# Some Quasi-metrics and Oriented Hypercubes

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# I. General quasi-semi-metrics

# Quasi-semi-metrics

Given a set $X$, a function $q: X \times X \rightarrow \mathbb{R}_{\geq 0}$ with $q(x,x)$=$0$ is a **quasi-distance** (or, in Topology, **prametric**) on $X$.

- A quasi-distance $q$ is a **quasi-semi-metric** if for $x, y, z \in X$ holds $q(x,y) \leq q(x,z) + q(z,y)$ (**oriented triangle inequality**).
- $q'$ given by $q'(x,y)$=$q(y,x)$ is **dual** quasi-semi-metric to $q$.
- $(X, q)$ can be partially ordered by the **specialization order**: $x \preceq y$ iff $q(x,y) = 0$. **Discrete quasi-metric** on poset $(X, \leq)$ is $q_{\leq}(x,y) = 1_{x>y}$; for $(X, q_{\leq})$, order $\preceq$ coincides with $\leq$.
- A **weak quasi-metric** is a quasi-semi-metric $q$ with **weak symmetry**: $q(x,y) = q(y,x)$ whenever $q(y,x) = 0$.
- An **Albert quasi-metric** is a quasi-semi-metric $q$ with **weak definiteness**: $x = y$ whenever $q(x,y) = q(y,x) = 0$.

# Quasi-metrics

A **quasi-metric** (or asymmetric, directed, oriented metric) is a quasi-semi-metric $q$ with **definiteness**: $x = y$ iff $q(x, y) = 0$.
A **quasi-metric space** $(X, q)$ is a set $X$ with a quasi-metric $q$.
Asymmetric distances were introduced in Hausdorff, 1914.
Real world examples: one-way streets mileages, travel time, transportation costs (up/downhill or up/downstream).

A quasi-metric $q$ is **non-Archimedean** (or **quasi-ultrametric**) if it satisfy strengthened oriented triangle inequality

$$q(x, y) \leq \max\{q(x, z), q(z, y)\} \text{ for all } x, y, z \in X.$$

Cf. symmetric: distance, semi-metric, metric, ultrametric.

For a quasi-metric $q$, the functions $\frac{(q^p(x,y)+q^p(y,x))^{\frac{1}{p}}}{2}$, $p \geq 1$, (usually, $p = 1$ and $\frac{q(x,y)+q(y,x)}{2}$ is called **symmetrization** of $q$), $\max\{q(x, y), q(y, x)\}$, $\min\{q(x, y), q(y, x)\}$ are **metrics**.

# Example: gauge quasi-metric

Given a compact convex region $B \subset \mathbb{R}^n$ containing origin, the **convex distance function** (or **Minkowski distance function, gauge**) is the quasi-metric on $\mathbb{R}^n$ defined, for $x \neq y$, by

$$q_B(x, y) = \inf\{\alpha > 0 : y - x \in \alpha B\}.$$

Equivalently, it is $\frac{||y-x||_2}{||z-x||_2}$, where $z$ is unique point of the boundary $\partial(x + B)$ hit by the ray from $x$ via $y$. It holds $B = \{x \in \mathbb{R}^n : q_B(0, x) \leq 1\}$ with equality only for $x \in \partial B$.

If $B$ is centrally-symmetric with respect to the origin, then $q_B$ is a **Minkowskian metric** whose unit ball is $B$.

# Examples: quasi-metrics on $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{S}^1$

- **Sorgenfrey quasi-metric** is a quasi-metric $q(x, y)$ on $\mathbb{R}$, equal to $y - x$ if $y \geq x$ and equal to 1, otherwise.
- Some similar quasi-metrics on $\mathbb{R}$ are:
  $q_1(x, y) = \max\{y - x, 0\}$ ($l_1$ **quasi-metric**),
  $q_2(x, y) = \min\{y - x, 1\}$ if $y \geq x$ and equal to $1$, else,
  Given a>0, $q_3(x, y) = y - x$ if $y \geq x$ and =$a(x - y)$, else.
  $q_4(x, y) = e^y - e^x$ if $y \geq x$ and equal to $e^{-y} - e^{-x}$, else.
- The **real half-line quasi-semi-metric** on $\mathbb{R}_{>0}$ is $\max\{0, \ln \frac{y}{x}\}$.
- The **circular-railroad quasi-metric** is a quasi-metric on the **unit circle** $\mathbb{S}^1 \subset \mathbb{R}^2$, defined, for any $x, y \in \mathbb{S}^1$, as the length of counter-clockwise circular arc from $x$ to $y$ in $\mathbb{S}^1$.

# Digression: quasi-metrizable spaces

A topological space $(X, \tau)$ is called **quasi-metrizable space** if $X$ admits a quasi-metric $q$ such that the set of open $q$-balls $\{B(x, r) : r > 0\}$ form a neighborhood base at each $x \in X$.

More general **$\gamma$-space** is a topological space admitting a **$\gamma$-metric** $q$ ( a function $q : X \times X \to \mathbb{R}_{\geq 0}$ with $q(x, z_n) \to 0$ if $q(x, y_n) \to 0$ and $q(y_n, z_n) \to 0$) such that the set of open **forward** $q$-balls $\{B(x, r) : r > 0\}$ form a base at each $x \in X$.

The **Sorgenfrey line** is the topological space $(\mathbb{R}, \tau)$ defined by the base $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. It is not metrizable, 1st (not 2nd) countable paracompact (not locally compact) $T_5$**-space**. But it is quasi-metrizable by **Sorgenfrey quasi-metric**:

$q(x, y) = y - x$ if $y \geq x$, and $q(x, y) = 1$, otherwise.

# Digraph quasi-metric and metrics

- A **directed graph** (or **digraph**) is a pair $G = (V, A)$, where $V$ is a set of vertices and $A$ is a set of arcs.
- The **path quasi-metric** $q_{dpath}$ in digraph $G$=$(V, A)$ is, for any $u, v \in V$, the length of a shortest $(u - v)$ path in $G$.
  **Exp.**: **Web hyperlink quasi-metric** (or **click count**) is $q_{dpath}$ between two web pages (vertices of Web digraph).
- The **circular metric** (in digraph) is $q_{dpath}(u, v) + q_{dpath}(v, u)$.
- Chartrand-Erwin-Raines-Zhang, 1999: the **strong metric** between $u, v \in V$ is the minimum number of edges of strongly connected subdigraph of $G$ containing $u$ and $v$.
- Chartrand-Erwin-Raines-Zhang, 2001: the **orientation metric** between $2$ orientations $D$ and $D'$ of a graph is the minimum number of arcs of $D$ whose directions must be reversed to produce an orientation isomorphic to $D'$.

# Examples at large

- In Psychophysics, the **probability-distance hypothesis**: the probability with which one stimulus is discriminated from another is a (continuously increasing) function of some subjective quasi-metric between these stimuli.
- Østvang, 2001, proposed a quasi-metric framework for relativistic gravity.
- The **Thurston quasi-metric** on the **Teichmüller space** $T_g$ is $\frac{1}{2}\inf_h \ln ||h||_{Lip}$ for any $R_1^*, R_2^* \in T_g$, where $h : R_1 \rightarrow_2$ is a quasi-conformal homeomorphism, homotopic to the identity, and $||.||_{Lip}$ is the **Lipschitz norm** on the set of all injective functions $f : X \rightarrow Y$ defined by $||f||_{Lip} = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x),f(y))}{d_X(x,y)}$.

# Point-set distance and its applications

- In a (quasi)-metric space $(X, d)$, the **point-set distance** between $x \in X$ and $A \subset X$ is $d(x, A) = \inf_{y \in A} d(x, y)$, The function $f_A(x) = d(x, A)$ is **distance map**. Distance maps are used in MRI ($A$ is gray/white matter interface) as cortical maps, in Image Processing ($A$ is image boundary), in Robot Motion ($A$ is obstacle points set).
- $A \subset X$ is **Chebyshev set** if for each $x \in X$, there is unique **element of best approximation**: $y \in A$ with $d(x, y) = d(x, A)$. If $A \subset X$ (usually, $A$ is the boundary of a solid $X \subset \mathbb{R}^3$), **skeleton** of $X$ is $\{x \in X : |\{y \in A : d(x, y) = d(x, A)\}| > 1\}$, i,e. all boundary points of **Voronoi regions** of points of $A$.
- The **directed Hausdorff distance** (on compact subspaces of $(X, d)$) is $q_{dHaus}(B, A) = \sup_{x \in B} d(x, A)$. The **Hausdorff metric** is $d_{Haus}(A, B) = \max\{q_{dHaus}(A, B), q_{dHaus}(B, A)\}$.

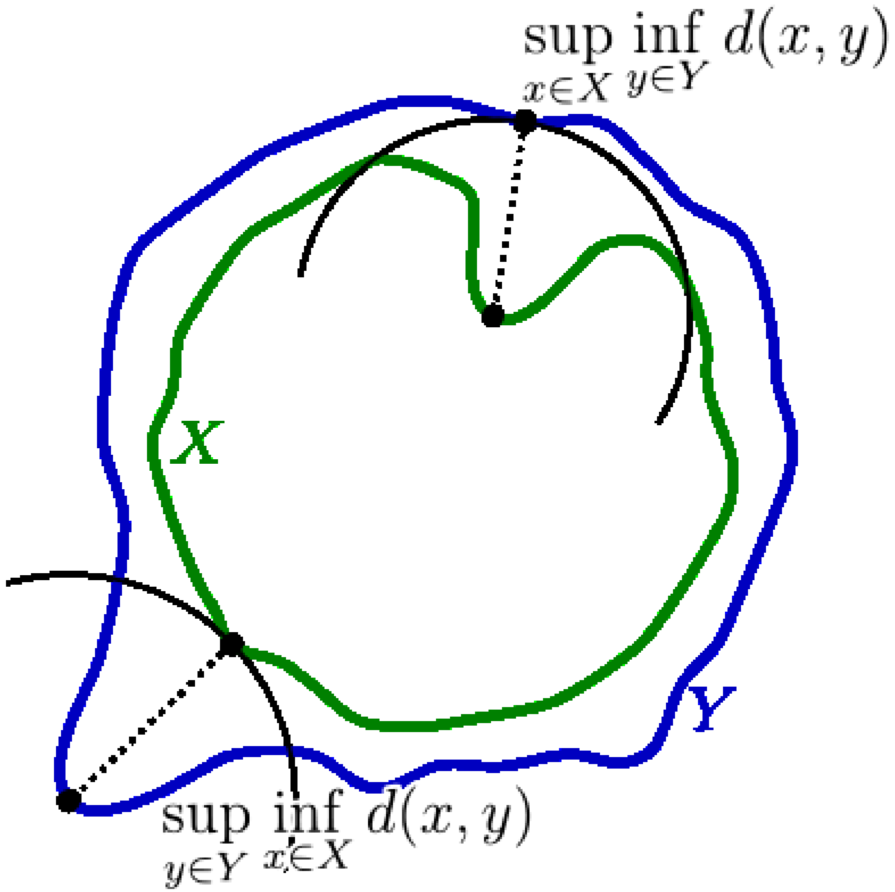
# A generalization: approach space

An **approach space** (Lowe, 1989) is a pair $(X, D)$, where $X$ is a set, and $D$ is a **point-set function**, i.e., a function $D : X \times P(X) \to [0, \infty]$ (where $P(X)$ is the set of all subsets of $X$) satisfying, for all $x \in X$ and all $A, B \subset X$, to:

1. $D(x, \{x\}) = 0$;
2. $D(x, \{\emptyset\}) = \infty$;
3. $D(x, A \cup B) = \min\{D(x, A), D(x, B)\}$;
4. $D(x, A) \leq D(x, A^\epsilon) + \epsilon$, for any $\epsilon \geq 0$
(here $A^\epsilon = \{x : D(x, A) \leq \epsilon\}$ is "$\epsilon$-ball" with the center $x$).

