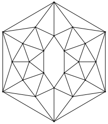
$a{=}3$ D_{2h}



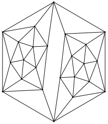
$a{=}3$ D_{2h}



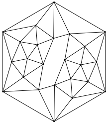
$a{=}3$ D_{2h}



$a{=}3$ D_{2h}



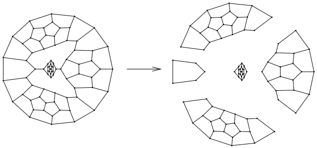
$a{=}3$ C_{2h}



$a{=}3$ D_2

Proof method: elementary (a, k)-polycycles

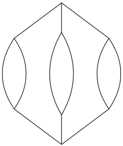
- A (a, k)-polycycle is a 2-connected plane graph with faces partitioned in a-gonal proper faces and holes, exterior face among them, so that vertex degrees are in $\{2, \dots, k\}$ and can be $< k$ only for a vertex lying on the boundary of a hole.
- Any (a, k)-polycycle decomposes uniquely along its bridges (non-boundary going hole-to-hole, possibly, same, edges) into elementary ones. Cf. integer factorisation into primes.
- We listed them for $\kappa_a = 1 + \frac{a}{k} - \frac{a}{2} \geq 0$. Othervise, continuum.



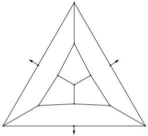
This $(\{5, 15\}, 3)$-sphere with $p_{15} = 3$ is a 3-holes $(\{5\}, 3)$-polycycle
It decomposes into five 1-hole elementary $(\{5\}; k)$-polycycles.

$(\{a,b\},3)$-spheres with $p_b = 3 \leq a$

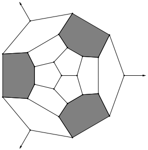
- $(\{a,b\};k)$-sphere with $p_b = 3 \leq a$ exists if and only if $b \equiv 2, a, 2a-2 \pmod{2a}$ and $b \equiv 4, 6 \pmod{10}$ if a=5.
- There are 7 such spheres with $t = \lfloor \frac{b}{6} \rfloor = 0$ and $3+4+5+17$ of them for any $t \geq 1$.
- Such sphere are unique if b is not $\equiv a \pmod{2a}$ and then their symmetry is D_{3h}, except $(a,k)=(3,5)$, when it is D_3.



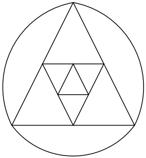
8, D_{3h}
$(\{2,6\};3)$-
$\vec{p}=(3,3)$



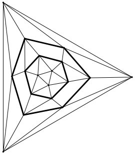
14, D_{3h}
$(\{4,6\};3)$-
$\vec{p}=(6,3)$



26, D_{3h}
$(\{5,6\};3)$-
$\vec{p}=(12,3)$



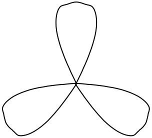
9, D_{3h}
$(\{3,4\};4)$-
$\vec{p}=(8,3)$



18, D_3
$(\{3,4\};5)$-
$\vec{p}=(26,3)$

$(\{a, b\}, k)$-spheres with $a = 1, 2$ and $p_b = 1$

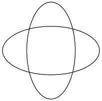
- There are no $(\{a, b\}; k)$-spheres with $a \geq 2$, having $p_b = 1$.
- The only $(\{1, b\}; k)$-spheres with $p_b{=}1$ are:
  1-vertex *b*-foliums (K_1 with b 1-gons); so, $k{=}2b{\geq}4$, $p_1{=}b$ and
  2-vertex *b*-dumbbells (K_2 with $\frac{b-2}{2}$ 1-gons on each vertex);
  so, having odd $k{=}b-1{\geq}3$ and $p_1{=}b-2$.
  2-folium and 4-dumbbell are elliptic, 3-folium is parabolic.



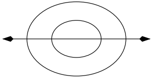
3-folium

$(\{a,b\},k)$-spheres with $a=1,2$ and $p_b=2$

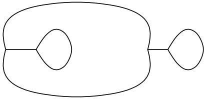
- An $(\{2,b\};k)$-$\mathbb{S}^2$ with $p_b{=}2$ exists if and only if bk is even, and then it has $\vec{p}{=}(\frac{b(k-2)}{2},2)$ and $v{=}b$ vertices. It is either, for odd b, b-cycle with edges repeated $\frac{k}{2}$ times; or, for even b and any integer $m\in[1,\frac{k}{2}]$, b-cycle with edges repeated, alternatively, m and $k-m$ times.
- An $(\{1,b\};k)$-sphere with $p_b{=}2$ exists iff $v{=}\frac{4b}{k+2}{\in}\mathbb{N}$, and then it has v vertices and $\vec{p}{=}(2(b-v),2)$. It is either, for $k=3$, a $\frac{2b}{5}$-cycle with matches from each cycle's vertex, so that the same number of them goes inside and outside. or, for $k\geq 4$, a $\frac{4b}{k+2}$-cycle with $\frac{k-2}{2}$ 1-gons from each vertex, so that the same number of them goes inside and outside.



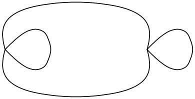
4, D_{4h}
$(\{2,4\};4)$-
$\vec{p}{=}(4,2)$



4, D_{2h}
$(\{2,4\};4)$-
$\vec{p}{=}(4,2)$



4, C_{2h}
$(\{1,5\};3)$-
$\vec{p}{=}(2,2)$



2, C_{2h}
$(\{1,3\};4)$-
$\vec{p}{=}(2,2)$

Symmetry groups of $(\{a, b\}; k)$-spheres

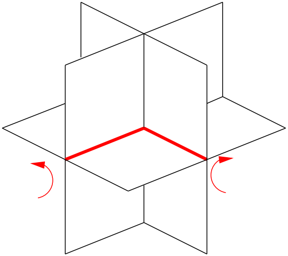
Finite isometry groups

All finite groups of isometries of 3-space $\mathbb{E}^3$ are classified.
In Schoenflies notations, they are:

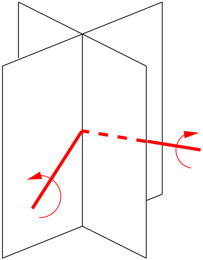
- C_1 is the trivial group
- C_s is the group generated by a plane reflexion
- $C_i = \{I_3, -I_3\}$ is the inversion group
- C_m is the group generated by a rotation of order m of axis Δ
- C_{mv} ($\simeq$ dihedral group) is the group generated by C_m and m reflexion containing Δ
- $C_{mh} = C_m \times C_s$ is the group generated by C_m and the symmetry by the plane orthogonal to Δ
- S_{2m} is the group of order $2m$ generated by an antirotation, i.e. commuting composition of a rotation and a plane symmetry

Finite isometry groups D_m, D_{mh}, D_{md}

- D_m ($\simeq$ dihedral group) is the group generated of C_m and m rotations of order 2 with axis orthogonal to Δ
- D_{mh} is the group generated by D_m and a plane symmetry orthogonal to Δ
- D_{md} is the group generated by D_m and m symmetry planes containing Δ and which does not contain axis of order 2



D_{2h}



D_{2d}

Remaining 7 finite isometry groups

- $I_h = H_3$ is the group of isometries of Dodecahedron; $I_h \simeq Alt_5 \times C_2$
- $I \simeq Alt_5$ is the group of rotations of Dodecahedron
- $O_h = B_3$ is the group of isometries of Cube
- $O \simeq Sym(4)$ is the group of rotations of Cube
- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of Tetrahedron
- $T_h = T \cup -T$

While (point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group), Mani, 1971: for any 3-polytope P, there is a map-isomorphic 3-polytope P' (so, with the same skeleton $G(P') = G(P)$), such that the group $Isom(P')$ of its isometries is isomorphic to $Aut(G)$.

8 parabolic families: symmetry groups

1. 28 for $\{5,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_5, D_{5h}, D_{5d}; D_6, D_{6h}, D_{6d}; T, T_d, T_h; I, I_h (Fowler–Manolopoulos, 1995)
2. 16 for $\{4,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; O, O_h (Deza–Dutour, 2005)
3. 5 for $\{3,6\}_v$: D_2, D_{2h}, D_{2d}; T, T_d (Fowler–Cremona,1997)
4. 2 for $\{2,6\}_v$: D_3, D_{3h} (Grűnbaum–Zaks, 1974)
5. 18 for $\{3,4\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_4, D_{4h}, D_{4d}; O, O_h (Deza-Dutour-Shtogrin, 2003)
6. 5 for $\{2,4\}_v$: D_2, D_{2h}, D_{2d}; D_4, D_{4h}, all in $[D_2, D_{4h}]$ (same)
7. 3 for $\{1,3\}_v$: C_3, C_{3v}, C_{3h} (Deza–Dutour, 2010)
8. 22 for $\{2,3\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; T, T_d, T_h (same)

38 for icosahedrites $(\{3,4\},5)$- (same, 2011).

8 families: Goldberg–Coxeter construction $GC_{k,l}(.)$

With $\mathbf{T}=\{T, T_d, T_h\}$, $\mathbf{O}=\{O, O_h\}$, $\mathbf{I}=\{I, I_h\}$, $\mathbf{C_1}=\{C_1, C_s, C_i\}$, $\mathbf{C_m}=\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}=\{D_m, D_{mh}, D_{md}\}$, we get

1. for $(\{5,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
2. for $(\{2,3\},6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
3. for $(\{4,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{O}$
4. for $(\{3,4\},4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
5. for $(\{3,6\},3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
6. for $(\{2,4\},4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
7. for $(\{2,6\},3)$-: $\mathbf{D_3}\backslash D_{3d} = \{D_3, D_{3h}\}$
8. for $(\{1,3\},6)$-: $\mathbf{C_3}\backslash S_6 = \{C_3, C_{3v}, C_{3h}\}$

if $(\{3,4\},5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.

8 families: Goldberg–Coxeter construction $GC_{k,l}(.)$

With $\mathbf{T}=\{T, T_d, T_h\}$, $\mathbf{O}=\{O, O_h\}$, $\mathbf{I}=\{I, I_h\}$, $\mathbf{C_1}=\{C_1, C_s, C_i\}$, $\mathbf{C_m}=\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}=\{D_m, D_{mh}, D_{md}\}$, we get

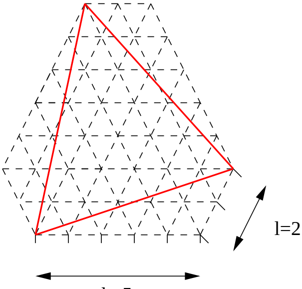
1. for $(\{5,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
2. for $(\{2,3\},6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
3. for $(\{4,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\setminus S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{O}$
4. for $(\{3,4\},4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
5. for $(\{3,6\},3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
6. for $(\{2,4\},4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
7. for $(\{2,6\},3)$-: $\mathbf{D_3}\setminus D_{3d}= \{D_3, D_{3h}\}$
8. for $(\{1,3\},6)$-: $\mathbf{C_3}\setminus S_6=\{C_3, C_{3v}, C_{3h}\}$

if $(\{3,4\},5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.
Spheres of blue symmetry are $GC_{k,l}$ from 1st such; so, given by one complex (Gaussian for k=4, Eisenstein for k=3,6) parameter.
Goldberg, 1937 and Coxeter, 1971: $\{5,6\}_v(I, I_h)$, $\{4,6\}_v(O, O_h)$, $\{3,6\}_v(T, T_d)$. Dutour-Deza, 2004 and 2010: for other cases.

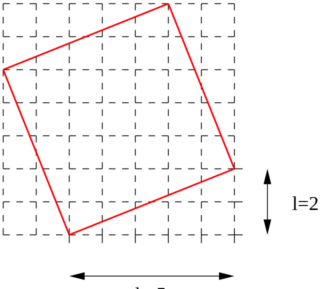
Goldberg–Coxeter construction and parameterizing

Goldberg–Coxeter (1 parameter) construction $GC_{k,l}(.)$

- Take a 3- or 4-regular plane graph G. The faces of dual graph G^* are triangles or squares, respectively.
- Break each face into pieces according to parameter (k, l). *Master polygons* below have area $\mathcal{A}(k^2+kl+l^2)$ or $\mathcal{A}(k^2+l^2)$, where $\mathcal{A}$ is the area of a small polygon.