Any **quasi-semi-metric space** $(X, q)$ is an approach space with $D(x, A) = \min_{y \in A} q(x, y)$ (usual point-set distance).

# Hausdorff distance




http://en.wikipedia.org/wiki/User:Rocchini

# II. Weightable q-s-metrics

# and

# eqivalent notions

# Weightable quasi-semi-metrics

- A **weightable quasi-semi-metric** is a q-s-metric $q$ on $X$ admitting a weight function $w(x) \in \mathbb{R}$ on $X$ with $q(x,y) - q(y,x) = w(y) - w(x)$ for all $x, y \in X$, i.e., $q(x,y) + \frac{1}{2}(w(x) - w(y))$ is its **symmetrization semi-metric** $\frac{q(x,y)+q(y,x)}{2}$.
- $w(x) + C$ is also such weight function for any constant $C$. If the set $\{q(x,y_0) - q(y_0,x)\}$ is bounded, then weight can be non-negative; then call $w'(x) = w(x)$-$\min_{y \in X} w(y) \geq 0$ **normalized weight function**.
- **Example**. Let $q$ be quasi-metric on $X = V_3 = \{1,2,3\}$ with $q_{21} = q_{23} = 2$ and $q_{ij} = 1$ for other $1 \leq i \neq j \leq 3$. Then $q$ is weightable with weight $w(i)$=$1,0,1$ for $i$=$1,2,3$.
- $q$ is weightable iff $q(x,y)$+$w(x)$ is **partial semi-metric**.

# Partial semi-metrics

A function $p: X \times X \to \mathbb{R}_{\geq 0}$ with $p(x,y) = p(y,x)$ is a **partial semi-metric** (Matthews, 1992) if for all $x, y, z \in X$, it holds **1)** $p(x,x) \leq p(x,y)$ and **2)** $p(x,y) \leq p(x,z)+p(z,y)-p(z,z)$ (**sharp triangle inequality**).

If **1)** is dropped, **weak partial semi-metric**. Example: $(\mathbb{R}_{\geq 0}, x+y)$. If, moreover, **2)** is weakened to $p(x,y) \leq p(x,z)+p(z,y)$, then $p$ is a **dislocated metric** (or Matthews **metric domain**).

Function $p$ is a partial semi-metric iff $q= p(x,y)-p(x,x)$ is a **weightable q-s-metric** with $w(x)=p(x,x)$ and $p$ is **partial metric** (i.e. $T_0$-separation holds: $x=y$ if $p(x,x)=p(x,y)=p(y,y)=0$) iff, moreover, $q$ is an **Albert quasi-metric**.

Güldürek and Richmond, 2005: every topology on a finite set $X$ is defined, for $x \in X$, by $cl\{x\}=\{y \in X : y \preceq x\}$, where $x \preceq y$ means $p(x,y)=p(x,x)$ for some partial semimetric $p$ on $X$. Not every one is so defined from a semimetric on $X$.

# Weak partial semi-metrics

A function $p : X \times X \to \mathbb{R}_{\geq 0}$ with $p(x,y) = p(y,x)$ is a **weak partial semi-metric** (Heckmann, 1997) if for all $x, y, z \in X$, it holds $p(x,y) \leq p(x,z)$+$p(z,y)$-$p(z,z)$. For $x$=$y$, it gives the weakening $p(x,z) \geq \frac{p(x,x)+p(z,z)}{2}$ of $p(x,z) \geq p(x,x)$.

On any set $X$, $d(x,y)$=$p(x,y)$-$\frac{p(x,x)+p(y,y)}{2}$, $w(x)$=$\frac{p(x,x)}{2}$ and $p(x,y)$=$d(x,y)$+$w(x)$+$w(y$ is a bijection between weak partial semi-metrics $p$ and weighted semi-metrics $(d,w)$ ($w : X \to \mathbb{R}_{\geq 0}$). Moreover, $p$ is partial metric iff $d$ is metric.

In weak partial semi-metric space $(X,p)$, define **open ball** $B(x,r)$=$\{y \in X : p(x,y) < r\}$. Call $U \subset X$ **open** if for all $x \in U$ there is $\epsilon > 0$ with $B(x,\epsilon) \subset U$. The open sets form topology with basis the balls $B(x,r)$; in general, not $T_2$ (Hausdorff).

Its **specialization preorder** induced by $p$ is $x \preceq y$ if and only if $p(x,y) \leq p(a,a)$. It is partial order iff $p$ is weak partial metric.

# Digression on Semantics of Computation

A poset $(X, x \preceq y)$ is **dcpo** if it has a smallest element and each **directed subset** $A \subset X$ (i.e. $A \neq \emptyset$ and for any $x, y \in A$, exists $z \in A$ with $x, y \preceq z$) has a supremum $\sup A$ in $X$.

Let $X^C$ be the set of **compact** $x \in X$, i.e. for each directed subset $A$ with $x \preceq \sup A$, there is $a \in A$ with $x \preceq a$.

A **Scott domain** is a dcpo where all sets $\{a \in X^C : a \preceq x\}$ are directed with $\sup$=$x$ and each **consistent** $A \subset X$ (i.e. there exists $x \in X$ with $a \preceq x$ for all $a \in A$) has supremum in $X$.

Main examples: all words over finite aphabet with prefix order, all *vague real numbers* (nonempty segments of $\mathbb{R}$) with reverse inclusion order, all subsets of $\mathbb{N}$ under inclusion

**Quantitative Domain Theory**: a "distance" between programs (points of a semantic domain) is used to quantify speed (of processing or convergence) or complexity of programs.

$x \preceq y$ (program $y$ contains all info from $x$) is **specialization order** ($x \preceq y$ iff $p(x, y)$=$p(x, x)$) for a **partial metric** $p$ on $X$.

# Quantale-valued partial metrics

Scott's domain theory gave partial order and non-Hausdorff topology on partial objects in computation.
In computation over a metric space of totally defined objects, partial metric models partially defined information: $p(x, x)$>$0$ (=$0$) mean that object $x$ is partially (**totally**) defined.

A **quantale** is a complete lattice $M$ with an associative binary operation $*$ with $x * \bigvee_{i\in I} y_i$=$\bigvee_{i\in I}(x * y_i)$, $\bigvee_{i\in I} y_i * x$=$\bigvee_{i\in I}(y_i * x)$.
Kooperman-Mattews-Rajoonesh, 2004: any topology can arise from a quantale-valued partial metric.

Another way to see: fuzzy non-reflexive equalities. Hohle, 1992: for a commutative quantale $M$=$(M, \leq, 1, 0, \vee, \wedge, *)$, **multivalued** ($M$**-valued**) set is a set $X$ equipped with a **fuzzy equality**, i.e., a map $E : X \times X \to M$ subject to $E(x, x) = 1$, $E(x, y)$=$E(y, x)$ and $E(x, y) * E(y, z) \leq E(x, z)$ for $x, y, z \in X$.

# $WQSMET_n$ and $PSMET_n$, $wPSMET_n$

Clearly, all weightable quasi-semi-metrics on n-set $X = [n] = \{1, 2, \ldots, n\}$ form a polyhedral convex cone of dimension $\binom{n}{2} + n = \binom{n+1}{2}$. Denote it by $WQSMET_n$. $WQSMET_n$ is the section of $QSMET_n$ by $\binom{n}{3}$ hyperplanes $xyzx = xzyx$ of **relaxed symmetry** defined next.

Denote by $PSMET_n$ and $wPSMET_n$ the cones of partial and weak partial semi-metrics on $n$-points. They have $3\binom{n}{3}+n^2$ and $3\binom{n}{3}+\binom{n+1}{2}$ facets, resp. They are relaxations of $\binom{n}{2}$-dimensional cone $SMET_n$ of all n-points semi-metrics.

# Relaxed and cyclic symmetry

- Quasi-semi-metric $q$ on $X$ has **relaxed symmetry** ($xyzx = xzyx$) if for different $x, y, z \in X$ it holds

  $$q(x,y) + q(y,z) + q(z,x) = q(x,z) + q(z,y) + q(y,x), \text{ i.e.}$$

  $q(x,y) - q(y,x) = (q(z,y) - q(y,z)) - (q(z,x) - q(x,z))$, Equivalently, $q$ is weightable: fix point $z_0$ and define $w(x) = q(z_0,x) - q(x,z_0)$.
- Given $k \geq 3$, quasi-semi-metric $q$ is **$k$-cyclically symmetric** if $x_1x_2x_3\dots x_kx_1 = x_1x_kx_{k-1}\dots x_2x_1$, for $x_1x_2\dots x_k \in X$. The case $k=3$ (relaxed symmetry) is equivalent to the general case of any $k \geq 3$. For example, for $k=4$, $(x_1x_2x_3x_1$-$x_1x_3x_2x_1)$+$(x_1x_3x_4x_1$-$x_1x_4x_3x_1)$= $x_1x_2x_3x_4x_1$-$x_1x_4x_3x_2x_1$ and, in other direction, $(x_1x_2x_3x_4x_1$-$x_1x_4x_3x_2x_1)$+$(x_1x_2x_4x_3x_1$-$x_1x_3x_4x_2x_1)$+ $(x_1x_4x_2x_3x_1$-$x_1x_3x_2x_4x_1)$=2 $(x_1x_2x_3x_1$-$x_1x_3x_2x_1)$.

# Realizations by weighted (di)graphs

- Any finite semi-metric $d$ is the shortest path semi-metric of a $\mathbb{R}_{\geq 0}$-weighted graph $G$.
  $G$ can be a tree iff $d$ satisfy to **$4$-points inequality**
  $d(x,y) + d(z,u) \leq \max\{d(x,z) + d(y,u), d(x,u) + d(y,z)\}$.
- Any finite quasi-semi-metric $q$ is the shortest path q-s-metric of a $\mathbb{R}_{\geq 0}$-weighted digraph $G$.
  **Patrinos-Hakimi, 1972**: $G$ can be a **bidirectional tree** (a tree with all edges replaced by 2 oppositely directed arcs) iff $q$ is **weightable** and $q(x,y) + q(y,x)$ is tree-realizable.

# Weigtable hitting time quasi-metric

Given connected graph $G = (V, E)$ with $|E| = m$, consider random walks on $G$, where at each step walk moves with uniform probability from current vertex a neighboring one.

The **hitting time quasi-metric** $H(u, v)$ from $u \in V$ to $v \in V$ is the expected number of steps (edges) for a random walk on $G$ beginning at $u$ to reach $v$ for the first time; put $H(u, u) = 0$. This quasi-metric is **weightable**.

The **commuting time metric** is $C(u, v) = H(u, v) + H(v, u)$. It holds $C((u, v) = 2m\Omega(u, v)$, where $\Omega(u, v)$ is the **effective resistance metric**: $0$ if $u = v$ and, else, $\frac{1}{\Omega(u,v)}$ is the current flowing into grounded $v$ when potential 1 volt is applied to $u$ (each edge is seen as a resistor of 1 ohm). $\Omega(u, v)$ is $\sup_{f:V\to\mathbb{R},\, D(f)>0} \frac{(f(u)-f(v))^2}{D(f)}$ with $D(f) = \sum_{st\in E}(f(s) - f(t))^2$.

# $z_0$-derivations of semi-metrics

Given semi-metric space $(X, d)$ and $z_0 \in X$, its **$z_0$-derivation** is q-s-metric $q(x,y)$=$\frac{1}{2}(d(x,y)$+$d(y,z_0)$-$d(x,z_0))$. So, $d$=$q+q'$, $q$ is weightable with $w(x)$=$d(x,z_0)$=$q(z_0,x)$ and $d(x,z_0))$$\equiv$$0$. Weightable q-s-metric $q$ is $z_0$-derivation of $q$+$q'$ iff $d(x,z_0))$$\equiv$$0$

Quasi-metric $q$ is **$z_0$-derivation** of a metric $d$ iff **partial metric** $p(x,y)$=$q(x,y)+w(x))$ is $\frac{1}{2}(d(x,y)$+$d(y,z_0)$**+**$d(x,z_0))$.

Clearly, $z_0$-derivations of semi-metrics $d \in SMET_n$ for fixed $z_0 = i \in X = [n]$ form a cone $\mathbf{D_iWQSMET_n} \subset WQSMET_n$.

Any inequality $\sum_{1 \le i,j \le n} a_{ij} dij \ge 0$, valid for $d \in SMET_n$, implies, valid for $q \in D_{z_0}WQSMET_n$, inequality $\sum_{1 \le i,j \le n} a_{ij} qij + \sum_{1 \le i,j \le n} a_{ij} d(j, z_0) - \sum_{1 \le i,j \le n} a_{ij} d(i, z_0) \ge 0$.

# III. $l_1$ quasi-metrics

# $l_p$-quasi-metrics

- On a **normed vector space** $(V, ||.||)$, **norm metric** is $||x-y||$. The **$l_p$-metric** is $||x-y||_p$ norm metric on $\mathbb{R}^m$ (or on $\mathbb{C}^m$): $||x||_p$=$(\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$ and $||x||_\infty$= $\max_{1 \leq i \leq m} |x_i|$. The **Euclidean metric** (or **Pythagorean distance**, **as-crow-flies distance**, **beeline distance**) is $l_2$-metric on $\mathbb{R}^m$.
- **$l_p$-quasi-metric** on $\mathbb{R}^m$ is $z_0$-derivation of $l_p$-metric with $z_0$=$(0, \dots, 0)$, i.e. it is **oriented $l_p$-norm** $||x-y||_{p,\,or}$= $(\sum_{i=1}^m |x_i - y_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^m |y_i|^p)^{\frac{1}{p}} - (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}$ and $l^m_{p,\,or}$ is the quasi-metric space $(\mathbb{R}^m, ||x-y||_{p,\,or})$,

  $l_p$-$QSMET_n$ is the set of all $l_p$ q-s-metrics on $n$ points; it is (as for semi-metrics) a cone exactly for $p = 1, \infty$.
- $(l_2\text{-}QSMET_n)^2$=$\{q^2 : q \in l_2\text{-}QSMET_n\}$ is a cone also.

# $l_1$ and $l_\infty$ quasi-metrics

- In particular, **$l_1$-quasi-metric** on $\mathbb{R}^m_{\geq 0}$ is $\sum_{i=1}^m (|x_i - y_i| + |y_i| - |x_i|) = 2\sum_{i=1}^m \max\{y_i - x_i, 0\}$ and **$l_\infty$-quasi-metric** is $2\max_{1\leq i\leq m}\max\{y_i - x_i, 0\}$.
- Any q-s-metric $q$ on $n$ points **embeds in $l^m_{1,\,or}$ for some $m$ iff** $q \in OCUT_n$ (cone generated by all oriented cuts on $[n]$).
- Any q-s-metric $q$ on $n$ points **embeds into $l^n_{\infty,\,or}$**. In fact, let $v_1, \dots, v_n \in \mathbb{R}^n$ be $v_i = (q(i,1), q(i,2), \dots, q(i,n))$. Then $||v_i - v_j||_{\infty,\,or} = max_k(q(j,k) - q(i,k), 0) \leq q(j,i)$, while $q(j,i) - q(i,i) = q(j,i)$; so, $||v_i - v_j||_{\infty,\,or} = q(j,i)$.

**Exp.**: on $\mathbb{R}_{\geq 0}$, to partial metric $p(x,y)$=$\max\{x,y\}$ correspond $l_1$ **quasi-metric** $q(x,y)$=$\max\{x,y\}$-$x$= $\max\{y$-$x, 0\}$ (with $w(x)$=$x$) and $d(x,y)$=$\frac{q(x,y)+q(y,x)}{2}$=$\frac{|x-y|}{2}$=$p(x,y)$- $\frac{x+y}{2}$ (twice $l_1$ **metric**).

# Embedding between $l_p$ quasi-metrics

Clearly, any isometric embedding $f$ of semi-metric spaces $(X, d_X)$ into $(Y, d_Y)$ is isometric embedding of $z_0$-derivations of $(X, d_X)$ into $f(z_0)$-derivation of $(Y, d_Y)$.
So (as well as for semi-metrics), it holds:

- Any $l_p$-quasi-metric with $1 \leq p \leq 2$ is a $l_1$-quasi-metric.
- Any $l_1$-quasi-metric is the square of a $l_2$-quasi-metric.
- Any quasi-metric is a $l_\infty$-quasi-metric.

So, $l_2$-$QSMET_n \subset l_1$-$QSMET_n \subset (l_2$-$QSMET_n)^2$ holds; it generalizes $l_2$-$SMET_n \subset l_1$-$SMET_n \subset (l_2$-$SMET_n)^2$, where, for semi-metrics, $(l_2$-$SMET_n)^2$ is the **negative type cone** $NEG_n$ and $l_1$-$SMET_n$ is the **cut cone** $CUT_n$.

# Measure quasi-semi-metric versus $l_1$

- Given a **measure space** $(\Omega, \mathcal{A}, \mu)$, the **symmetric difference** (or **measure**) **semi-metric** on the set $\mathcal{A}_\mu = \{A \in \mathcal{A} : \mu(A) < \infty\}$ is $\mu(A \triangle B)$ (where $A \triangle B$= $(A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$ is the **symmetric difference** of sets $A, B$) and $0$ if $\mu(A \triangle B) = 0$. Identifying $A, B \in \mathcal{A}_\mu$ if $\mu(A \triangle B) = 0$, gives the **measure metric**. If $\mu(A) = |A|$, then $\mu(A \triangle B) = |A \triangle B|$ is a metric.
- **Measure quasi-semi-metric** on the set $\mathcal{A}_\mu$ is $z_0$-derivation of the measure semi-metric for $z_0 = \emptyset$, i.e. it is $q(A, B) = \mu(A \triangle B) + \mu(B) - \mu(A) = \mu(B \backslash A)$.

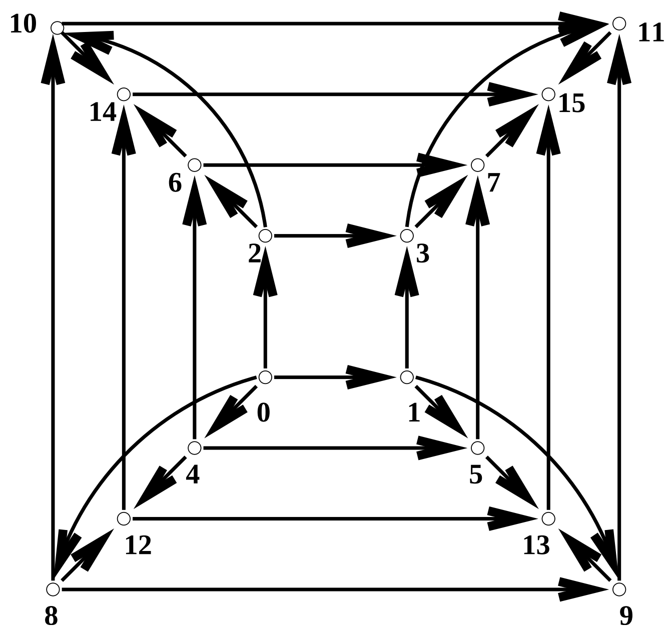
In fact (as well as in the metric case), a q-s-metric is $l_1$-quasi-metric **if and only if** it is a measure quasi-metric.

# $n$-cube: inclusion (Boolean) orientation

Label vertices of $n$-cube by numbers $0, \ldots, 2^n - 1$; their binary expansions label all subsets $A$ of $[n] = \{1, \ldots, n\}$.

**Hasse diagram** of the **Boolean lattice** $2^{[n]}$ is **inclusion-oriented $n$-cube**: do arc from $A$ to $B$ if $A \subset B$ and $|B \backslash A|$=1.

Its **path quasi-semi-metric** is $|B \backslash A|$ if $A \subset B$ and =$\infty$, else, while **Hamming semi-distance** is $l_1$ quasi-metric $|B \backslash A|$, i.e. $|B \backslash (B \cap A)|$=$\sum_{i=1}^{n} \max\{1_{i \in B} - 1_{i \in A}, 0\}$=$\sum_{i=1}^{n} 1_{i \in B}(1 - 1_{i \in A})$.



# IV. The cones under consideration

# The cones under consideration

$l_1SMET_n$=$CUT_n$=$MCUT_n$=$BSMET_n \subset SMET_n$=$l_\infty SMET_n$;

$l_1QSMET_n$=$OCUT_n \subset WQSMET_n \subset QSMET_n$=$l_\infty QSMET_n$,
and $OCUT_n \subset OMCUT_n \subset BQSMET_n \subset QSMET_n$, where $MCUT_n, OMCUT_n$ are generated by multicuts, o-multicuts, and $BSMET_n, BQSMET_n$ are generated by $\{0,1\}$-valued semi-metrics, quasi-semi-metrics.

Also, $l_1$-$PSMET_n$=$BPSMET_n \subset PSMET_n$, where
$PSMET_n$=$\{p = ((p_{ij} = q_{ij} + w_i))\} : q = ((q_{ij})) \in WQSMET_n$,
$l_1$-$PSMET_n$=$\{p = ((p_{ij} = q_{ij} + w_i))\} : q = ((q_{ij})) \in OCUT_n$,
and $BPSMET_n$ is generated by $\{0,1\}$-valued $p \in PSMET_n$.

# Oriented cut quasi-semi-metrics

Given a subset $S$ of $[n]=\{1,\dots,n\}$, the **oriented cut quasi-semi-metric** (or **o-cut**) $\delta(S)'$ is a quasi-semi-metric on $[n]$:

$$\delta'_{ij}(S)=|(S\cap\{i\})\backslash(S\cap\{j\})|=\begin{cases}1, & \text{if} \quad i\in S, j\notin S,\\ 0, & \text{otherwise.}\end{cases}$$

$\delta'(S)$ is, for any $z_0\in\overline{S}$, **$z_0$-derivation** of the **cut semi-metric** $\delta(S)$=$\delta'(S)+\delta'([n]\backslash S)$ (twice of symmetrization of $\delta'(S)$).

Quasi-semi-metric $\delta'(S)$ is **weightable** with $w(i)=1_{i\notin S}$.

**Oriented cut cone** $OCUT_n$ is $\binom{n+1}{2}$-dimensional subcone of $WQSMET_n$ generated by $2^n$-2 non-zero o-cuts $\delta'(S)$ of $[n]$.

$OCUT_n$=$l_1$-$QSMET_n$, the cone of $n$ points $l_1$ q-s-metrics.

# Oriented multicut quasi-semi-metrics

Given an **ordered partition** $\{S_1,\dots,S_t\}$, $t \geq 2$, of $[n]$, **oriented multicut quasi-semi-metric** (or **o-multicut**) $\delta^{'}(S_1,\dots,S_t)$ is:

$$\delta^{'}_{ij}(S_1,\dots,S_t) = \begin{cases} 1, & \text{if} \quad i \in S_h, j \in S_m, m > h, \\ 0, & \text{otherwise.} \end{cases}$$

The **multicut semi-metric** $\delta(S_1,\dots,S_t)$ is symmetrization $\delta^{'}(S_1,\dots,S_t)$+$\delta^{'}(S_t,\dots,S_1)$ of q-s-metric $2\delta^{'}(S_1,\dots,S_t)$.

An o-multicut $\delta^{'}(S_1,S_2)$ is exactly o-cut $\delta^{'}(S_1)$.