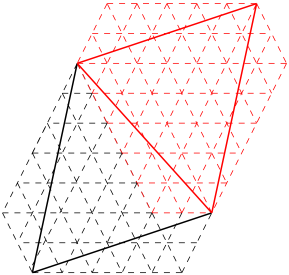
3−valent case



4−valent case

Gluing the pieces together in a coherent way

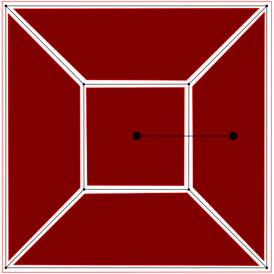
- Gluing the pieces so that, say, 2 non-triangles, coming from subdivision of neighboring triangles, form a small triangle, we obtain another triangulation or quadrangulation of the plane.



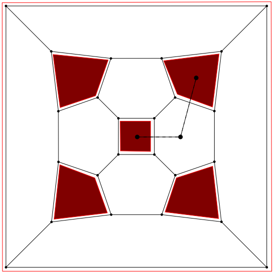
- The dual is a 3- or 4-regular plane graph, denoted $GC_{k,l}(G)$; we call it Goldberg–Coxeter construction.
- It works for any 3- or 4-regular map on oriented surface.

$GC_{k,l}(Cube)$ for $(k,l) = (1,0), (1,1), (2,0), (2,1)$

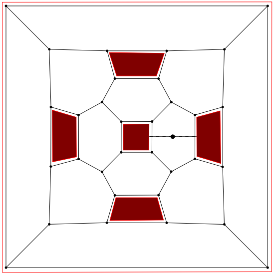
1,0



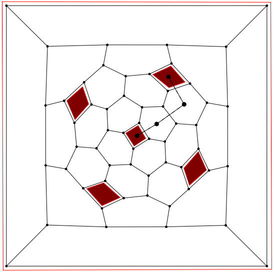
1,1



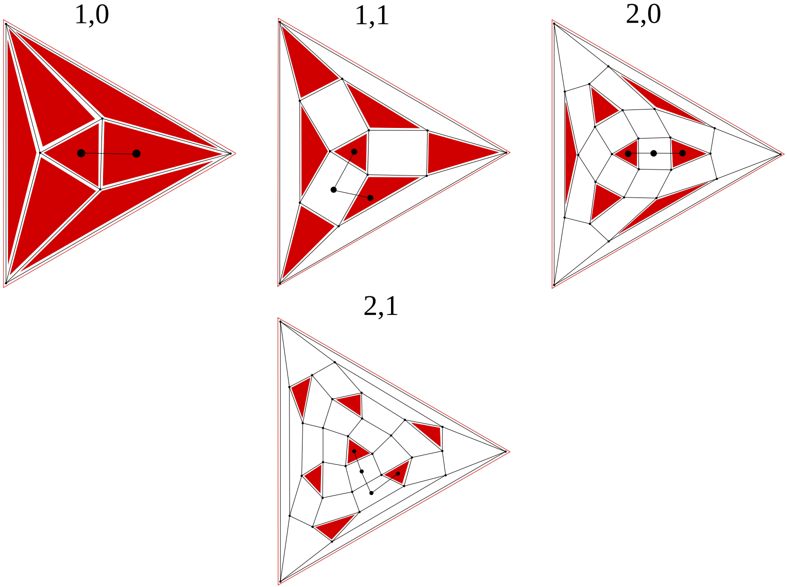
2,0



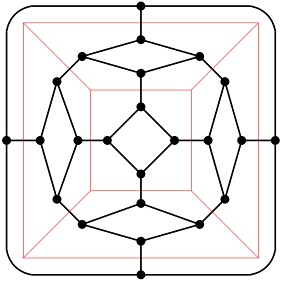
2,1



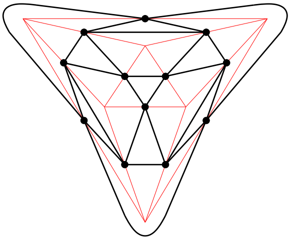
Goldberg–Coxeter construction from Octahedron



The case $(k, l)=(1, 1)$ of $GC_{k,l}(G)$



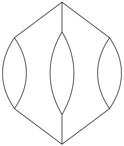
For 3-regular $G=(V, E)$,
$GC_{1,1}$ is called leapfrog
($\frac{1}{3}$-truncation of the dual),
$3|V|$ vertices.
Truncated Octahedron



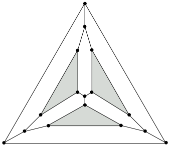
For 4-regular $G=(V, E)$,
$GC_{1,1}$ is called medial
($\frac{1}{2}$-truncation),
$|E|$ vertices
Cuboctahedron

The case $(k,l)=(k,0)$ of $GC_{k,l}(G)$: k-inflation

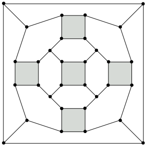
Chamfering (*quadrupling*) $GC_{2,0}(G)$ of smallest $(\{a,b\};k)$-spheres, $(a,b)=(2,6),(3,6),(4,6),(5,6)$ and $(2,4),(3,4),(1,3),(2,3)$, are:



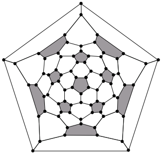
D_{3h} (12^2)



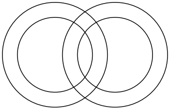
T_d (8^6)



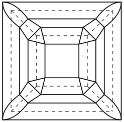
O_h (12^8)



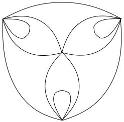
I_h (20^{12})



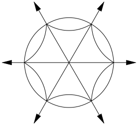
D_{4h} (4^4)



O_h (8^6)

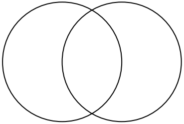


C_{3v} (6^2)

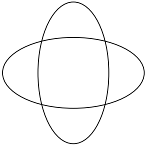


D_{6h} $(4^3,6^2)$

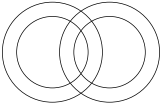
First four $GC_{k,l}(4\times K_2)$ and $GC_{k,l}(6\times K_2)$



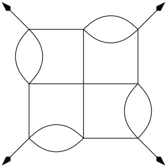
D_{4h} $4\times K_2$



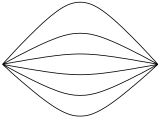
D_{4h} medial



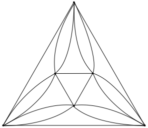
D_{4h} $G_{2,0}$



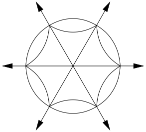
D_4 $G_{2,1}$



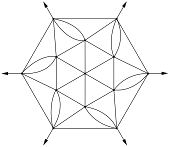
D_{6h} $6\times K_2$



D_{3d} $G_{1,1}$



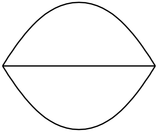
D_{6h} $G_{2,0}$



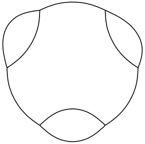
D_6 $G_{2,1}$

First four $GC_{k,l}(3\times K_2)$ and $GC_{k,l}(Trifolium = 3\times(aa))$

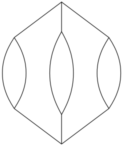
All $(\{2,6\},3)$-spheres are $G_{k,l}(3\times K_2)$: D_{3h}, D_{3h}, D_3 if l=0, k, else.



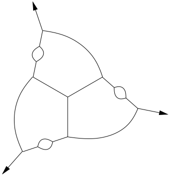
D_{3h} $3\times K_2$



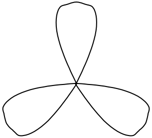
D_{3h} leapfrog



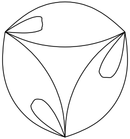
D_{3h} $G_{2,0}$



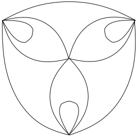
D_3 $G_{2,1}$



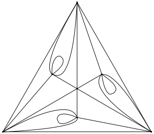
C_{3v} $3\times(aa)$



C_{3h} $G_{1,1}$



C_{3v} $G_{2,0}$



C_3 $G_{2,1}$

All $(\{1,3\},6)$-spheres are $G_{k,l}(3\times(aa))$: C_{3v}, C_{3h}, C_3 if l=0, k, else

Plane tilings $\{4^4\}$, $\{3^6\}$ and complex rings $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$

- The vertices of regular plane tilings $\{4^4\}$ and $\{3^6\}$ form each, convenient algebraic structures: lattice and ring. Path-metrics of those graphs are l_1- 4-*metric* and *hexagonal* 6-*metric*, resp.
- $\{4^4\}$: square lattice $\mathbb{Z}_2$ and ring $\mathbb{Z}[i]=\{z=k+li : k,l \in \mathbb{Z}\}$ of Gaussian integers with norm $N(z)=z\overline{z}=k^2+l^2=||(k,l)||^2$.
- $\{3^6\}$: hexagonal lattice $A_2=\{x \in \mathbb{Z}_3 : x_0+x_1+x_2=0\}$ and ring $\mathbb{Z}[\omega]=\{z=k+lw : k,l \in \mathbb{Z}\}$, where $\omega=e^{i\frac{\pi}{3}}=\frac{1}{2}(1+i\sqrt{3})$, of Eisenstein integers with norm $N(z)=z\overline{z}=k^2+kl+l^2=||(k,l)||^2$. We identify points $x=(x_0,x_1,x_2) \in A_2$ with $x_0+x_1\omega \in \mathbb{Z}[\omega]$.
- Both, $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ are unique factorization rings.
- A natural number $n=\prod_i p_i^{\alpha_i}$ is of form $n=k^2+l^2$ iff any α_i is even, whenever $p_i \equiv 3 \pmod 4$ (Fermat Theorem). It is of form $n=k^2+kl+l^2$ if and only if $p_i \equiv 2 \pmod 3$.
- The first cases of non-unicity with $gcd(k,l)=gcd(k_1,l_1)=1$ are $91=9^2+9+1^2=6^2+30+5^2$ and $65=8^2+1^2=7^2+4^2$. The first cases with $l=0$ are $7^2=5^2+15+3^2$ and $5^2=4^2+3^2$.

The bilattice of vertices of hexagonal plane tiling $\{6^3\}$

- We identify again the *hexagonal lattice* A_2 of the vertices of the plane tiling $\{3^6\}$ with *Eisenstein ring* $\mathbb{Z}[\omega]$.
- The hexagon centers of $\{6^3\}$ form $\{3^6\}$. Also, with vertices of $\{6^3\}$, they form $\{3^6\}$, rotated by 90° and scaled by $\frac{1}{3}\sqrt{3}$.
- The complex coordinates of vertices of $\{6^3\}$ are given by vectors $v_1{=}1$ and $v_2{=}\omega$. The lattice $L{=}\mathbb{Z}v_1{+}\mathbb{Z}v_2$ is $\mathbb{Z}[\omega]$.
- The vertices of $\{6^3\}$ form bilattice $L_1 \cup L_2$, where the bipartite complements, $L_1{=}(1{+}\omega)L$ and $L_2{=}1{+}(1{+}\omega)L$, are stable under multiplication. Using this,

$GC_{k,l}(G)$ for 6-regular graph G can be defined similarly to 3- and 4-regular case, but only for $z{=}k{+}l\omega{\in}L_2$, i.e. $k \equiv l \pm 1 \pmod 3$. If $z \in L_1$, then $z{=}(1+\omega)s(k'{+}l'\omega)\omega$, where $k' \equiv l \pm 1' \pmod 3$ and $s{\geq}0$. Then $GC_{k,l}(G){:=}G_{k',l'}(Or^s(G))$ via oriented tripling $Or(G){:=}GC_{1,1}$, defined using vertex 2-coloring of bipartition of G^*.

Goldberg–Coxeter operation in ring terms

	3-regular G	4-regular G	6-regular G
the tiling	$\{3^6\}$	$\{4^4\}$	$\{6^3\}$
the lattice	A_2	Z_2	bilattice $L_1 \cup L_2$
the ring	Eisenstein $\mathbb{Z}[\omega]$	Gaussian $\mathbb{Z}[i]$	Eisenstein $\mathbb{Z}[\omega]$
Euler formula	$\sum_i (6-i)p_i = 12$	$\sum_i (4-i)p_i = 8$	$\sum_i (3-i)p_i = 6$
curvature 0	hexagons	quadrangles	triangles
$GC_{11}(G)$	leapfrog graph	medial graph	oriented tripling

Goldberg–Coxeter operation in ring terms

	3-regular G	4-regular G	6-regular G
the tiling	$\{3^6\}$	$\{4^4\}$	$\{6^3\}$
the lattice	A_2	Z_2	bilattice $L_1 \cup L_2$
the ring	Eisenstein $\mathbb{Z}[\omega]$	Gaussian $\mathbb{Z}[i]$	Eisenstein $\mathbb{Z}[\omega]$
Euler formula	$\sum_i(6-i)p_i{=}12$	$\sum_i(4-i)p_i{=}8$	$\sum_i(3-i)p_i{=}6$
curvature 0	hexagons	quadrangles	triangles
$GC_{11}(G)$	leapfrog graph	medial graph	oriented tripling

- If $GC_z(G){:=}GC_{k,l}(G)$, then $GC_z(GC_{z'}(G)){=}GC_{zz'}(G)$, i.e. in ring terms, $GC_z(G)$ corresponds to scalar multiplication by z. Example: $GC_{2k^2,0}(G){=}GC_{k,k}(GC_{k,k}(G))$ by $(k{+}ki)^2{=}2k^2i$.
- G has v vertices, then $GC_{k,l}(G)$ has $vN(z)$ vertices.
- $GC_z(G)$ has all rotational symmetries of G in 3- and 4-regular case, and all symmetries if $l{=}0, k$ in general case.
- $GC_z(G){=}GC_{\overline{z}}(\overline{G})$, where $\overline{G}$ differs by a plane symmetry only.