**Lemma**: o-cuts are exactly weightable o-multicut q-s-metrics

In fact, let $i{\in}S_1$, $j{\in}S_2$, $k{\in}S_3$ in q-s-metric $q$=$\delta^{'}_{ij}(S_1,\dots,S_q)$.

If $q$ is weightable, then $q(i,j) = w(j) - w(i) = 1$. Impossible, since $q(i,k) = w(k) - w(i) = 1, q(j,k) = w(k) - w(j) = 1$.

# Oriented cuts with $n = 3$

There are $7$ oriented cut q-s-metrics on $3$ points, given by binary $\binom{3}{2}$-vectors indexed as $(12, 13; 21, 23; 31, 32)$:

$$\delta^{'}(\{\emptyset\}) = \delta^{'}(\{1, 2, 3\}) = (0, 0; 0, 0; 0, 0),$$
$$\delta^{'}(\{1\}) = (1, 1; 0, 0; 0, 0),$$
$$\delta^{'}(\{2\}) = (0, 0; 1, 1; 0, 0),$$
$$\delta^{'}(\{3\}) = (0, 0; 0, 0; 1, 1),$$
$$\delta^{'}(\{1, 2\}) = (0, 1; 0, 1; 0, 0),$$
$$\delta^{'}(\{1, 3\}) = (1, 0; 0, 0; 0, 1),$$
$$\delta^{'}(\{2, 3\}) = (0, 0, 1, 0, 1, 0).$$

**Example**. Let again $q$ be quasi-metric on $X = V_3 = \{1, 2, 3\}$ with $q_{21} = q_{23} = 2$ and $q_{ij} = 1$ for other $1 \leq i \neq j \leq 3$. Then $q = \delta^{'}(\{1\}) + 2\delta^{'}(\{2\}) + \delta^{'}(\{3\})$, i.e. $q \in OCUT_3$.

# Oriented multicuts versus oriented cuts

There are 6 oriented multicuts on 3 points, in addition to 7 oriented cuts, listed above:

$$\delta'(\{1\},\{2\},\{3\}) = (1,1;0,1;0,0),$$
$$\delta'(\{2\},\{1\},\{3\}) = (0,1;1,0;0,0),$$
$$\delta'(\{1\},\{3\},\{2\}) = (1,1;0,0;0,1),$$
$$\delta'(\{2\},\{3\},\{1\}) = (0,0;1,1;1,0),$$
$$\delta'(\{3\},\{1\},\{2\}) = (1,0;0,1;1,1),$$
$$\delta'(\{3\},\{2\},\{1\}) = (0,0;1,0;1,1).$$

Every **multicut** is $\mathbb{R}_{\geq 0}$-linear combination of cuts, while any **oriented multicut** with $t > 2$ is a $\mathbb{R}$-linear but not $\mathbb{R}_{\geq 0}$-linear combination of o-cuts, since it is non-weightable q-s-metric.

The number of oriented multicuts on $[n]$ is ordered Bell number $Bo(n)$ (the sequence A00670 in Sloan's OEIS).

# Linear description of $QSMET_n$

| cone | dim. | Nr. of ext. rays (orbits) | Nr. of facets (orbits) | diam. |
|---|---|---|---|---|
| $OMCUT_3$ =$QSMET_3$ | 6 | 12(2) | 12(2) | 2; 2 |
| $OMCUT_4$ | 12 | 74(5) | 72(4) | 2; 2 |
| $QSMET_4$ | 12 | 164(10) | 36(2) | 3; 2 |
| $OMCUT_5$ | 20 | 540(9) | 35320(194) | 2; 3 |
| $QSMET_5$ | 20 | 43590(229) | 80(2) | 3; 2 |
| $OMCUT_6$ | 30 | 4682(19) | $> 2.1 \cdot 10^9 (> 1.6 \cdot 10^6)$ | 2; ? |
| $QSMET_6$ | 30 | $> 1.8 \cdot 10^9 (> 1.2 \cdot 10^6)$ | 150(2) | ?; 2 |

The orbits are under the symmetry group $Z_2 \times Sym(n)$: $n!$ permutations of $[n] = \{1, \dots, n\}$ and the reversal $(ij) \to (ji)$.

$QSMET_n$ has $n(n-1)^2$ **facets** in 2 orbits: $6\binom{n}{3}$ oriented triangle inequalities and $n(n-1)$ inequalities $q(x,y) \geq 0$. Moreover, they are also facets of $OCUT_n$ and so, of cones $WQSMET_n$, $OMCUT_n$ and $BQSMET_n$ containing $OCUT_n$.

# Cones on $3$ points (all $6$-dimensional)

The cone $OCUT_3$ of $l_1$ q-s-metrics on $3$ points **coincides** with the cone of weightable quasi-semi-metrics $WQSMET_3$.
It has $6$ extreme rays in 2 orbits of sizes $3,3$ represented by o-cuts $\delta^{'}(\{1\})$=$(1,1;0,0;0,0)$ and $\delta^{'}(\overline{\{1\}})$=$(0,0;1,0;1,0)$,
and $9=6+3$ facets represented by $q_{ij}\geq 0$ and $Tr_{ij,k}\geq 0$.

Larger cone $OMCUT_3=BQSMET_3=QSMET_3$ has $12$ extreme rays in $3$ orbits represented by two above o-cuts and the **o-multicut** $\delta^{'}(\{1\},\{2\},\{3\})=(1,1;0,1;0,0)$,
and $12=6+6$ facets represented by $q_{ij}\geq 0$ and $Tr_{ij,k}\geq 0$.

Cone $l_1$-$PSMET_3$=$PSMET_3$ has $13$=$1$+$3$+$3$+$3$+$3$ extreme rays represented by $(1,1;1,1;1,1)$, $P(\delta^{'}(\{1\}))$, $P(\delta'(\overline{\{1\}}))$,
$P(\delta(\{1\}))=\delta(\{1\})=\delta'(\{1\})+\delta'(\overline{\{1\}}$, $P(\delta'(\{1\})+\delta'(\{2\})$,
and $12$=$6$+$3$+$3$ facets repr. by $p_{ij}\geq p_{ii}$, $Tr_{ij,k}\geq p_{kk}$, $p_{ii}\geq 0$.

# Anti-o-multicut quasi-semi-metrics

Given proper partition $\{S_1, \ldots, S_t\}$, $2 \le t \le n$, of $\{1, \ldots, n\}$, **anti-o-multicut q-s-metric** (or **anti-o-multicut**) $\alpha^{'}(S_1, \ldots, S_t)$ is $1 - \delta^{'}_{ij}(S_1, \ldots, S_t)$ if $1 \le i \ne j \le n$ and $= 0$, else.

It is a $\{0, 1\}$-valued q-s-metric, which is weightable iff $t$=$2$ (i.e. for **anti-o-cut** $\alpha^{'}(S, \overline{S})$) with weight function $w(x) = 1_{x \in S}$.

**Anticut semi-metric** $\alpha(S_t, \ldots, S_1) = \alpha^{'}(S_1, \ldots, S_t)$+$\alpha^{'}(S_t, \ldots, S_1)$ (twice symmetrization) is graph path-metric $d(K_{|S_1|, \ldots, |S_t|})$.

For **semi-metrics**, $SMET_n = CUT_n$ if $n \le 4$, and all extreme rays of $SMET_5$ are all $2^4 - 1$ non-zero cuts and all $\binom{5}{2}$ anticuts $\alpha(\{a_1, a_2\}, \{a_3, a_4, a_5\})$ (permutations of $d(K_{2,3})$).

Are $\alpha^{'}$, except $\alpha^{'}(\{1\}, [n] \backslash \{1\})$= $\sum_{s=2}^{n} \delta^{'}(\{s\}, [n] \backslash \{s\})$ and $\alpha^{'}(\{1\}, \ldots, \{n\})$=$\delta^{'}(\{n\}, \ldots, \{1\})$ , extreme in $QSMET_n$?

# Extreme rays of $QSMET_4$, $QSMET_5$

$QSMET_4$ has $164$ extreme rays in $10$ orbits. Among $8$ $\{0,1\}$**-valued ones** ($116$ ext. rays of $BQSMET_4$), $5$ are of $\neq 0$ **o-multicuts** ($74$ ext. rays of $OMCUT_4$), incl. o-cuts $\delta^{'}(\{1\})$, $\delta^{'}(\{1,2\})$ ($14$ ext. rays of $OCUT_4$), and $3$ of **anti-o-multicuts** $\alpha^{'}(\{1,2\},\{3,4\})$, $\alpha^{'}(\{1\},\{2\},\{3,4\})$, $\alpha^{'}(\{1\},\{2,3\},\{4\})$.

$QSMET_5$ has $229$ orbits of extreme rays. Among $29$ $\{0,1\}$**-valued ones**, $9$ are of all **o-multicuts** $\delta^{'}(S_1,\dots,S_t)\neq 0$ (including $\delta^{'}(\{1\})$, $\delta^{'}(\{1,2\})$) and $7$ are of **anti-o-multicuts**. Only $3$ $\{0,1\}$-valued ones consist of **weightable q-s-metrics**: $2$ above orbits of o-cuts and one of anti-o-cuts $\alpha^{'}(\{1,2\})$.

# Cones $PSMET_n$ and $l_1$-$PSMET_n$

| cone | dim. | Nr. of ext. rays (orbits) | Nr. of facets (orbits) | diam. |
|---|---|---|---|---|
| $CUT_3$=$SMET_3$ | 3 | 3(1) | 3(1) | 1; 1 |
| $CUT_4$=$SMET_4$ | 6 | 7(2) | 12(1) | 1; 2 |
| $CUT_5$ | 10 | 15(2) | 40(2) | 1; 2 |
| $SMET_5$ | 10 | 25(3) | 30(1) | 2; 2 |
| $CUT_6$ | 15 | 31(3) | 210(4) | 1; 3 |
| $SMET_6$ | 15 | 296(7) | 60(1) | 2; 2 |
| $l_1$-$PSMET_3$=$PSMET_3$ | 6 | 13(5) | 12(3) | |
| $l_1$-$PSMET_4$ | 10 | 44(9) | 46(5) | |
| $PSMET_4$ | 10 | 62(11) | 28(3) | |
| $l_1$-$PSMET_5$ | 15 | 166(14) | 585(15) | |
| $PSMET_5$ | 15 | 1696(44) | 55(3) | |
| $l_1$-$PSMET_6$ | 21 | 705(23) | | |
| $PSMET_6$ | 21 | 337092(734) | 96(3) | |

# $\{0, 1\}$-valued partial semi-metrics

All such elements of $PSMET_n$ are $\sum_{0 \le i \le n} \binom{n}{i} B(n-i)$ elements ($\sum_{0 \le i \le n} Q(i)$ orbits under $Sym(n)$) of the form $J(S_0) + \delta(S_0, S_1, \dots, S_t) = P(\sum_{1 \le i \le t} \delta'(S_i))$, where $S_0$ is any subset of $[n] = \{1, \dots, n\}$ and $S_1, \dots, S_t$ is any partition of $\overline{S_0}$.

$2^{n-1} + \sum_{1 \le i \le n-1} \binom{n}{i} B(n-i)$ among them ($1 + \lfloor \frac{n}{2} \rfloor + \sum_{1 \le i \le n-1} Q(i)$ orbits) represent extreme rays: ones with $t = 2$ if $S_0 = \emptyset$ (w.l.o.g. suppose $S_i \neq \emptyset$ for $1 \le i \le t$).

Here **partition number** $Q(i)$ is the number of ways to write $i$ as a sum of positive integers;

**Bell number** $B(i)$ is the number of partitions (multicuts) of $[i]$, while the numbers of cuts $=2^{i-1}$, of o-cuts $=2^i$, of o-multicuts is **ordered Bell number** $Bo(i)$ of ordered partitions of $[i]$.

# $\{0,1\}$-valued partial semi-metrics

See below $p$=$((p_{ij}))$=$J(\{\mathbf{67}\})$+$\delta(\{\mathbf{1}\},\{\mathbf{23}\},\{\mathbf{45}\},\{\mathbf{67}\})$=$P(q)$ ($\{0,1\}$-valued extreme ray of $PSMET_7$) and its q-s-metric $q$=$((q_{ij}$=$p_{ij}$-$p_{ii}))$=$\delta(\{\mathbf{1}\})$+ $\delta(\{\mathbf{23}\})$+$\delta(\{\mathbf{45}\})$+$\delta(\{\mathbf{67}\})$ ($\{0,1\}$-valued non-extreme ray of $WQSMET_7$).

$$
\begin{array}{ccccccc}
\mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 \\
1 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & 1 \\
1 & 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} \\
1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1}
\end{array}
\qquad
\begin{array}{ccccccc}
\mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & \mathbf{0} & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \mathbf{0} & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & \mathbf{0} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}
\end{array}
$$

Uniqie orbit of **simplicial** (belong to $\binom{n+1}{2}$-$1$ facets) $0,1$-valued extreme rays of $PSMET_n$ consists of $n$ rays $\sum_{1,i\neq j}^{n}\delta'(\{i\})$, $1\leq j\leq n$, i.e. $J(\{j\})$+$\delta(\{j\},S_1,\dots,S_{n-1})$ with all $|S_i|$=$1$.

# Facets of $l_1$-$PSMET_n$

Let $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ and $\sum(b) = \sum_{i=1}^n b_i \in \{0, 1\}$. Then **hypermetric inequality** $Hyp_p(b) : \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1}^n b_i p_{ii}$ and, for $\max_{1 \le i \le n} |b_i| \le 2$, **modular inequality**

$$A_p(b) : \sum_{1 \le i,j \le n} b_i b_j p_{ij} \le \sum_{i=1, b_i \ne 0}^n (2 - |b_i|) p_{ii}$$

are valid, for any $p = ((p_{ij})) \in l_1$-$PSMET_n$.

$PSMET_n$ has $3$ orbits of facets, represented by $p_{ii} \ge 0$, $Hyp_p(1, -1, 0, \dots, 0)$ and $Hyp_p(1, 1, -1, 0, \dots, 0)$.
$l_1$-$PSMET_3$=$PSMET_3$.
$l_1$-$PSMET_4$, besides $3$ orbits of $PSMET_4$ has $2$ orbits of facets, represented by $Hyp_p(1, 1, -1, -1)$, $A_p(2, 1, -1, -1)$.
$l_1$-$PSMET_5$, besides $3$ orbits of $PSMET_5$, has $12$ orbits of facets includ. represented by $Hyp_p(b)$ with $b = (1, 1, 1,\text{-}1,\text{-}1)$, $(1, 1,\text{-}1,\text{-}1, 0)$, $(1, 1, 1,\text{-}1,\text{-}2)$, $(2, 1,\text{-}1,\text{-}1,\text{-}1)$ and $A_p(b)$ with $b = (2, 1,\text{-}1,\text{-}1, 0)$, $(2, 2,\text{-}1,\text{-}1,\text{-}1)$, $(2, 1, 1,\text{-}1,\text{-}2)$, $(3, 1,\text{-}1,\text{-}1,\text{-}1)$.

# V.Path quasi-metrics

# of

# oriented hypercubes

# Generalities on oriented $n$-cubes

We consider only **oriented** (or **unidirectional**) $n$**-cubes**, since there is no bidirectional electrical/optical converter and full-duplex transmission in optical fiber networks is costly.
The number of all orientations of $n$-cube $H(n)$ is $2^{n2^{n-1}}$.

Robbins, 1939: connected graph has **strong orientation** (i.e. strongly connected) if and only if it is bridgeless.
The number of strong orientations of $n$-cube is unknown.

In $n$-cube (as in any oriented bipartite graph), any 2 directed paths joining two fixed points have lengths equal modulo 2.
So, **symmetrization** $\frac{q(x,y)+q(y,x)}{2}$ of quasi-metric $q$=$q(Q(n))$ of any its strong orientation $Q(n)$ is integer-valued.

A vertex $i$ in a $n$-cube is called **even** if its binary expansion has even number of ones and **odd**, otherwise.

# O-diameter of oriented $n$-cube

Given a graph of diameter $d$ and its strong orientation $O$, **oriented diameter** (or **o-diameter**) $D_O$ is maximal length of shortest directed $(u, v)$-path.

Clearly, $D_O \geq d$; orientation $O$ called **tight** if $D_O = d$.

Chvatal-Thomassen, 1978: $2d^2 + 2d \leq \max_O D_O \leq 5d^2 + d$.

Among strong orientations $O$ of $n$-cube, $\min_O D_O = \infty, 3, 5$ and $n$ for $n = 1, 2, 3$ and (McCanna, 1988) $n \geq 4$, resp.

For strong orientation $O$, $d(u, v)$=$n$ implies $q_O(u, v)$=$n$. It suffice to show $q_O(0, 2^n - 1)$$\leq$$n$. For $1$$\leq$$i$$<$$n$, exists $\geq$$1$ arc $(u, v)$ with $i, i$+$1$ ones in label $\{0, 1\}$-expansions of $u, v$.

Everett-Gupta, 1989: there exists an acyclic (not strong) orientation of $n$-cube with finite length of shortest directed $(u, v)$-path $\geq F_{n+1}$ (Fibonacci number), i.e. $> (\frac{3}{2})^{n-1}$.

# Connectivity

Given a digraph $D=(V,A)$, its **vertex-connectivity** $\kappa$ (resp. **arc-connectivity** $\lambda$) is the minimum number of vertices (resp. arcs) needed to disconnect it. By Menger's theorem (max-flow-min-cut), $\kappa$ (resp. $\lambda$) is minimum over $u,v \in V$ of the number of vertex- (resp. arc-) disjoint $(u,v)$-paths.

High connectivity of network $D$ improve its fault-tolerance and communication performance (routing, broadcasting).

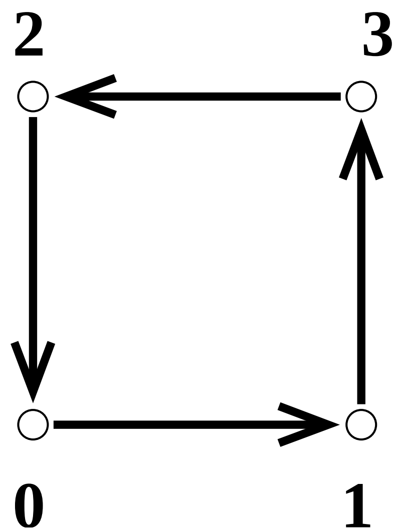
An **Hamilton** $(u,v)$**-path** in a graph is $(u,v)$-path visiting any vertex exactly once. In $n$-cube, it exists iff $d(u,v)$ is odd. A graph is $k$**-vertex** (resp. $k$**-edge Hamiltonian**) if it remains Hamiltonian after deleting any $k$ vertices (resp. edges).

A (di)graph is **Eulerian** if exists a (directed) circuit visiting any (arc) edge exactly once; eqv., it is (strongly) connected and any vertex $v$ has (indegree(v)=outdegree(v)) even degree.

# Mini-cubes $Q(n)$

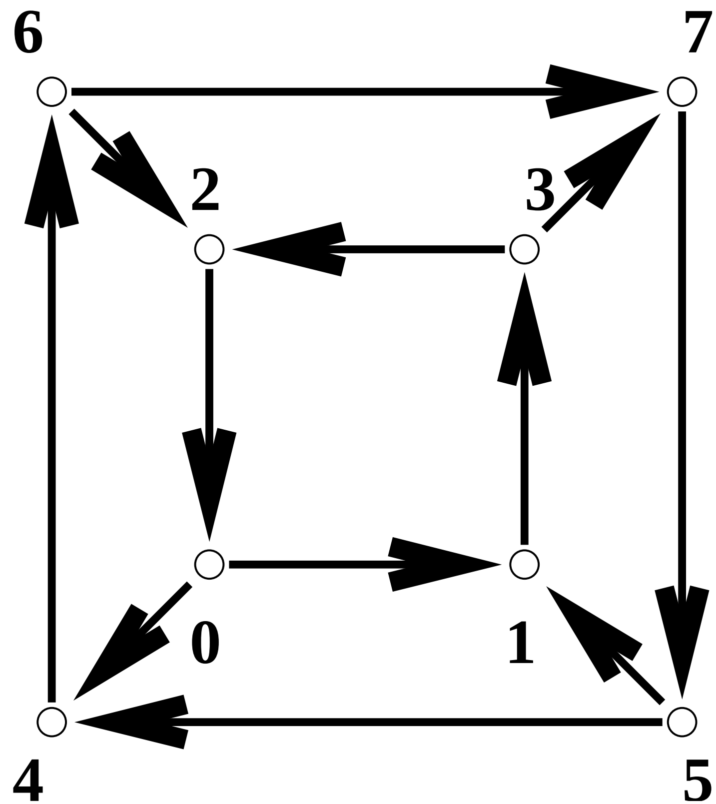
$1$**-cube** $Q(1)$ has two orientations.


$2$**-cube** $Q(2)$ has two strongly connected orientations.



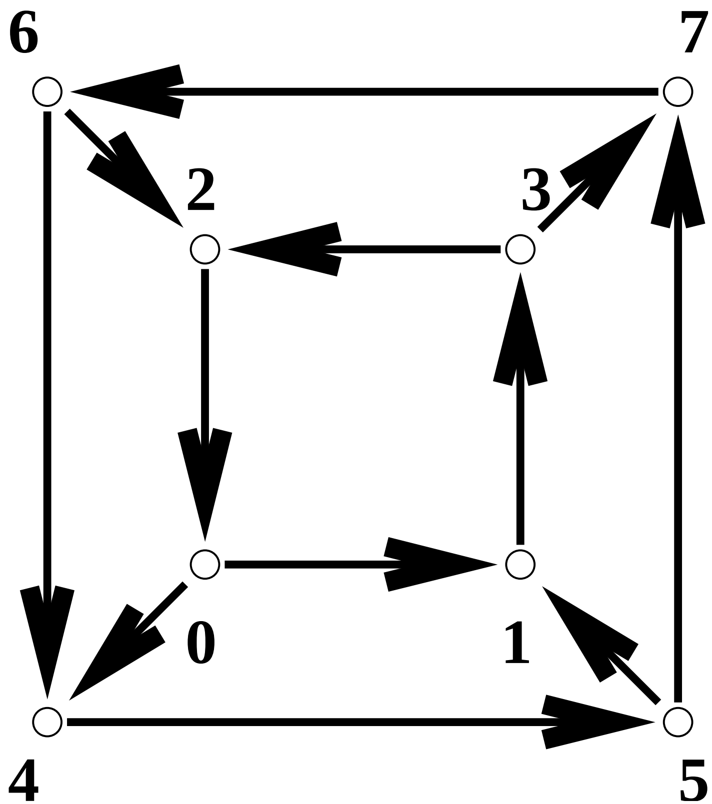

The **symmetrization** $D(Q(2)) = ((D_{ij}))$=$((\frac{1}{2}(q_{ij} + q_{ji})))$ of its quasi-metric $q = ((q_{ij}))$ is $2d(K_4)$, while $d(H(2)) = d(C_4)$.

# 3-cube: Chou-Du orientation $Q_{CD}(3)$




**Chou-Du orientation** $Q_{CD}(n)$ come from $2$ factors $Q_{CD}(n-1)$ with mutually reversed orientations (above inside, outside squares $Q_{CD}(2)$) and, on remaining matching, arcs from each even vertex to its odd match. The symmetrization of its quasi-metric $q_{CD}(3)$ is $2d(K_8 - C_{0527} - C_{6341})$.