Parameterizing parabolic ($\kappa_b = 0$) $(\{a, b\}; k)$-spheres

Thurston, 1993, implies: $(\{a, b\}; k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m{=}p_a - 2 \geq 2$.
Idea: since b-gons are of zero curvature, it suffices to give relative positions of a-gons having curvature $\kappa_i{=}1{+}\frac{a}{k} - \frac{a}{2}$.
At most $p_a - 1$ vectors will do, since one position can be taken 0.
But once $p_a - 1$ a-gons are specified, the last one is constrained.
The number of m-parametrized spheres with at most v vertices is $O(v^m)$ by direct integration. The number of such v-vertex spheres is $O(v^{m-1})$ if $m > 1$, by a Tauberian theorem.

Parameterizing parabolic $(\kappa_b = 0)$ $(\{a,b\};k)$-spheres

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- Goldberg, 1937: $\{a,6\}_v$ (highest 2 symmetries): 1 parameter
  Fowler and al., 1988: $\{5,6\}_v$ (D_5, D_6 or T): 2 parameters.
- Grűnbaum–Motzkin, 1963: $\{3,6\}_v$: 2 parameters.
  Deza–Shtogrin, 2003: $\{2,4\}_v$; 2 (Gaussian int.) parameters.
- Thurston, 1993: $\{5,6\}_v$: 10 (Eisenstein integers) parameters
  Graver, 1999: $\{5,6\}_v$: 20 integer parameters.
- Rivin, 1994: $\{5,6\}_v$: parametrization by 18 dihedral angles.

Parameterizing (R, k)-spheres with $\min_{i \in R} \kappa_i \geq 0$

Thurston, 1998 (actually, 1993) parametrized (dually) all 19 series of $(\{3, 4, 5, 6\}, 3)$-spheres. In general, such (R, k)-spheres are given by $m = \sum_{3 \leq i < \frac{2k}{k-2}} p_i - 2$ complex parameters $z_1, \dots, z_m$.
The number of vertices is expressed as a non-degenerate Hermitian form $q = q(z_1, \dots, z_m)$ of signature $(1, m-1)$.
Let H^m be the cone of $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ with $q(z) > 0$.
Given (R, k)-sphere is described by different parameter sets; let $M = M(\{p_3, \dots, p_m\}; k)$ be the discrete linear group preserving q.
For $k=3$, the quotient $H^m / (\mathbb{R}_{>0} \times M)$ is of finite covolume. Sah, 1994, deduced: the number of corresp. spheres grows as $O(v^{m-1})$
Dutour partially generalized above for other k and surface maps.

8 families: number of complex parameters by groups

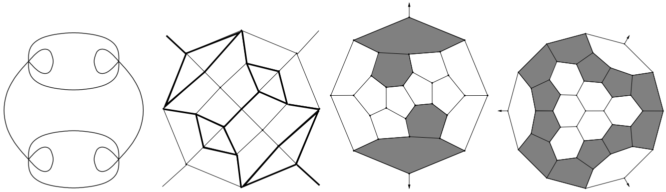
1. $\{5,6\}_v$ $\mathbf{C_1}(10)$, $\mathbf{C_2}(6)$, $\mathbf{C_3}(4)$, $\mathbf{D_2}(4)$, $\mathbf{D_3}(3)$, $\mathbf{D_5}(2)$, $\mathbf{D_6}(2)$, $\mathbf{T}(2)$, $\{I, I_h\}(1)$
2. $\{4,6\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}\backslash S_4(3)$, $\mathbf{D_2}(2)$, $\mathbf{D_3}(2)$, $\{D_6, D_{6h}\}(1)$, $\{O, O_h\}(1)$
3. $\{3,4\}_v$ $\mathbf{C_1}(6)$, $\mathbf{C_2}(4)$, $\mathbf{D_2}(3)$, $\mathbf{D_3}(2)$, $\mathbf{D_4}(2)$, $\{O, O_h\}(1)$
4. $\{2,3\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}(3)$, $\mathbf{C_3}(3)$, $\mathbf{D_2}(2)$, $\mathbf{D_3}(2)$, $\mathbf{T}(1)$, $\{D_6, D_{6h}\}(1)$
5. $\{3,6\}_v$ $\mathbf{D_2}$ (2) (also, 3 natural parameters), $\{T, T_d\}(1)$
6. $\{2,4\}_v$ $\mathbf{D_2}(2)$ (also, 3 natural parameters), $\{D_4, D_{4h}\}(1)$
7. $\{2,6\}_v$ $\{D_3, D_{3h}\}(1)$
8. $\{1,3\}_v$ $\{C_3, C_{3v}, C_{3h}\}(1)$

Thurston, 1998 implies: $(\{a,b\};k)$-$\mathbb{S}^2$ have $p_a - 2$ parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m{=}p_a - 2 > 1$.

LEGO-LIKE $(\{a, b\}; k)$-SPHERES AND TORI

Let all faces be partitioned into isomorphic clusters

- lego-like maps: $(\{a,b\};k)$-$\mathbb{F}^2$ with $1\leq a<b$ and all faces partitioned into $\min(p_a,p_b)$ legos (isomorphic disjoint clusters of faces); they are called ab^f lego-like or $a^f b$ lego-like, resp.
- m-reducible maps: $(R;k)$-$\mathbb{F}^2$ with all faces partitioned into $m\geq 2$ legos (isomorphic disjoint clusters of faces). Clearly, $m\leq \min_{a\in R} p_a$ holds with equality exactly for lego-like maps.

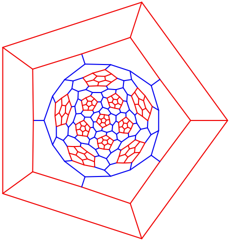


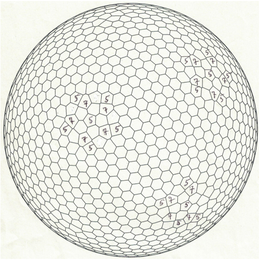
$4,\ D_{2h}$	$22,\ D_{3d}$	$28,\ D_2$	$44,\ D_{3d}$
$(\{1,4\};5)$-	$(\{3,4\};4)$-	$(\{5,6\};3)$-	$(\{5,6\};3)$-
$\vec{p}=(4,4)$	$\vec{p}=(8,16)$	$\vec{p}=(12,4)$	$\vec{p}=(12,12)$

2-reducible $(\{a,b\};k)$-$\mathbb{S}^2$ with $2<\min(p_a,p_b)$. All but 1-st are lego-like

Another generalization: c-near-parabolic maps

A c-near-parabolic map is $(\{a,b,c\};k)$-$\mathbb{F}^2$ with $1 \le a < b = \frac{2k}{k-2}$ and all a- and c-gonal faces partitioned into $\min(p_c, \frac{b\chi}{b-a})$ legos.
They are exactly parabolic maps $(\{a,b\};k)$-$\mathbb{F}^2$ if $c=a$ (clusters are a-gons) and parabolic lego-like maps $(\{a,b\};k)$-$\mathbb{F}^2$ if $c=b$.
They are some hyperbolic maps if $\kappa_c = 1 + \frac{c}{k} - \frac{c}{2} < 0$, i.e., $c > b$.





Each of above two 7-near-fullerenes $(\{5,6,7\},3)$-$\mathbb{S}^2$ (with $\vec{p} = (p_5, p_6, p_7) = (72,0,60)$ and $(72,1460,60)$ has 12 legos, consisting of six 5-gons and five 7-gons. Only 1-st is lego-like.

New frontier: to enumerate c-near-fullerenes

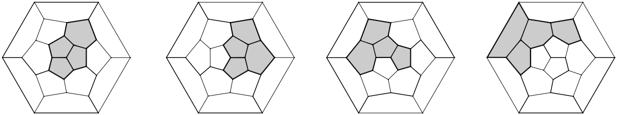
- c-near-fullerenes exist iff $c \geq 5$; they are fullerenes (clusters are 5-gons) for $c=5$ and 56^f lego-like fullerenes for $c=6$.
- The spherical Voronoi polyhedra of many energy potential minimizers (say, in Thomson problem for v unit-charged particles on sphere $\mathbb{S}^2$) and maximizers (say, in Tammes problem of minimum distance between v points on $\mathbb{S}^2$) are fullerenes or, for large v, 7-near-fullerenes.
- Haeckel, 1887, represented skeletons of zooplankton *Aulonia* by near-fullerene-looking $(\{5, 6, 7\}, 3)$- and $(\{5, 6, 8\}, 3)$-$\mathbb{S}^2$. Same holds for some basket's patterns.
- But needed computations are too hard; so, we considered lego-likeness only, but for any $(\{a, b\}; k)$-spheres and tori.

Enumeration of lego-like fullerenes

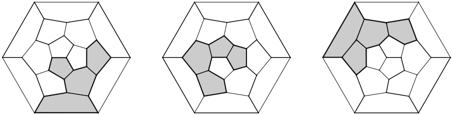
A fullerene is lego-like if all its $12 + p_6$ faces are partitioned into $\min(p_6, 12)$ legos (isomorphic clusters). So, $\frac{12}{p_6}$ or $\frac{p_6}{12}$ is an integer.

- All $1, 1, 2, 6, 89$ of, resp., $24, 26, 28, 32, 44$-vertex fullerenes are 5^f6 lego-like with $f = \frac{12}{p_6} = 6, 4, 3, 2, 1$, respectively.
- Larger such fullerenes have $v{=}20{+}2p_6{\equiv}20 \pmod{24}$ vertices. $4,281$ of $6,332$ 68-vertex and $5,520$ of $126,409$ 92-vertex fullerenes are 56^f lego-like with $f{=}\frac{p_6}{12}{=}2, 3$, respectively.
- Any Goldgerg–Coxeter $GC_{s,s-1}(Dodecahedron)$ fullerene has $v{=}20{+}120\binom{s}{2}$ and it is lego-like. Its $12{+}60\binom{s}{2}$ faces form 12 legos: 5-gon surrounded by s-1 coronas of 6-gons.