# 3-cube: Chou-Du orientation $Q_{CD'}(3)$




For odd $n \geq 3$, **2nd Chou-Du orientation** $Q_{CD'}(n)$ come from two factors $Q_{CD}(n-1)$ with the same orientation (above inside and outside squares $Q_{CD}(2)$) and, on remaining mathching, again arcs from each even vertex to its odd match. For even $n$, $Q_{CD'}(n) = Q_{CD}(n)$.

# Chou-Du orientations $CD, CD'$

- Chou-Du, 1990: both $Q(n)$, as communication network (for high-speed computing using optical fibers as links), have efficient routing and short delay since are small:

  **oriented diameter**: $n+1$ for even $n$ and $n$+$2$ for odd $n > 1$ (for $CD$), $5$ for $n$=$3$ and $n$+$1$ for other $n > 1$ (for $CD'$) and

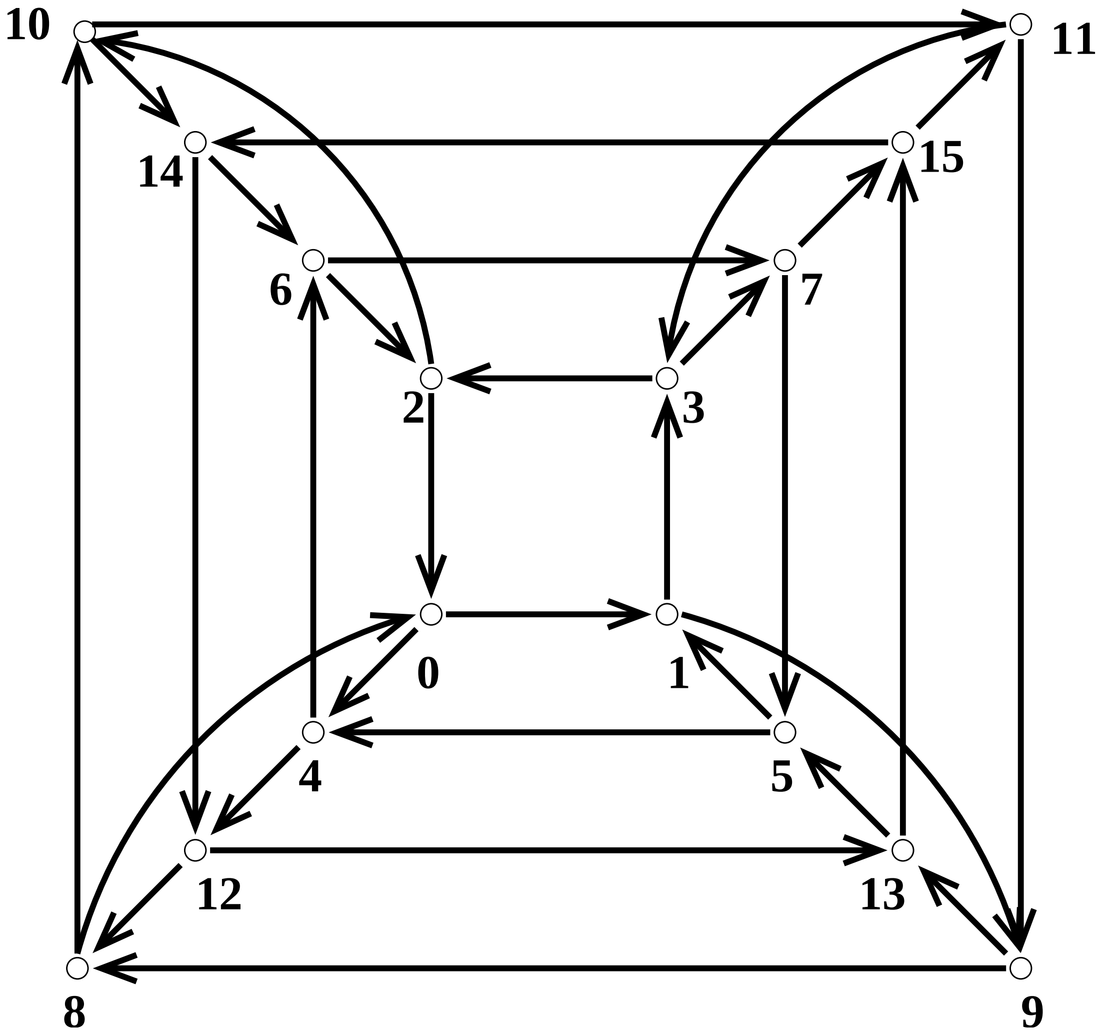
  **mean distance** $\frac{n2^{n-1}+2n\binom{n-1}{\lfloor n/2 \rfloor}}{2^n-1}$, $\frac{n2^{n-1}+(n-1)\binom{n-1}{\lfloor n/2 \rfloor}+2}{2^n-1}$ ($n$ odd).

- Let $C(x,y)$ be a largest set of vertex-disjoint $(x,y)$-paths (**max-container**), $L(C(x,y))$: longest path length in $C(x,y)$.
  **Wide-diameter**: $\max_{(x,y)} \min_{C(x,y)} L(C(x,y))$; $\geq$ o-diameter

- Jwo-Tuan, 1998: $CD$, $CD'$ are maximally fault-tolerant, since $|C(x,y)| \leq \min(out(x), in(y))$ become equality.

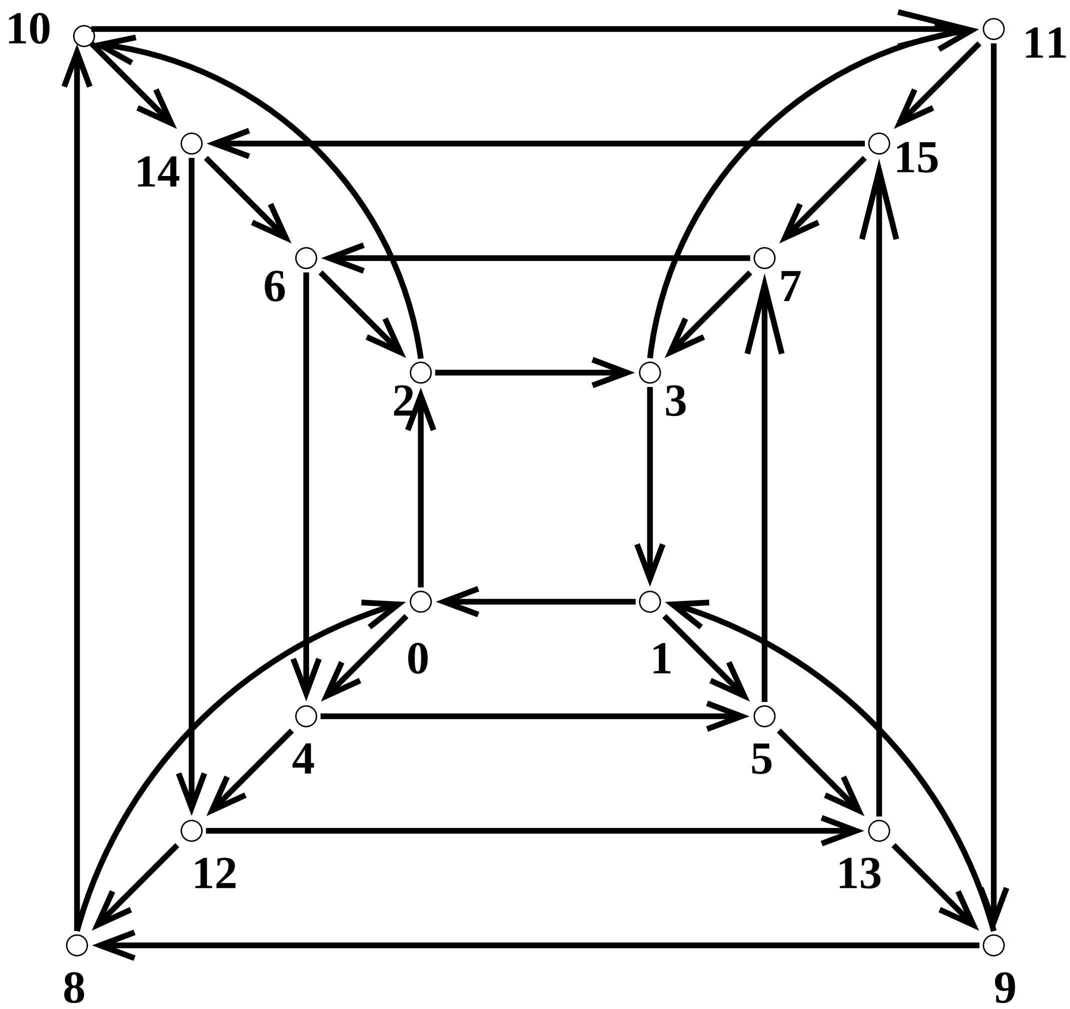
  Lu-Zhang, 2002: wide-diameters of $CD$, $CD'$ are $n+2$.

# Chou-Du orientation $Q_{CD}(4)$=$Q_{CD'}(4)$



# 4-cube: McCanna orientation $Q_{MC}(4)$

McCanna, 1988, gave this **tight** (i.e. with oriented diameter $n = 4$) orientation of $4$-cube.



# Generalized McCanna orientation

For $n \geq 4$, **generalized McCanna orientation** $Q_{MC}(n)$ come from 2 factors $Q_{MC}(n\text{-}1)$ with same orientation and, on remaining matching, arcs from each even vertex to its odd match.
A vertex $i$ in a n-cube is called **even** if its binary expansion has even number of ones and **odd**, otherwise.

- Its **oriented diameter** is minimal: $n$, i.e. $Q_{MC}(n)$ is **tight**.
- Its **vertex**- and **arc-connectivity** are maximal: $\kappa$=$\lambda$=$\lfloor \frac{n}{2} \rfloor$.
- Fraigniaud-König-Lazard, 1992: it is **Hamiltonian** iff $n \geq 5$.

# $n$-cube: signature-defined orientations

Given an orientation $O$ of $n$-cube, its **signature** is $\pm 1$-valued $n$-vector $a_O = (a_1, a_2, \dots, a_n)$ with $a_i = +1$ if the edge $(0, 2^i)$ is oriented in $O$ by arc $(0, 2^i)$ and $a_i = -1$ if this edge is oriented by (incoming to $0$) arc $(2^i, 0)$.
**Excess** of signature is the difference $e$ between number of $1$'s and $-1$'s in it. $0$ is **source** if $e = n$ and **sink** if $e = -n$.

An orientation is **signature-defined** if any its arc is uniquely defined by arcs involving $0$.
It is **||-defined** if any its arc has the same orientation (from even to odd vertex) as the parallel edge involving $0$.
Cariolaro: ||-defined orientation is str. connected iff $|e| < n$.

Chou-Du orientation $CD$ is ||-defined, while $CD'$, McCanna and Hamiltonian orientations are only signature-defined.