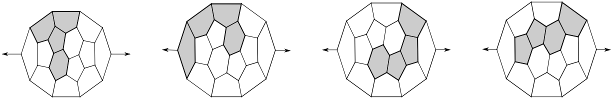
All 11 possible lego's kinds in 28-vertex fullerenes



28 $T_d(S_4)$ 13 $T_d(S_4)$ 13 $T_d(S_4)$ 13 $T_d(S_4)$ 13



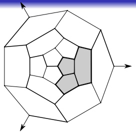
28 $T_d(T)$ 13 $T_d(D_2)$ 13 $T_d(D_2)$ 13



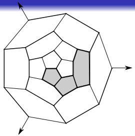
28 $D_2(D_2)$ 25 $D_2(D_2)$ 25 $D_2(D_2)$ 25 $D_2(D_2)$ 25

Representatives of all kinds of lego tilings in $F_{28}(T_d)$ and $F_{28}(D_2)$ having lego-wise, $2,1,1,1,1,4,2,0,1,0,0$ and $3,1,3,3,0,5,5,1,1,2,1$ orbits

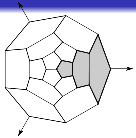
All possible lego's kinds in 32-, 44-, 68-vertex fullerenes



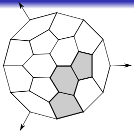
32 $D_{3d}(D_3)$ **2**+6+1+0



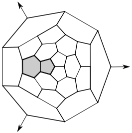
$D_{3d}(D_3)$ 2+**6**+1+0



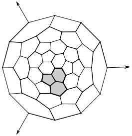
$D_{3d}(D_{3d})$ 2+6+**1**+0



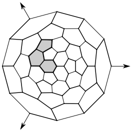
$D_{3h}(D_3)$ 1+4+0+**1**



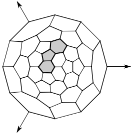
44 $D_{3h}(D_3)$ **69**



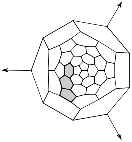
68 $T_d(T)$ **1**+1+40+0



$T_d(T)$ 1+**1**+40+0

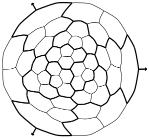


$T_d(T)$ 1+1+**40**+0

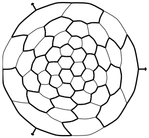


$D_{3d}(D_3)$ 0+0+0+**1**

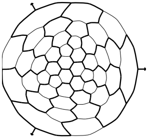
All possible lego's kinds in 92-vertex fullerenes



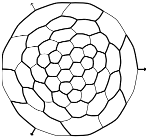
92, $T_h(T)$



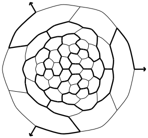
92, $T_h(T)$



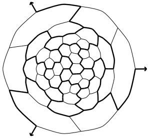
92, $T_h(S_6)$



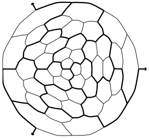
92, $T_h(S_6)$



92, $T_d(T)$



92, $T_d(T)$



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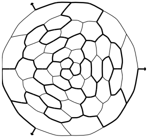

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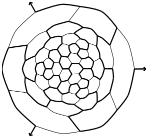

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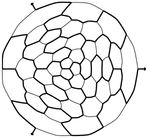
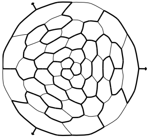
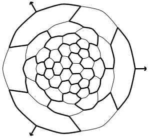


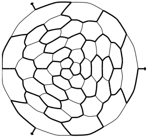
92, $T_d(T)$



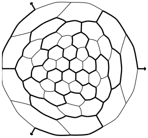
92, $T_d(T)$



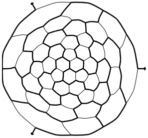
92, $T_d(T)$



92, $T_d(T)$



92, $T(T)$



92, $T(T)$

Parabolic lego-like $(\{a,b\};k)$-$\mathbb{S}^2$: computations

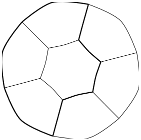
For (3, 6; 3), (2, 6; 3), (2, 4; 4), all computed spheres are lego-like.

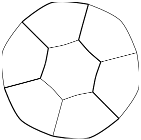
k	lego	(p_a, p_b)	v	nbG/real.	nbCases/real.	nbCasesRed/real.	Max./Min.	total
3	$4^3 6$	(6,2)	12	1/1	9/3	7/3	3/3	3
3	$4^2 6$	(6,3)	14	1/1	4/2	4/2	2/2	2
3	46	(6,6)	20	3/3	1/1	1/1	9/2	13
3	46^2	(6,12)	32	8/8	5/5	4/4	18/3	59
3	46^3	(6,18)	44	14/14	21/20	13/13	36/2	132
3	46^4	(6,24)	56	23/20	103/86	57/53	60/1	324
3	$5^6 6$	(12,2)	24	1/1	628/31	328/31	31/31	31
3	$5^4 6$	(12,3)	26	1/1	62/6	36/6	6/6	6
3	$5^3 6$	(12,4)	28	2/2	18/16	11/11	25/13	38
3	$5^2 6$	(12,6)	32	6/6	5/5	4/4	13/4	45
3	56	(12,12)	44	89/89	1/1	1/1	627/1	11846
3	56^2	(12,24)	68	6332/4281	5/5	4/4	128/1	36760
3	56^3	(12,36)	92	126409/5520	25/25	15/15	287/1	18691
4	$3^4 4$	(8,2)	8	1/1	20/5	13/5	5/5	5
4	$3^2 4$	(8,4)	10	2/2	4/4	3/3	8/4	12
4	34	(8,8)	14	8/8	1/1	1/1	11/1	27
4	34^2	(8,16)	22	51/43	4/4	3/3	14/1	268
4	34^3	(8,24)	30	218/69	16/16	10/10	20/1	311
4	34^4	(8,32)	38	650/118	59/54	33/32	30/1	412
4	34^5	(8,40)	46	1653/327	229/157	121/94	77/1	1312
6	$2^3 3$	(6,2)	3	1/1	4/2	3/2	2/2	2
6	23	(6,6)	5	2/2	1/1	1/1	2/1	3
6	23^2	(6,12)	8	12/10	3/3	2/2	4/1	22
6	23^3	(6,18)	11	16/9	7/6	4/4	5/1	19
6	23^4	(6,24)	14	42/18	22/18	12/10	10/1	52
6	23^5	(6,30)	17	48/11	61/27	32/17	28/1	55
6	23^6	(6,36)	20	100/26	180/89	93/57	29/1	179

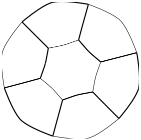
Parabolic lego-like $(\{a,b\};k)$-$\mathbb{S}^2$: computations

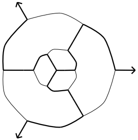
- A parabolic $(\{a,b\};k)$-$\mathbb{S}^2$ is lego-admissible if and only if:
  for fullerenes $(\{5,6\};3)$- $p_6 \mid 12$ or $12 \mid p_6$, i.e., either $v = 24, 26, 28, 32$, or $v \equiv 20 \pmod{24}$;
  for $(\{4,6\};3)$- $p_6 \mid 6$ or $6 \mid p_6$: $v{=}12, 14$ or $v \equiv 8 \pmod{12}$;
  for $(\{3,6\};3)$- $p_6 \mid 4$ or $4 \mid p_6$: $v = 8$ or $v \equiv 4 \pmod{8}$;
  for $(\{2,6\};3)$- $p_6 \mid 3$, impossible, or $3 \mid p_6$: $v \equiv 2 \pmod{6}$;
  for $(\{3,4\};4)$- $p_4 \mid 8$ or $8 \mid p_4$: $v{=}8, 10$ or $v \equiv 6 \pmod{8}$;
  for $(\{2,4\};4)$- $p_4 \mid 4$ or $4 \mid p_4$: $v = 4$ or $v \equiv 2 \pmod{4}$;
  for $(\{2,3\};6)$- $p_3 \mid 6$ or $6 \mid p_3$: $v = 3$ or $v \equiv 2 \pmod{3}$;
  for $(\{1,3\};6)$- $p_3 \mid 3$, impossible, or $3 \mid p_3$, impossible.
- All 126 lego-admissible parabolic $(\{a,b\};k)$-$\mathbb{S}^2$ with $p_b \le p_a$ (and all 22 $(\{4,6\};3)$-$\mathbb{S}^2$ with $\frac{p_6}{p_4} = 2, 3$) are lego-like.
- For $(a,b;k) = (4,6;3), (5,6;3), (3,4;4), (2,3;6)$, the vertex numbers, for which a lego-admissible, but not lego-like, parabolic $(\{a,b\};k)$-$\mathbb{S}^2$ is known, are all v, not as above. For $(3,6;3), (2,6;3), (2,4;4)$, all computed spheres are lego-like.

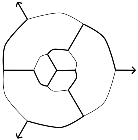
$(\{4, 6\}, 3)$-$\mathbb{S}^2$: all legos for $v<44$ and $2, 3$ for $v=44, 56$

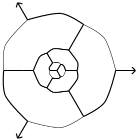

12 $D_{6h}(C_{2h})$

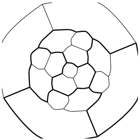

12 $D_{6h}(C_2)$

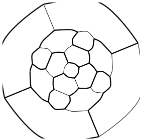

12 $D_{6h}(D_{3d})$

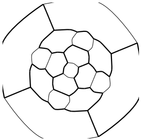

14 $D_{3h}(C_{3h})$

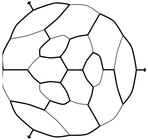

14 $D_{3h}(D_3)$

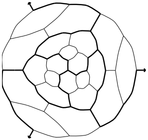

20 $D_{3d}(S_6)$

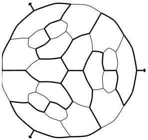

32 $O_h(S_6)$

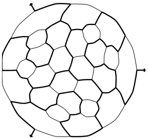

32 $O_h(S_6)$

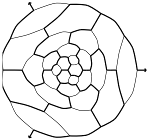

32 $O_h(T_h)$

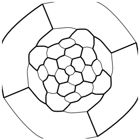

32 $D_{3h}(D_3)$


44, $D_3(D_3)$

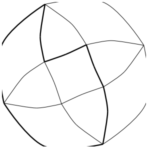
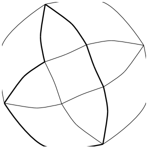
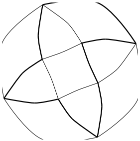
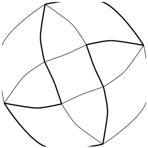
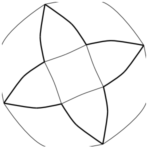

44 $D_{3h}(D_3)$


56, $D_3(D_3)$

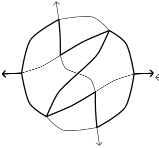
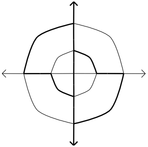
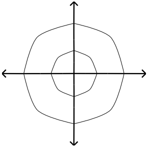
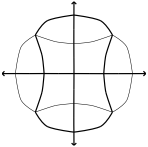
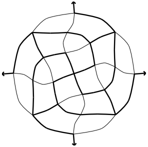

56 $D_{3d}(S_6)$


56 $O(T)$

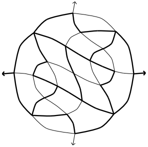
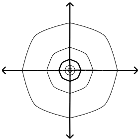
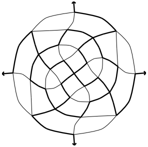
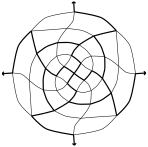
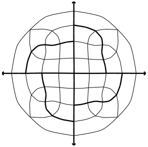
$(\{3, 4\}, 4)$-$\mathbb{S}^2$: all legos for $v < 30$ and 1 for $v = 30, 38, 46$

8 $D_{4d}(C_2)$ 8 $D_{4d}(C_2)$ 8 $D_{4d}(C_2)$ 8 $D_{4d}(D_2)$ 8 $D_{4d}(D_{4d})$

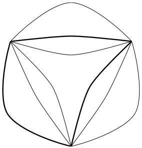






10 $D_2(D_2)$ 10 $D_{4h}(D_2)$ 10 $D_{4h}(D_{4h})$ 14 $D_{4h}(D_{2d})$ 22 $D_4(D_4)$

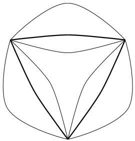






22 $D_{2d}(D_2)$ 22 $D_{4h}(D_{4h})$ 30 $O(O)$ 38 $D_4(D_4)$ 46 $D_{4h}(D_4)$

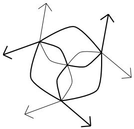
$(\{2,3\},6)$-$\mathbb{S}^2$: all legos for $v<14$ and 2 for $v=14,17,20$



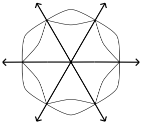
3, $D_{3h}(C_2)$



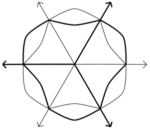
3, $D_{3h}(D_{3h})$



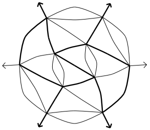
5, $D_{3h}(D_3)$



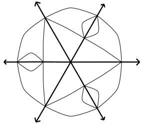
8, $D_{6h}(D_{6h})$



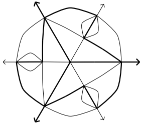
8, $D_{6h}(D_3)$



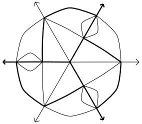
11, $C_2(C_2)$



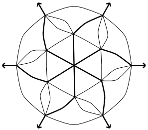
11, $D_{3h}(D_{3h})$



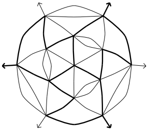
11, $D_{3h}(D_3)$



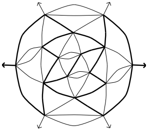
11, $D_{3h}(D_3)$



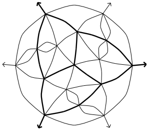
14, $D_6(D_6)$



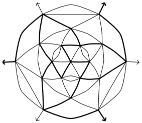
14, $D_3(D_3)$



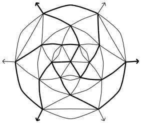
17, $C_2(C_2)$



17, $D_3(D_3)$



20, $D_{3d}(S_6)$



20, $D_{3h}(D_3)$

Goldberg–Coxeter series $GC_z(G_0)$: lego-admissibility

- Such $(\{a,b\};k)$-$\mathbb{S}^2$ are parameterized by one $z\in\mathbb{C}$: Gaussian integer $s+ti$, $||z||=z\overline{z}=s^2+t^2$ for $k=4$ and Eisenstein integer $s+t\omega$, $\omega=e^{\frac{2\pi}{6}i}=\frac{1+i\sqrt{3}}{2}$, $||z||=z\overline{z}=s^2+st+t^2$ for $k=3,6$.
- We have $GC_z(G_{z'}(G_0))=G_{z''}(G_0)$, where $z''=zz'$ is multiplication in the rings $\mathbb{Z}[i]=\mathbb{Z}^2$ and $\mathbb{Z}[\omega]$ of such integers.
- Given $z\in\mathbb{Z}[i]$ or $\in\mathbb{Z}[\omega]$ and a parabolic $(\{a,b\};k)$-sphere G_0 with p_a a-gons, p_b b-gons and so, $v=\frac{a}{k}p_a+\frac{b}{k}p_b$ vertices, the parabolic $(\{a,b\};k)$-sphere $GC_z(G_0)$ has $v'=v||z||$ vertices, $p'_a=p_a$ and $p'_b=\frac{k}{b}(v||z||-\frac{a}{k}p_a)=\frac{||z||a}{b}p_a+||z||p_b-\frac{a}{b}p_a$.
- So, $\frac{p'_b}{p'_a}=(||z||-1)\frac{a}{b}+||z||\frac{p_b}{p_a}\in\mathbb{N}$ if $\frac{p_b}{p_a}\in\mathbb{N}$ and for $(a,b;k)=$ $(5,6;3)$, $(3,4;4)$, $(2,3;6)$: $||z||\equiv 1 \pmod{b}$, $(3,6;3)$, $(2,4;4)$: $||z||\equiv 1 \pmod{2}$ and $(4,6;3)$, $(2,6;3)$: $||z||\equiv 1 \pmod{3}$.
- Each of 7 sets of all such z form a multiplicative submonoid of $\mathbb{Z}(i)$ or $\mathbb{Z}(\omega)$ (submonoids, by multiplication and addition, of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, respectively, with $s\geq t\geq 0,(s,t)\neq(0,0)$).

7 $||\cdot||$-defined monoids of Eisenstein and Gaussian integers

The submonoids $\mathbb{Z}(i)$, $\mathbb{Z}(\omega)$ (of $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, respectively, with $s \geq \max(t,1) \geq 0$) admit following three partitions into 2 monoids:

$||s{+}t\omega|| = s^2{+}st{+}t^2 \equiv 0$ or $1,3 \pmod 4$ iff $s,t \equiv 0 \pmod 2$ or not
$M{=}\{z \in Z(\omega) : ||z|| \equiv 1 \pmod 2\}$ and $\overline{M}{=}\mathbb{Z}(\omega)\backslash M$ are monoids.

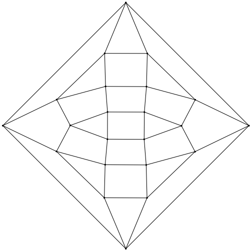
$||s{+}t\omega|| = 3st{+}(s\text{-}t)^2 \equiv 0$ or $1 \pmod 3$ iff $s\text{-}t \equiv 0$ or $1,2 \pmod 3$.
$N{=}\{z \in Z(\omega) : ||z|| \equiv 1 \pmod 3\}$ and $\overline{N}{=}\mathbb{Z}(\omega)\backslash N$ are monoids, since $(s+t\omega)(s'+t'\omega) = (S = ss' - tt') + (T = tt' + st' + s't)\omega$ and $s\text{-}t, s'\text{-}t' \equiv m \pmod 3$ imply $S\text{-}T \equiv m^2 \pmod 3$.

$L{=}M \cap N{=}\{z \in Z(\omega) : ||z|| \equiv 1 \pmod 6\}$ is also monoid.

$||s{+}ti|| {=} 2st{+}(s\text{-}t)^2 \equiv 0,2$ or $1 \pmod 4$ iff $s\text{-}t \equiv 0$ or $1 \pmod 2$.
$R{=}\{z \in Z(i) : ||z|| \equiv 1 \pmod 4\}$ and $\overline{R}{=}\mathbb{Z}(i)\backslash R$ are monoids.

Two series of lego-admissible $GC_z(G_0)$ with G_0's $\frac{p_b}{p_a} \notin \mathbb{N}$

- (i) $(\{4,6\},3)$-$\mathbb{S}^2$: $v \equiv 2 \pmod{12}$, $z \equiv 4 \pmod{12}$. Smallest case: $v = 14, z = 2 + 0\omega$; unique G_0 has $\frac{p_b}{p_a} = \frac{3}{6}$, it is 4^26; 56-vertex $GC_{2,0}(G_0)$ is lego-admissible but not lego like.
- (ii) $(\{3,4\},4)$-$\mathbb{S}^2$: $v \equiv 3 \pmod{4}$, $z \equiv 2 \pmod{4}$. Smallest case: $v{=}11, z{=}1{+}i$; both, G_0 and $G_{1,1}(G_0)$ are not lego-like.



22, $C_{2\nu}$; $\vec{p} = (p_3, p_4) = (8, 16)$
$G_{1,1}(G_0{=}\textit{unique}\ 11\textit{-vertex}\ \{3,4\};4)$-$\mathbb{S}^2)$

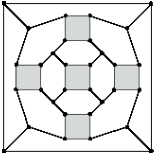
It is lego-admissible ($p_4 = 2p_3$) but not lego-like, i.e., not 34^2

Infinite series of lego-like Goldberg–Coxeter $GC_z((\{a\}; k))$

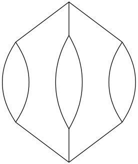
- Theorem: If $||z = s + t\omega|| \equiv 1 \mod 6$, then $GC_z((\{a\}, 3)\text{-}\mathbb{S}^2)$ is a lego-like $(\{a, 6\}, 3)\text{-}\mathbb{S}^2$ for $a = 2, 3, 4, 5$. Moreover:
  (i) $GC_z(Dodecahedron)$ is lego-like iff $||z|| \equiv 1 \mod 6$.
  (ii) $GC_{s,s-1}((\{a\}, k)\text{-}\mathbb{S}^2)$ is lego-like iff $(\{a\}; k)\text{-}\mathbb{S}^2$ lego-like, i.e. for each of 7 (all but $(\{1, 3\}, 6)\text{-}\mathbb{S}^2$) parabolic families.
  In fact, $||s + (s-1)\omega|| = s^2 + s(s-1) + (s-1)^2 = 6\binom{s}{2} + 1$ and $||s + (s-1)i|| = s^2 + (s-1)^2 = 4\binom{s}{2} + 1$ for $k = 4$.
- Conjecture: lego-admissible $GC_{s,t}((\{a\}; k)\text{-}\mathbb{S}^2)$ are lego-like. Moreover: (i) One of possible legos is a-gon, surrounded, in some a-gonal symmetry, by layers (not necessarily complete) of b-gons. It holds for above $t = s - 1$, when $\frac{p_b}{p_a} = a\binom{s}{2}$.
  (ii) If the number of vertices is large enough, no other lego-like parabolic spheres exist.

All parabolic ab^f-spheres $GC_{2,0}$ (1, 2, 3-rd) and all 7 $GC_{2,1}$

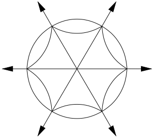
Unique $GC_{1,1}$: Trunc. Tetrahedron, 12, T_d; $(\{3, 6\}; 3)$-, $\vec{p}$=$(4, 4)$.



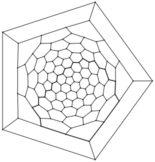
32, O_h
$(\{4, 6\}; 3)$-
$\vec{p}$=$(6, 12)$



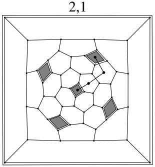
8, D_{3h}
$(\{2, 6\}; 3)$-
$\vec{p}$=$(3, 3)$



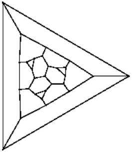
8, D_{6h}
$(\{2, 3\}; 6)$-
$\vec{p}$=$(6, 12)$



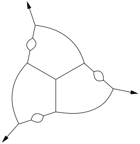
140, I
$(\{5, 6\}; 3)$-
$\vec{p}$=$(12, 60)$



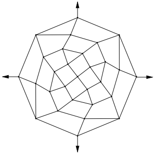
56, O
$(\{4, 6\}; 3)$-
$\vec{p}$=$(6, 24)$



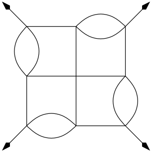
28, T
$(\{3, 6\}; 3)$-
$\vec{p}$=$(4, 12)$



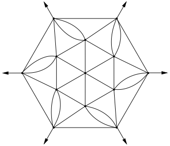
14, D_3
$(\{2, 6\}; 3)$-
$\vec{p}$=$(3, 6)$



30, O
$(\{3, 4\}; 4)$-
$\vec{p}$=$(8, 24)$

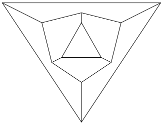


10, D_4
$(\{2, 4\}; 4)$-
$\vec{p}$=$(4, 8)$

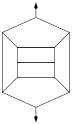


14, D_6
$(\{2, 3\}; 6)$-
$\vec{p}$=$(6, 24)$

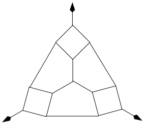
All (13 and 1 infinite series) elliptic lego-like $(\{a, b\}; k)$-$\mathbb{S}^2$



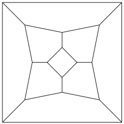
12, D_{3d} $(\{3, 5\}; 3)$-
$\vec{p}$=(2, 6)



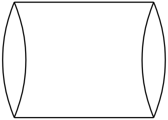
12, D_{2d} $(\{4, 5\}; 3)$-
$\vec{p}$=(4, 4)



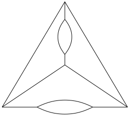
14, D_{3h} $(\{4, 5\}; 3)$-
$\vec{p}$=(3, 6)



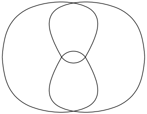
16, D_{4d} $(\{4, 5\}; 3)$-
$\vec{p}$=(2, 8)



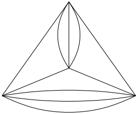
4, D_{2h} $(\{2, 4\}; 3)$-
$\vec{p}$=(2, 2)



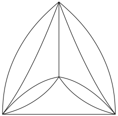
8, D_{2d} $(\{2, 5\}; 3)$-
$\vec{p}$=(2, 4)



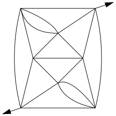
4, D_{2d} $(\{2, 3\}; 4)$-
$\vec{p}$=(2, 4)



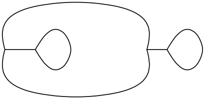
4, D_{2d} $(\{2, 3\}; 5)$-
$\vec{p}$=(4, 4)



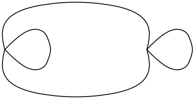
4, D_{2d} $(\{2, 3\}; 5)$-
$\vec{p}$=(4, 4)



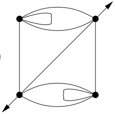
8, D_{2d} $(\{2, 3\}; 5)$-
$\vec{p}$=(2, 12)



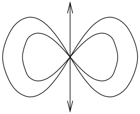
4, C_{2h} $(\{1, 5\}; 3)$-
$\vec{p}$=(2, 2)



2, C_{2h} $(\{1, 3\}; 4)$-
$\vec{p}$=(2, 2)



4, C_{2h} $(\{1, 3\}; 5)$-
$\vec{p}$=(2, 6)