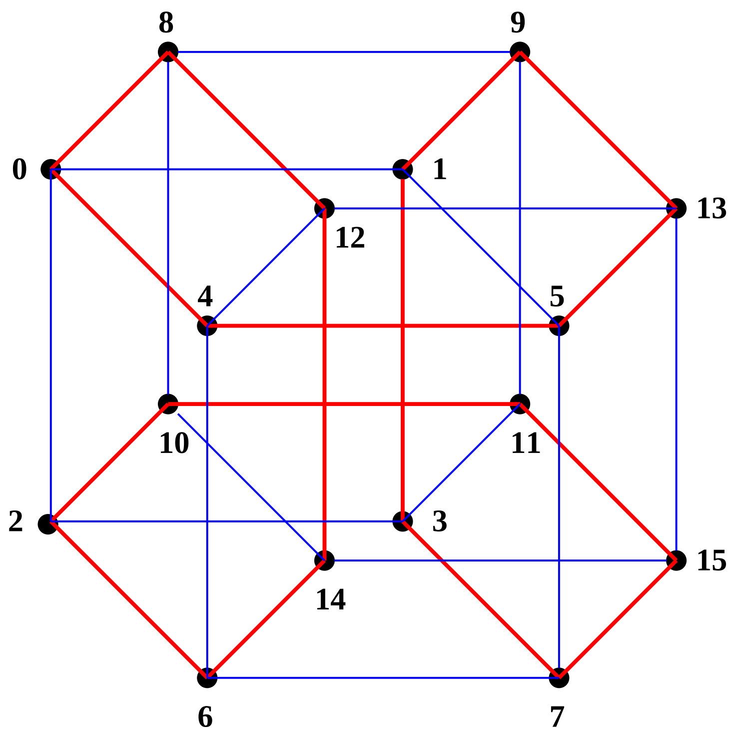
# VI. Hamiltonian orientations of hypercubes

# Hamiltonian decomposition of $H(n)$

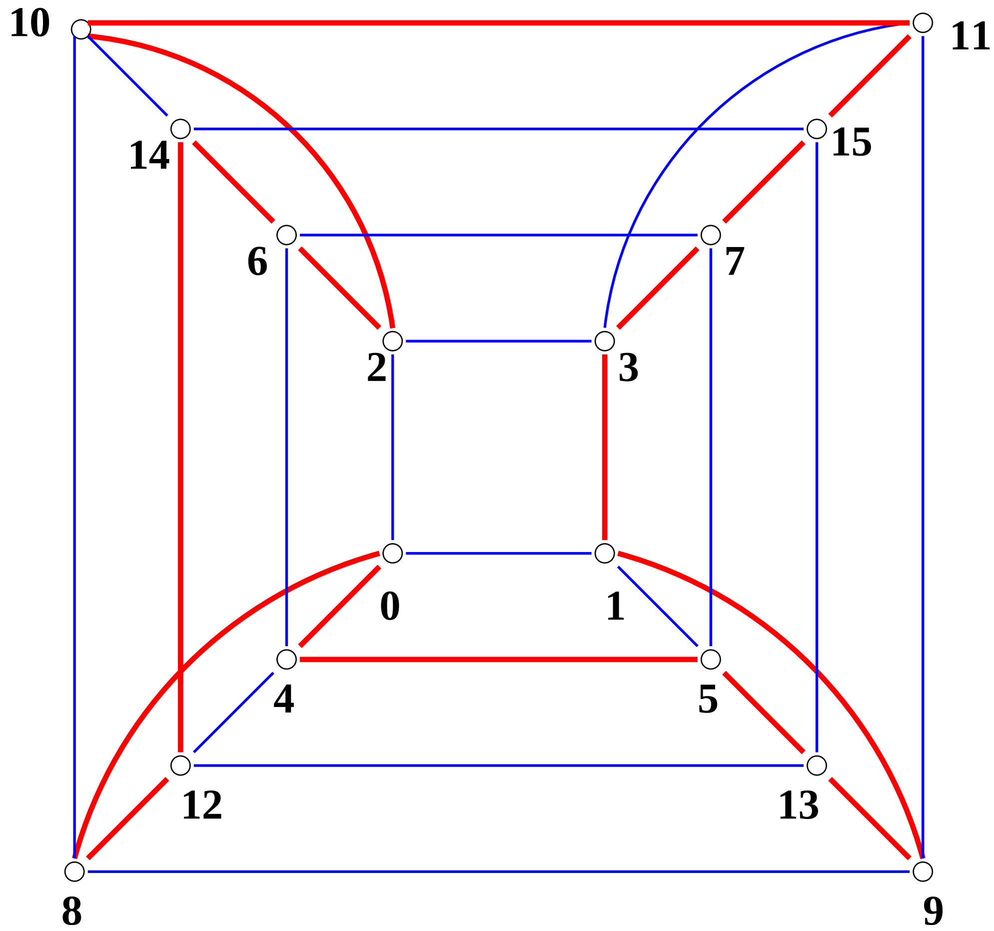
Alspach-Bermond-Sotteau, 1990: edge-set of $H(n)$ can be decomposed into $\frac{n}{2}$ disjoint Hamilton cycles, if $n$ is even, and into $\frac{n-1}{2}$ Hamilton cycles and a perfect mathching, else.

For even $n$, $H(n)$=$C_4 \times \ldots \times C_4$ ($\frac{n}{2}$ times) $\sim 4$-ary $\frac{n}{2}$-cube.

Stong, 2006: for odd $n$, **bidirected** $Q_n$ decomposes into $n$ **directed** Hamilton cycles.



# Hamiltonian decomposition of $H(4)$



# All Hamilton cycles of $H(4)$

Parkhomenko, 2001: $4$-cube has 1344 Hamilton cycles.

See Hamilton cycle $V$=$\{v_i\}$, $1 \le i \le 2^n$, as sequence $t(V)$= $\{1 + \lg_2 |t_i - t_{i+1}|\}$, $1 \le i \le 2^n$, where $t_i$ is label of $v_i$. Then (up to $Sym(4)$, reversals and cyclyc shifts) all cycles are:

A $\{8, 4, 2, 2\}$: 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 4;

**B1** $\{6, 6, 2, 2\}$: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4,

B2 $\{6, 6, 2, 2\}$: 1 2 1 3 1 2 1 4 2 1 2 3 2 1 2 4;

C1 $\{6, 4, 4, 2\}$: 1 2 1 3 2 1 2 4 3 1 3 2 1 3 1 4,

**C2** $\{6, 4, 4, 2\}$: 1 2 1 3 1 2 4 3 1 2 1 3 1 2 4 3,

**C3** $\{6, 4, 4, 2\}$: 1 2 1 3 2 1 2 4 1 3 1 2 3 1 3 4,

**C4** $\{6, 4, 4, 2\}$: 1 2 1 3 1 2 1 4 2 3 1 3 2 3 1 4,

C5 $\{6, 4, 4, 2\}$: 1 2 1 3 1 2 4 2 1 3 1 2 1 3 4 3;

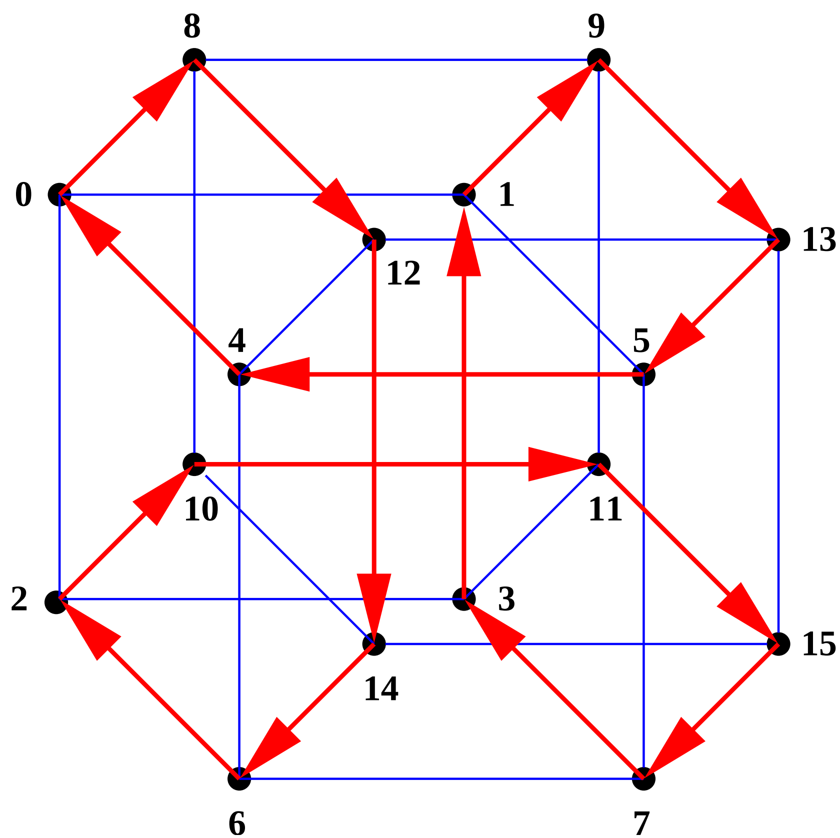
**D** $\{4, 4, 4, 4\}$: 1 2 1 3 1 4 3 2 3 4 1 4 2 3 2 4.

Above **class** $\{a_1, \ldots, a_n\}$ lists numbers $a_i$ of $i$ in a cycle.

The edges **not** belonging to Hamilton cycle form $C_8$+$C_4$+$C_4$, $C_6$+$C_6$+$C_4$, $C_{10}$+$C_6$ and $C_8$+$C_4$+$C_4$ for A, B2, C1 and C5.

# Exp.: complementary Hamilton cycles

The sequence $t(V) = \{1 + \lg_2 |t_i - t_{i+1}|\}$, $1 \leq i \leq 2^4$, of red Hamilton cycle is given by: 4 3 2 4 3 4 1 3 4 3 2 4 3 4 1 3;
its permutation $(4, 3, 1, 2)$ is: 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4 1,
a cyclic shift of which is B1: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4.
Remaining edges form ∼B1: 1 3 2 1 2 4 1 2 1 3 2 1 2 4 1 2.



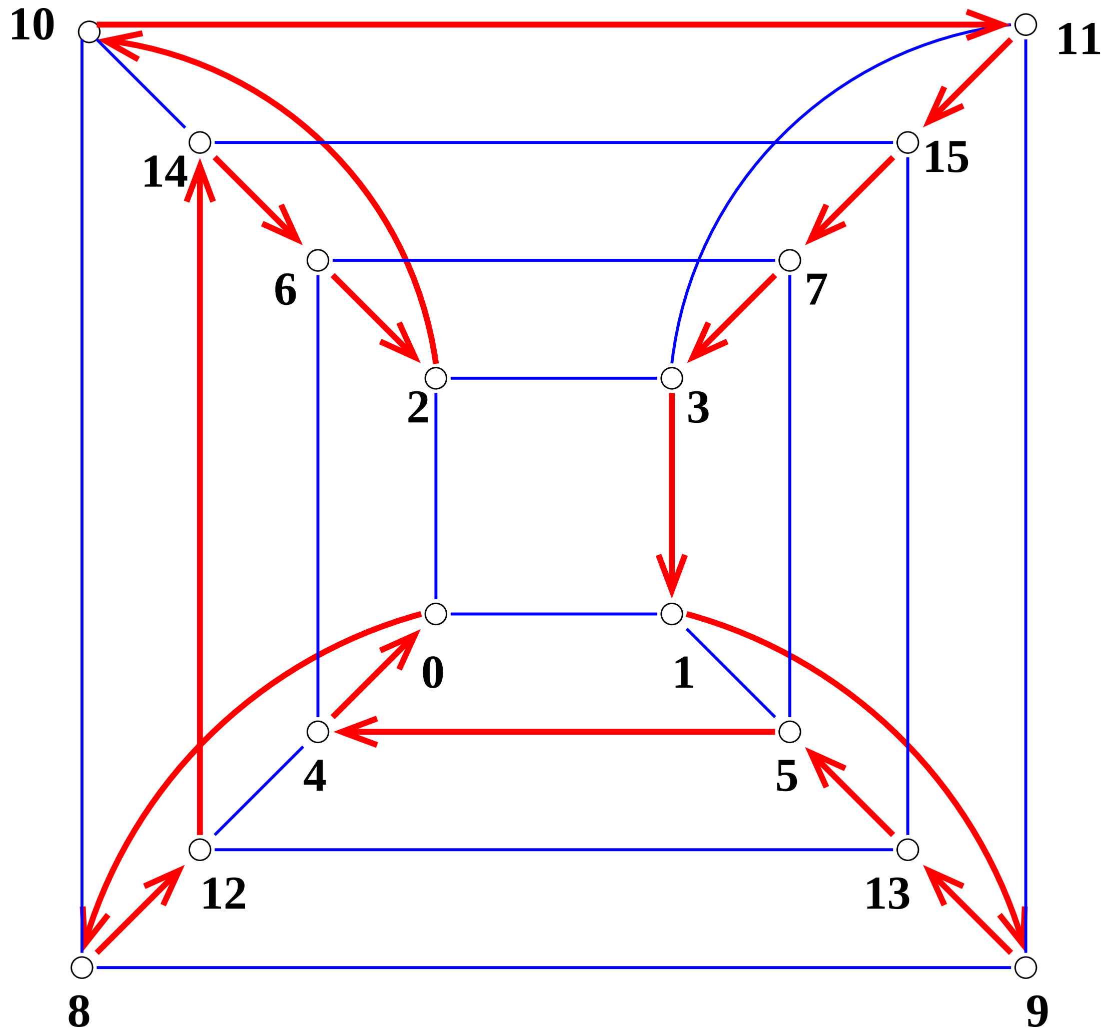
# Hamilton orientations of $n$=$2m$-cube

For any $n = 2m$ and a decomposition of the edge-set of $2m$-cube into $m$ disjoint Hamilton cycles, call **Hamilton orientation** any of $2^{m-1}$ orientations obtained by cyclically orienting those $m$ cycles. W.l.o.g. orient 1st cycle arbitrary.