1, $C_{2\nu}$ $(\{1, 2\}; k)$-
$\vec{p}$=$(2, 2\frac{k-2}{4})$

Hyperbolic lego-like $(\{a, b\}; k)$-$\mathbb{S}^2$: computations

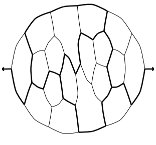
k	lego	(p_a, p_b)	v	nbG/real.	nbCases/real.	nbCasesRed/real.	Max./Min.	total
3	37^2	(12,24)	68	$\geq 105/\geq 101$	5/5	3/3	$\geq 120/1$	≥ 2625
3	37	(6,6)	20	4/4	1/1	1/1	6/2	15
3	$3^2 8$	(6,3)	14	1/1	4/2	4/2	2/2	2
3	38	(12,12)	44	298/203	1/1	1/1	104/3	4812
3	$3^3 9$	(6,2)	12	1/1	9/4	6/4	4/4	4
3	$3^2 9$	(8,4)	20	3/3	4/4	4/4	6/4	15
3	$4^2 7$	(8,4)	20	2/2	4/4	4/4	7/4	11
3	47	(12,12)	44	127/78	1/1	1/1	224/2	3440
3	$4^4 8$	(8,2)	16	2/2	34/16	24/16	11/6	17
3	$4^3 8$	(9,3)	20	0/0	14/0	10/0	0/0	0
3	$4^2 8$	(12,6)	32	32/17	5/5	5/5	11/1	61
3	$4^3 9$	(12,4)	28	3/3	18/18	12/12	18/12	46
3	$5^5 7$	(15,3)	32	0/0	276/0	146/0	0/0	0
3	$5^4 7$	(16,4)	36	2/2	79/54	45/37	53/45	98
3	$5^3 7$	(18,6)	44	13/11	21/21	13/13	27/1	103
3	$5^2 7$	(24,12)	68	6556/1122	5/5	4/4	303/1	10976
3	$5^6 8$	(18,3)	38	1/1	1316/20	682/20	20/20	20
3	$5^5 8$	(20,4)	44	3/3	374/148	196/105	89/30	191
3	$5^4 8$	(24,6)	56	27/15	103/84	59/55	75/1	343
4	$3^5 5$	(10,2)	10	1/1	59/11	34/11	11/11	11
4	$3^3 5$	(12,4)	14	2/2	12/10	8/8	10/6	16
4	$3^2 5$	(16,8)	22	52/13	4/4	3/3	27/1	157
5	$3^7 4$	(28,4)	20	5/5	803/233	407/171	86/24	300
5	$3^6 4$	(30,5)	22	12/3	305/3	159/2	2/1	4
5	$3^4 4$	(40,10)	32	45460/66	39/25	22/15	8/1	115

All hyperbolic lego-admissible $(\{a, b\}; k)$-$\mathbb{S}^2$ with $a \geq 3$:

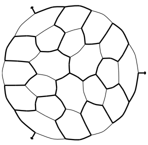
- For $(\{5, b \geq 7\}; 3)$-$\mathbb{S}^2$: $\vec{p} =$ $(2b, 2), (3(b-2), 3), (4(b-3), 4), (6(b-4), 6), (12(b-5), 12)$.
- For $(\{4, b \geq 7\}; 3)$-$\mathbb{S}^2$:
  $\vec{p} = (b, 2), (3\frac{b-2}{2}; 3), (3(b-4), 6)$ if b is even,
  $\vec{p} = (2(b-3), 4), (6(b-5), 12)$ if b is odd.
- For $(\{3, b \geq 7\}; 3)$-$\mathbb{S}^2$:
  $\vec{p} = (2\frac{b}{3}, 2), (4\frac{b-3}{3}; 4)$ if $b \equiv 0 \pmod 3$,
  $\vec{p} = (b-2, 3), (4(b-5), 12)$ if $b \equiv 2 \pmod 3$,
  $\vec{p} = (2(b-4), 6)$ if $b \equiv 1 \pmod 3$ and
  exceptional case of $\vec{p} = (12, 24)$ for $(\{3, 7\}; 3)$-$\mathbb{S}^2$.
- For $(\{3, b \geq 5\}; 4)$-$\mathbb{S}^2$: $\vec{p} =$ $(2b, 2), (4(b-2), 4), (8(b-3), 8)$.
- For $(\{3, b \geq 4\}; 5)$-$\mathbb{S}^2$: $\vec{p} = (6b, 2)$, $(4(3b-5), 4)$, $(5(3b-6), 5), (10(3b-8), 10), (20(3b-9), 20)$

Table presents lego-likeness data for smallest b in all above cases.

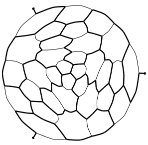
All hyperbolic lego-like $(\{a, b\}; k)$-$\mathbb{S}^2$ with $a \geq 3$: examples



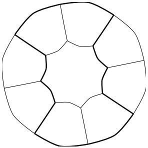
36 $D_{2d}(S_4)$
$(\{5, 7\}; 3\text{-}\mathbb{S}^2$



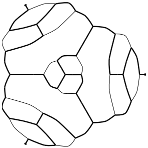
44 $D_3(D_3)$
$(\{5, 7\}; 3\text{-}\mathbb{S}^2$



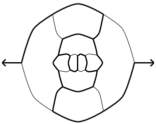
68 $T(T)$
$(\{5, 7\}; 3\text{-}\mathbb{S}^2$



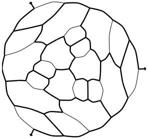
16 $D_{8h}(D_{2d})$
$(\{4, 8\}; 3\text{-}\mathbb{S}^2$



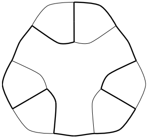
32 $T_d(T)$
$(\{4, 8\}; 3\text{-}\mathbb{S}^2$



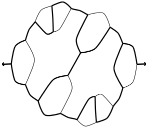
20 $D_{2d}(S_4)$
$(\{4, 7\}; 3\text{-}\mathbb{S}^2$



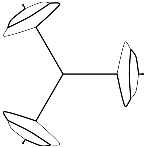
44 $T_h(T_h)$
$(\{4, 7\}; 3\text{-}\mathbb{S}^2$



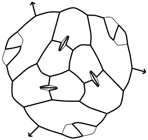
12 $D_{3h}(D_3)$
$(\{3, 9\}; 3\text{-}\mathbb{S}^2$



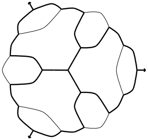
20 $D_{2d}(S_4)$
$(\{3, 9\}; 3\text{-}\mathbb{S}^2$



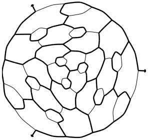
14 $D_{3h}(D_3)$
$(\{3, 8\}; 3\text{-}\mathbb{S}^2$



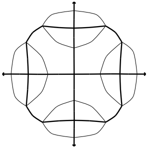
44 $T_h(T)$
$(\{3, 8\}; 3\text{-}\mathbb{S}^2$



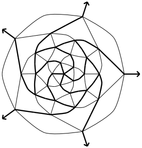
20 $D_{3d}(D_{3d})$
$(\{3, 7\}; 3\text{-}\mathbb{S}^2$



68 $T(T)$
$(\{3, 7\}; 3\text{-}\mathbb{S}^2$



22 $D_{4h}(D_{4h})$
$(\{3, 5\}; 4\text{-}\mathbb{S}^2$



32 $D_{5d}(S_{10})$
$(\{3, 4\}; 5\text{-}\mathbb{S}^2$

Lego-like $(\{a,b\};k)$-spheres with $a \geq 3$: synopsis

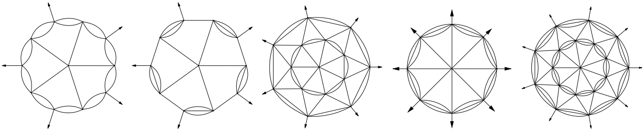
There are 4 elliptic ones and 4 infinite subseries: of parabolic series $(\{5,6\};3)$-, $(\{4,6\};3)$-, $(\{3,6\};3)$-, $(\{3,4\};4)$-$\mathbb{S}^2$. For hyperbolic:

- All possible (a,k) are $(5,3),(4,3),(3,3),(3,4)$ and $(3,5)$ with any integer $b > \frac{2k}{k-2}$ for each of possible five (a,k).
- The number of such spheres is finite for each fixed b.
- $1 \leq \frac{p_a}{p_b} \leq 3b$, except the case $\vec{p} = (12,24)$ for $(\{3,7\};3)$-$\mathbb{S}^2$. $\frac{p_a}{p_b}=1$ only in 3 cases with $k=3$; $\frac{p_a}{p_b}=2$ only in 13 cases $k=3,4$. $\frac{p_a}{p_b} = 3b$ only for $(\{3,b\};5)$-$\mathbb{S}^2$; otherwise, $\frac{p_a}{p_b} \leq 2b$.
- Any lego-admissible $(\{a,b\};k)$-$\mathbb{S}^2$ with $p_b=2\leq a$ is lego-like. All such lego-non-admissible ones are odd prisms and $(\{2,b\};k)$-$\mathbb{S}^2$ with odd $\frac{b(k-2)}{2}$. We list also all lego-like ones.

Lego-like $(\{2,b\};k)$-spheres: synopsis

There are 6 elliptic ones and 3 infinite subseries: of parabolic series $(\{2,6\};3)$-, $(\{2,4\};4)$-, $(\{2,3\};6)$-$\mathbb{S}^2$. For hyperbolic ones:

- There are double infinity of $(b>\frac{2k}{k-2}, k)$ for lego-admissible, but the number of such spheres is finite for each fixed (b,k). It holds $p_b \mid 4k$; for $k{=}3$, all $(2,3,4,6,12)$ are lego-admissible.
- $1 \leq \frac{p_2}{p_b} \leq \frac{b(k-2)}{4}$, except the cases $\vec{p} = (6,12),(12,36)$ for $(\{2,7\};3)$-$\mathbb{S}^2$ and $\vec{p} = (14,28),(28,84)$ for $(\{2,3\};7)$-$\mathbb{S}^2$.
- $(\{2,b\};k)$-$\mathbb{S}^2$ with $p_b{=}4k, 2k, \frac{4k}{3}, k$ is lego-agmissible iff, resp., $(b\text{-}2)(k\text{-}2){\equiv}3,2,1,0 \pmod 4$. Exp. of lego-like (b,p_b) are $(3,4k{=}16t{+}4)$, $(3,2k{=}8t)$, $(4,2k{=}4t{+}2)$, $(4t{+}2,k{=}3)$.

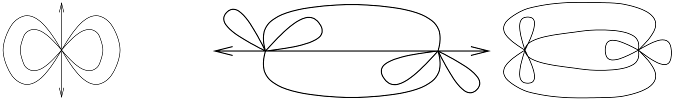


$12, D_{5d}$	$12, D_5$	$16, D_{7d}$	$10, D_{8h}$	$20, D_{9d}$
$\{2,4\};5$	$(\{2,4\};5)$-	$(\{2,3\};7)$-	$(\{2,3\};8)$-	$(\{2,3\};9)$-
$(10,10)$	$(10,10)$	$(14,28)$	$(16,16)$	$(36,36)$

Lego-like $(\{1,b\};k)$-spheres: synopsis

There are no parabolic ones. For elliptic: 3 and unique infinite series $(\{1,2\};k{=}4f{+}2)$-$\mathbb{S}^2$, $v{=}1$, with $\vec{p}{=}(2,2f)$. For hyperbolic:

- $\frac{p_1}{p_b}{\le}b$-2, except $\vec{p}{=}(4,2)$ for 1-vertex $(\{1,3\};10)$-$\mathbb{S}^2$, and $1{\le}\frac{p_1}{p_b}$, except 16 cases $(\{1,b\};k)$-$\mathbb{S}^2$ with $2{\le}\frac{p_b}{p_1}{\le}5$.
- For any $b{>}2{\le}p_b$ with even bp_b, series $(\{1,b\};k{=}p_b(b\text{-}1))$-, $v{=}2$, with $\vec{p}{=}(p_b(b\text{-}2),p_b)$. It is $p_b{\times}K_2$ with added, inside of each of p_b 2-gons: $\frac{b-2}{2}$ and $\frac{b-2}{2}$ 1-gons if b is even, or, alternating, $\frac{b-1}{2}$ and $\frac{b-3}{2}$ 1-gons if b is odd but p_b is even.
- For $\frac{p_a}{p_b}{=}1,2$, above series with $b{=}3,4$ are unique infinite ones



1, $C_{2\nu}$ — $(\{1,2\};k{=}10)$- $\vec{p}{=}(2,2\frac{k-2}{4})$

2, D_2 — $(\{1,3\},k{=}8)$- $\vec{p}{=}(\frac{k}{2},\frac{k}{2})$

2, D_{2d} — $(\{1,3\},k{=}8)$- $\vec{p}{=}(\frac{k}{2},\frac{k}{2})$

Lego-admissible $(\{a,b\};k)$-tori $\mathbb{T}^2$ and $\mathbb{K}^2$, $\mathbb{P}^2$

Any $(\{a,b\};k)$-$\mathbb{T}^2$ has $v=\frac{2}{k-2}p_a(\frac{p_b}{p_a}+1)$ and, if $p_b>0$, is hyperbolic
We have $a<\frac{2k}{k-2}\leq 6$ and, for $a\geq 3$, it holds $k<\frac{2a}{a-2}\leq 6$. For given a,k, the number of triples $(a,b;\frac{p_a}{p_b})$ with $\frac{p_a}{p_b}\in\mathbb{N}$ is infinite (say, $(\{5,b\};3)$-$\mathbb{T}^2$ with $p_5=(b\text{-}6)p_b$), while with $\frac{p_b}{p_a}\in\mathbb{N}$ it is finite (27).

The parameters of putative $(\{a,b\};k)$-$\mathbb{T}^2$ with $\frac{p_b}{p_a}\in\mathbb{N}$, $a\geq 3$. Also, 10 cases with $a=2$ ($k=3,\dots,8,10$) and 11 ($3\leq k\leq 14$) with $a=1$.

k	a,b	v	$\frac{p_b}{p_a}$
3	3,7	$8p_3$	3
3	3,9	$4p_3$	1
3	4,7	$6p_4$	2
3	4,8	$4p_4$	1
3	5,7	$4p_5$	1
4	3,5	$2p_3$	1

Lego-like maps $(\{a,b\};k)$ on the projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$ are the antipodal quotients of the centrally symmetric lego-like maps $(\{a,b\};k)$ on $\mathbb{S}^2$ and $\mathbb{T}^2$, resp., having $p_a,p_b\geq 4$.

Lego-like $(\{3, b\}; 3)$-tori with $\frac{p_a}{p_b} \leq 2$

$3, 4, 5$ are only possible a in a $(\{a, b\}; 3)$-torus with $a \geq 3$.

k	lego	(p_a, p_b)	v	nbG/real.	nbCases/real.	nbCasesRed/real.	Max./Min.	total
3	37^3	(1,3)	8	1/1	30/8	17/8	8/8	8
3	37^3	(2,6)	16	6/6	30/29	17/17	34/9	145
3	37^3	(3,9)	24	5/5	30/17	17/12	21/5	66
3	37^3	(4,12)	32	153/128	30/30	17/17	58/1	1735
3	37^3	(5,15)	40	219/74	30/17	17/12	28/1	276
3	37^3	(6,18)	48	6625/2165	30/30	17/17	81/1	11007
3	39	(1,1)	4	1/1	1/1	1/1	1/1	1
3	39	(2,2)	8	1/1	1/1	1/1	2/2	2
3	39	(3,3)	12	5/5	1/1	1/1	4/2	12
3	39	(4,4)	16	21/20	1/1	1/1	6/2	60
3	39	(5,5)	20	36/28	1/1	1/1	8/2	110
3	39	(6,6)	24	180/132	1/1	1/1	18/2	741
3	39	(7,7)	28	574/315	1/1	1/1	31/2	2194
3	39	(8,8)	32	2561/1296	1/1	1/1	49/2	11821
3	39	(9,9)	36	9402/3703	1/1	1/1	78/2	40284
3	$3^2 12$	(2,1)	6	1/1	6/2	6/2	2/2	2
3	$3^2 12$	(4,2)	12	5/4	6/6	6/6	5/4	18
3	$3^2 12$	(6,3)	18	14/12	6/4	6/4	4/1	21
3	$3^2 12$	(8,4)	24	217/96	6/6	6/6	14/1	299
3	$3^2 12$	(10,5)	30	245/60	6/5	6/5	4/1	89

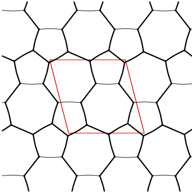
Lego-like $(\{4, b\}; 3)$-tori with $\frac{p_a}{p_b} \leq 2$

k	lego	(p_a, p_b)	v	nbG/real.	nbCases/real.	nbCasesRed/real.	Max./Min.	total
3	47^2	(1,2)	6	0/0	6/0	4/0	N/A	0
3	47^2	(2,4)	12	4/4	6/6	4/4	13/4	32
3	47^2	(3,6)	18	8/8	6/6	4/4	8/3	45
3	47^2	(4,8)	24	48/46	6/6	4/4	25/1	569
3	47^2	(5,10)	30	114/98	6/6	4/4	18/1	676
3	47^2	(6,12)	36	692/581	6/6	4/4	69/1	7145
3	47^2	(7,14)	42	2751/2013	6/6	4/4	66/1	17983
3	47^2	(8,16)	48	16970/11117	6/6	4/4	226/1	131136
3	48	(1,1)	4	1/1	1/1	1/1	1/1	1
3	48	(2,2)	8	3/3	1/1	1/1	1/1	3
3	48	(3,3)	12	5/5	1/1	1/1	3/1	7
3	48	(4,4)	16	25/23	1/1	1/1	10/1	79
3	48	(5,5)	20	21/15	1/1	1/1	7/1	41
3	48	(6,6)	24	158/115	1/1	1/1	30/1	858
3	48	(7,7)	28	161/89	1/1	1/1	29/1	634
3	48	(8,8)	32	1619/905	1/1	1/1	100/1	13918
3	48	(9,9)	36	1768/719	1/1	1/1	100/1	11751
3	48	(10,10)	40	19891/8269	1/1	1/1	360/1	236964
3	$4^2 10$	(2,1)	6	1/1	6/4	6/4	4/4	4
3	$4^2 10$	(4,2)	12	4/3	6/6	6/6	8/6	22
3	$4^2 10$	(6,3)	18	21/14	6/6	6/6	6/1	44
3	$4^2 10$	(8,4)	24	90/39	6/6	6/6	21/1	226
3	$4^2 10$	(10,5)	30	274/42	6/6	6/6	8/1	121
3	$4^2 10$	(12,6)	36	2450/435	6/6	6/6	24/1	1819

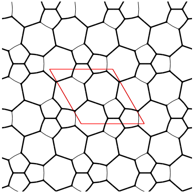
Lego-like $(\{5, b\}; 3)$-tori with $\frac{p_a}{p_b} \leq 3$

k	lego	(p_a, p_b)	v	nbG/real.	nbCases/real.	nbCasesRed/real.	Max./Min.	total
3	57	(1,1)	4	0/0	1/0	1/0	N/A	0
3	57	(2,2)	8	1/1	1/1	1/1	1/1	1
3	57	(3,3)	12	1/1	1/1	1/1	3/3	3
3	57	(4,4)	16	8/8	1/1	1/1	10/4	46
3	57	(5,5)	20	3/3	1/1	1/1	11/8	29
3	57	(6,6)	24	43/43	1/1	1/1	30/1	440
3	57	(7,7)	28	17/16	1/1	1/1	47/1	357
3	57	(8,8)	32	304/275	1/1	1/1	100/1	5866
3	57	(9,9)	36	229/191	1/1	1/1	234/1	8118
3	57	(10,10)	40	2698/2088	1/1	1/1	428/1	92030
3	57	(11,11)	44	2948/2109	1/1	1/1	829/1	154348
3	57	(12,12)	48	30625/19541	1/1	1/1	1514/1	1538904
3	$5^2 8$	(2,1)	6	1/1	6/4	5/4	4/4	4
3	$5^2 8$	(4,2)	12	4/4	6/6	5/5	9/6	31
3	$5^2 8$	(6,3)	18	10/8	6/6	5/5	7/2	37
3	$5^2 8$	(8,4)	24	46/46	6/6	5/5	28/1	370
3	$5^2 8$	(10,5)	30	118/65	6/6	5/5	17/1	228
3	$5^2 8$	(12,6)	36	670/414	6/6	5/5	75/1	2594
3	$5^2 8$	(14,7)	42	2613/763	6/6	5/5	58/1	3271
3	$5^2 8$	(16,8)	48	16162/4670	6/6	5/5	237/1	30743
3	$5^3 9$	(3,1)	8	0/0	30/0	18/0	N/A	0
3	$5^3 9$	(6,2)	16	4/4	30/27	18/18	35/12	108
3	$5^3 9$	(9,3)	24	7/6	30/15	18/12	12/1	27
3	$5^3 9$	(12,4)	32	120/94	30/30	18/18	57/1	1345
3	$5^3 9$	(15,5)	40	215/61	30/17	18/14	10/1	134
3	$5^3 9$	(18,6)	48	4601/1467	30/30	18/18	106/1	8673

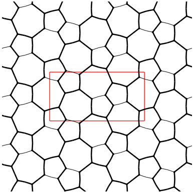
All but 1 $\leq$ 28-vertex azulenoids ($(\{5,7\};3)$-$\mathbb{T}^2$): lego-like



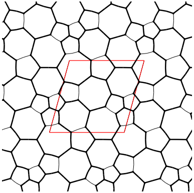
8, $c2mm$ (p2)
$\vec{p}=(2,2)$



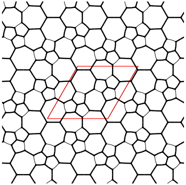
12, $p31m$ (p31m)
$\vec{p}=(3,3)$



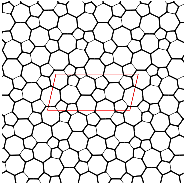
16, $p2gg$ (p2gg)
$\vec{p}=(4,4)$



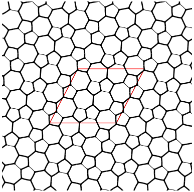
20, cm (p1)
$\vec{p}=(5,5)$



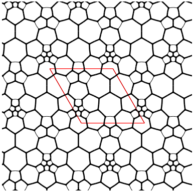
24, $p31m$ (p3)
$\vec{p}=(6,6)$



28, cm (cm)
$\vec{p}=(7,7)$



32, $p2gg$ (p2gg)
$\vec{p}=(8,8)$



36, $p3m1$ (p3m1)
$\vec{p}=(9,9)$

Representatives of unique kind of lego tiling of $(\{5,7\};3)$-$\mathbb{T}^2$ with $v \leq 36$

$(\{a, b\}; k)$-maps on general surfaces

(R,k)-maps on general surface $\mathbb{F}^2$

- Given $R \subset \mathbb{N}$ and a surface $\mathbb{F}^2$, an (R,k)-$\mathbb{F}^2$ is a k-regular map on surface $\mathbb{F}^2$ whose faces have gonalities $i \in R$.
- The Euler characteristic $\chi(\mathbb{F}^2)$ is v-e+f= $\sum_i p_i\kappa_i$, where κ_i=$1+\frac{i}{k}-\frac{i}{2}$ and p_i is the number of i-gons. So, elliptic and, with $|R|>1$, parabolic (R,k)-maps exist only on $\mathbb{S}^2$ and $\mathbb{P}^2$.
- In fact, all connected *closed* (compact and without boundary) irreducible surfaces $\mathbb{F}^2$ with $\chi(\mathbb{F}^2)\geq 0$ are (with $\chi = 2,0,1,0$, respectively): orientable: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ and non-orientable: real projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$.
- Again, let our (R,k)-maps be parabolic, i.e., $\min_{i\in R}\kappa_i = 0$. Then M=:$\max\{i\in R\}$=$\frac{2k}{k-2}$, and (M,k)=$(6,3)$, $(4,4)$, $(3,6)$.
- Also, there are infinity of parabolic maps (R,k)-$\mathbb{F}^2$, since the number p_M of *flat* (κ_M=0) faces is not restricted.
- Also, if $\chi(\mathbb{F}^2)$=$\sum_i p_i\kappa_i$= 0, i.e. $\mathbb{F}^2$ is $\mathbb{T}^2$ or $\mathbb{K}^2$, then R=$\{M\}$

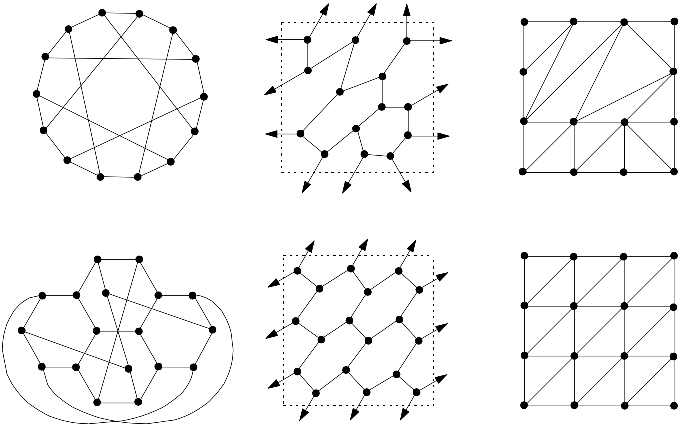
Parabolic $(\{a, b\}; k)$-maps on torus and Klein bottle

So, $\{a, b\}; k)$-$\mathbb{T}^2$ and $(\{a, b\}; k)$-$\mathbb{K}^2$ have $a = b = \frac{2k}{k-2}$ and $(a = b, k)$ should be $(6, 3), (3, 6)$ or $(4, 4)$.

We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

Smallest such $\mathbb{T}^2$- and $\mathbb{K}^2$-maps for $(a{=}b, k){=}(4, 4), (6, 3), (3, 6)$:
as 4-regular quadrangulations: K_5 and $K_{2,2,2}$ $(p_4 = 5, 6)$;
as 6-regular triangulations: K_7 and $K_{3,3,3}$ $(p_3 = 14, 18)$;
as 3-regular polyhexes: Heawood graph (dual K_7) and dual $K_{3,3,3}$ $(p_6{=}7, 9)$. Two those graphs are the smallest $\mathbb{T}^2$- and $\mathbb{K}^2$-fullerenes

Smallest $\mathbb{T}^2$- and $\mathbb{K}^2$-fullerenes: dual K_7 and dual $K_{3,3,3}$



3-regular polyhexes on $\mathbb{T}^2$, cylinder, Möbius surface, $\mathbb{K}^2$ are $\{6^3\}$'s quotients by fixed-point-free group of isometries, generated by: two translations, a transl., a glide reflection, transl. *and* glide reflection.

8 parabolic families on the projective plane

(R, k)-maps on the projective plane are the antipodal quotients of centrally symmetric (R, k)-$\mathbb{S}^2$; so, halving their p-vector and v.

The point symmetry groups with inversion operation are: T_h, O_h, I_h, C_{mh}, D_{mh} with even m and D_{md}, S_{2m} with odd m. So, they are

1. 9 for $\{5, 6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h, D_{5d}, I_h
2. 7 for $\{2, 3\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h
3. 6 for $\{4, 6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
4. 6 for $\{3, 4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
5. 2 for $\{2, 4\}_v$: D_{2h}, D_{4h}
6. 1 for $\{3, 6\}_v$: D_{2h}
7. 0 for $\{2, 6\}_v$ and $\{1, 3\}_v$
8. Cf. 12 for icosahedrites $((\{3, 4\}, 5)$-spheres): C_i, C_{2h}, C_{4h}, D_{2h}, D_{4h}, D_{3d}, D_{5d}, S_6, S_{10}, T_h, O_h, I_h

6 parabolic families $(\{a,b\};k)$-$\mathbb{P}^2$: 1-parameterization

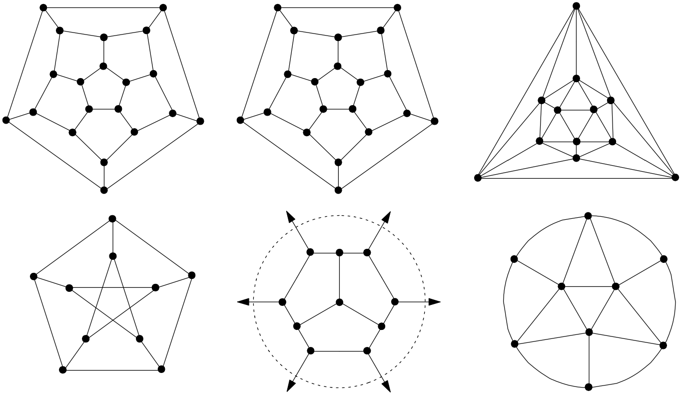
1. $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h, D_{5d}, I_h
2. $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h
3. $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
4. $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
5. $\{2,4\}_v$: D_{2h}, D_{4h}
6. $\{3,6\}_v$: D_{2h}

$(\{2,3\},6)$-spheres T_h and D_{6h} are $GC_{k,k}(2 \times \mathit{Tetrahedron})$ and, for $k \equiv 1,2 \pmod 3$, $GC_{k,0}(6 \times K_2)$, respectively. Other spheres of blue symmetry are $GC_{k,l}$ with $l = 0, k$ from the first such sphere.

So, each of 7 blue-symmetric families is described by one natural parameter k and contains $O(\sqrt{v})$ spheres with at most v vertices.

Petersen graph is the smallest projective plane's fullerene

The smallest maps for $(\{a,b\};k)$=$(\{5,6\},3),(\{3,4\},5),(\{4,6\},3)$ are: Petersen graph (dual K_6), K_6 (half-Icosahedron; smallest $\mathbb{P}^2$-triangulation), K_4 (smallest $\mathbb{P}^2$-quadrangulation), i.e., the antipodal quotients of Dodecahedron, Icosahedron and Cube.



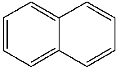
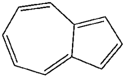
Relatives: plane fullerenes, azulenoids, schwartzites

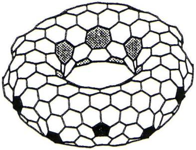
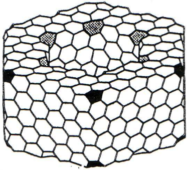
(Euclidean) plane fullerenes $(\{5, 6\}, 3)$-$\mathbb{E}^2$

- An $(\{a, b\}; k)$-$\mathbb{E}^2$ is a k-regular tiling of $\mathbb{E}^2$ by a- and b-gons.
- $(\{a, b\}; k)$-$\mathbb{E}^2$ have $p_a \leq \frac{b}{b-a}$ and $p_b = \infty$. It follows from Alexandrov, 1958: any metric on $\mathbb{E}^2$ of non-negative curvature can be realized as a metric of convex surface on $\mathbb{E}^3$. In fact, consider plane metric such that all faces became regular in it. Its curvature is 0 on all interior points (faces, edges) and ≥ 0 on vertices. A convex surface is at most half-$\mathbb{S}^2$.
- There are ∞ of $(\{a, b\}; k)$-$\mathbb{E}^2$ if $2 \leq p_a \leq \frac{b}{b-a}$ and 1 if $p_a = 0, 1$.
- For plane fullerenes (or *nanocones*) $(\{5, 6\}, 3)$-$\mathbb{E}^2$, the number of *equivalence* (isomorphic up to a finite induced subgraph) classes is (Klein–Balaban, 2007) $2, 2, 2, 1$ if $p_5 = 2, 3, 4, 5$, resp.
- Nanotubes (case $p_5 = 6$) come by rolling up the graphite $\{6^3\}$.
- There are 7 (with $b = 7, 7, 7, 7, 8, 8, 12$) plane fulleroids, i.e. $(\{5, b\}, 3)$-$\mathbb{E}^2$, which are 2-isohedral (symmetry $G \approx Aut$ and faces form 2 orbits under comb. automorphisms group Aut).

Two other $(\{5, 6, c\}, 3)$-$\mathbb{F}^2$ used in Chemistry

- Azulenoids: $(\{5, 6, 7\}, 3)$-$\mathbb{T}^2$; so, $g{=}1$, $p_5{=}p_7$ (Kirby–Diudea, 2003, et al.), since *naftalen* and *azulen* are $C_{10}H_8$ isomers.

- Schwartzits: $(\{6, c \geq 7\}, 3)$-$\mathbb{F}^2$ on minimal surfaces $\mathbb{F}^2$ of const. negative curvature ($g \geq 2$) (Terrones–MacKay, 1997). Knor et al., 2015: such polyhedral $(\{6, c\}, 3)$-maps exist for any $g{\geq}2$, $p_6{\geq}0$ and $c{=}7, 8, 9, 10$; with 1 undecided subcase. Analog of icos. fullerenes: $(\{6, 7\}, 3)_v$ on D-surface, $g{=}3$, with $v{=}56(p^2{+}pq{+}q^2)$, starting with Klein regular map $\{7^3\}$.

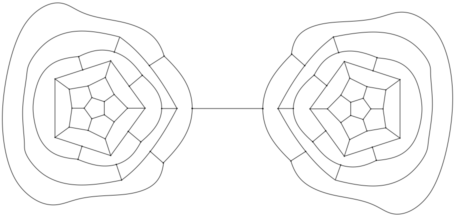
c-disk fullerenes

$(\{5,6,c\},3)$-spheres

- Clearly, a v-vertex $(\{5,6,c\},3)$-$\mathbb{S}^2$ is a fullerene if $c=5,6$ and $p_5=12+p_c(c-6)$, $\quad v=20+2(p_6+p_c(c-5))$, otherwise.
- In Haeckel, 1887, skeletons of radiolarian zooplankton *Aulonia hexagona* are represented by $(\{5,6,7\},3)$- and $(\{5,6,8\},3)$-spheres. Same holds for some basket's patterns.
- The spherical Voronoi polyhedra of many energy potential minimizers (say, in Thomson problem for v unit-charged particles on sphere $\mathbb{S}^2$) and maximizers (say, in Tammes problem of minimum distance between v points on $\mathbb{S}^2$) are fullerenes or, for large v, specific $(\{5,6,7\},3)$-$\mathbb{S}^2$.
- Behmaram, Doslic and Friedland, 2016, considered the number of perfect matchings in $(\{5,6,c\},3)$-$\mathbb{S}^2$ with $p_c=2$.
- We will consider in depth the case $p_c=1$, i.e., when 5- and 6-gons tile a c-disk, instead of a sphere as fullerenes do.

c-disk and c-multidisk fullerenes

- Call a $(\{5,6,c\},3)$-$\mathbb{S}^2$, p_c=1, c-disk-fullerene c-DF, if c-gon not self-intersects and c-multidisk-fullerene c-MDF, else.
- Any c-DF or c-MDF has p_5=c+6, v=$2(p_6+c+5)$ and there is an ∞ of c-DF's for any $c \geq 1$ and of c-MDF's for any $c \geq 8$
- Possible symmetry groups of a c-DF with $c \neq 5, 6$ or c-MDF: C_k, $C_{k\nu}$ with $k \in \{1, 2, 3, 5, 6\}$ and k dividing c (symmetries stabilize c-gon and axis pass by a vertex, edge or face),



8-$MDF_{78}(C_{2\nu})$: min. 8-MDF and c-MDF with smallest c

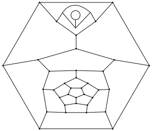
Fullerene c-disks: main notions

- Fullerene c-polycycle: an c-gon partitioned into 5- and 6-gons with vertices of degree 3 inside and 3 or 2 on the c-gon.
- c-disk fullerene: full. c-polycycle without degree 2 vertices; so, $p_5 = p_6 + 6$. If $c \in \{5, 6\}$, it is a fullerene without a face.
- Fullerene c-patch: fullerene c-polycycle, which is a fullerene's part; so, $p_5 \leq 12$. It is a c-disk fullerene if f $c \in \{5, 6\}$.
- c-thimble fullerene: a 3-connected c-disk fullerene with only 5-gons adjacent to the c-gon. It exists if and only if $c \geq 5$. Smallest c-thimble has $c - 6 \leq p_6 \leq \lfloor \frac{3(c-5)}{2} \rfloor$; conj.: $= \lfloor \frac{3(c-5)}{2} \rfloor$.

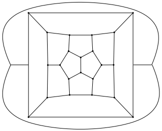
Connectivity of c-disk fullerenes

- Any c-*MDF* and 1-*DF* are 1-connected, but not 2-connected.
- Any c-*DF* is 2-connected; only 2-connected exist iff $c \geq 8$.
- Smallest such have $p_6 = 23, 17, 10, 8$ for $c = 8, 9, 10, 11$ and, for $c \geq 12$, $p_6 = 4, 5, 6$ if $c \equiv \pmod{10})$ to $4, 5, 6$ or $2, 3, 7, 8$ or $1, 9$
- Smallest 3-connected (i.e., polyhedral) ones have $m(c) := p_6 = 3, 2, 0, 1, 3, 4, 6, 7, 8$ for $3 \leq c \leq 11$ and (conj.) 6 for $c \geq 12$.
- Conjecture: 3-connected c-DF_v exists – except $(c, v) = (1, 42)$, $(3, 24)$, $(5, 22)$ – iff v is even and $v \geq 2(m(c) + c + 5)$.

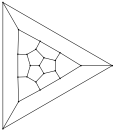
Minimal c-disk fullerenes



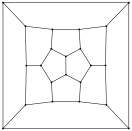
1 40, C_s



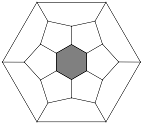
2 26, $C_{2\nu}$



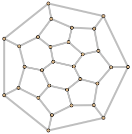
3 22, $C_{3\nu}$



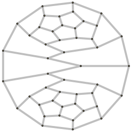
4 22, $C_{2\nu}$



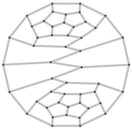
6 24 D_{6d}



7 30, C_s



13 48, C_s



14 50, C_2

For $v \neq 13, 14$ above are minimal, but minimal 13- and 14-*DF* are 2-connected and have $p_6 = 5, 4$ respectively, i.e. less than 6 above.
Conjecture: for $c \geq 13$, minimal 3-connected c-disk is c-pentatube $B + Hex_3 + Pen_{c-12} + Hex_3 + B$ (symmetry C_s/C_2 for odd/even c).
All minimal c-*DF*, $5 \leq c \leq 9$, and a minimal 10-*DF* are c-thimbles.