Any Hamilton orientation is signature-defined: number $a_i$ uniquely identifies outcoming (if $a_i$=1) or incoming (if $a_i$=-1) to $0$ Hamilton cycle and orientation on it. The number of $1$'s in its signature is $\frac{n}{2} = m$, i.e. its excess $e(a_O)$ is $0$.

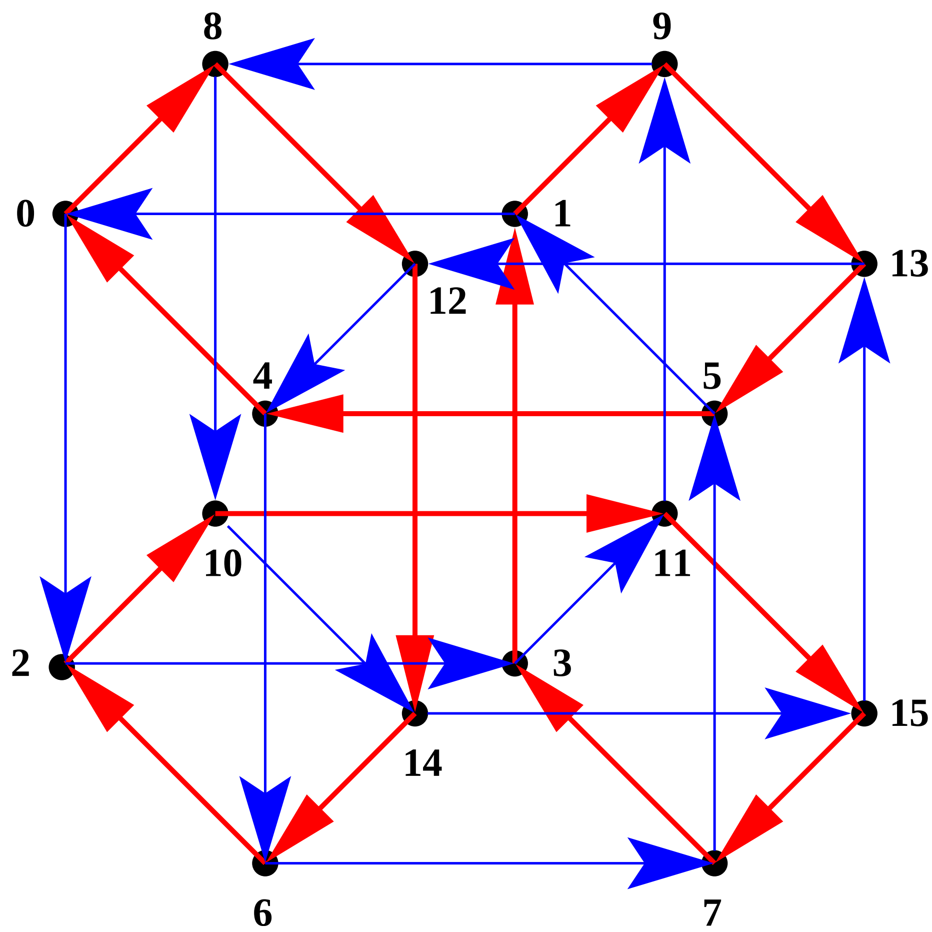
# Orient arbitrarily 1st Hamilton cycle

Fix orientation of 1st (red) cycle and define orientation of $4$-cube via orientation of 2nd (blue) Hamilton cycle.

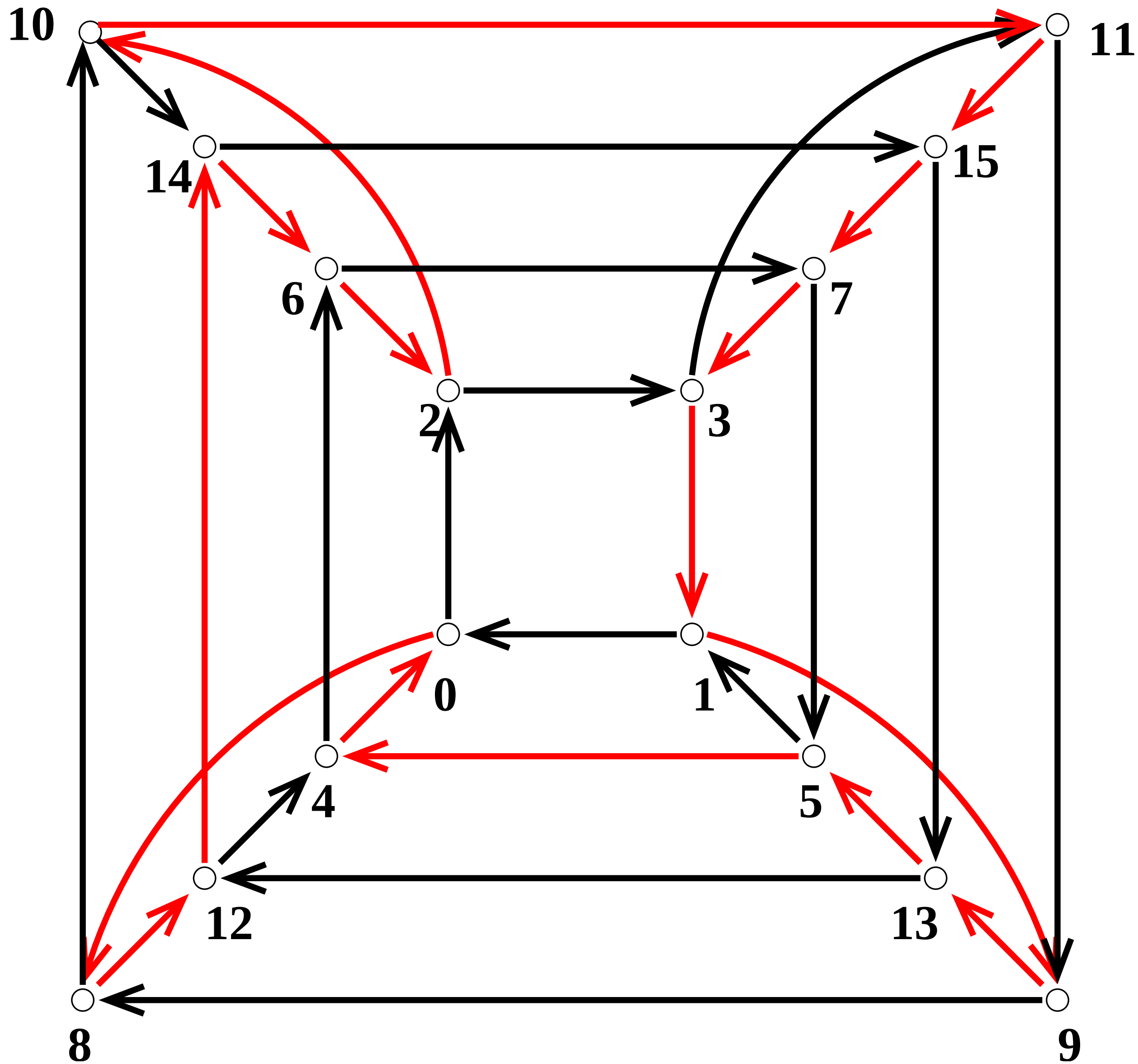


# Hamilton orientation $Q_{B1}(4)$

The edge-set of $H(4)$ decomposed into two complementary Hamilton cycles with one (so, both) of type B1.
Orientation $Q_{B1}(4)$ is defined by signature $(-1, 1 - 1, 1)$.

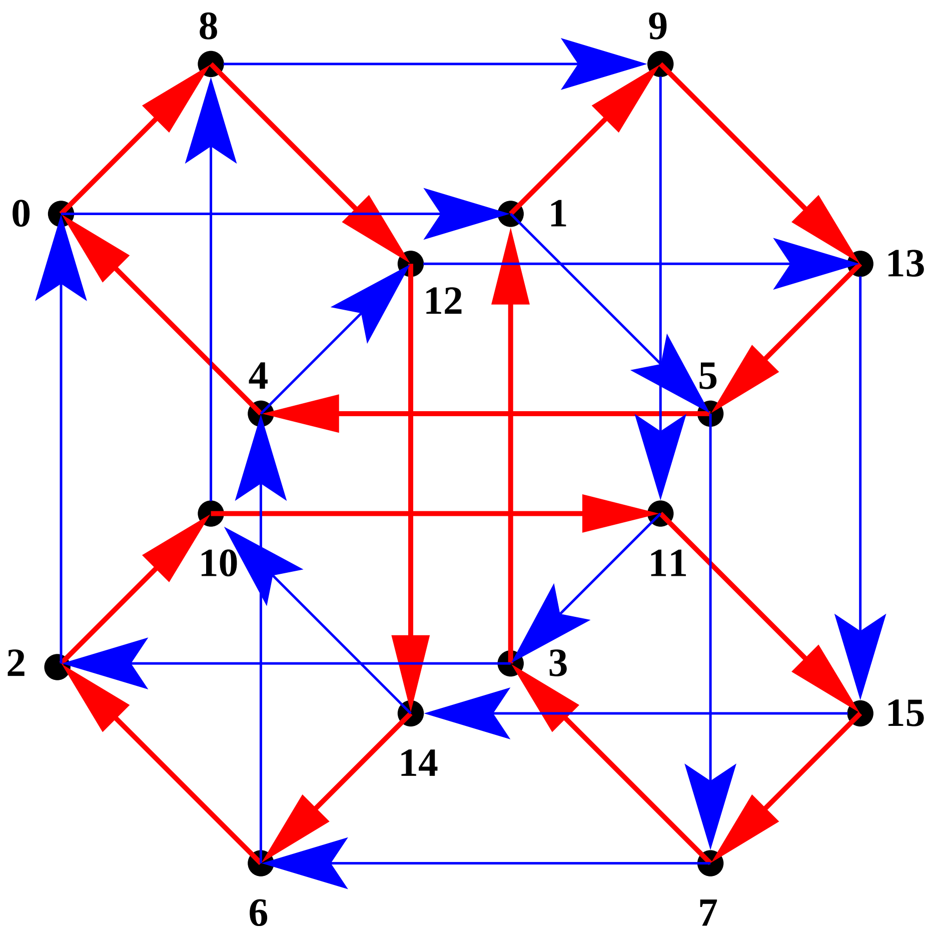


# Hamilton orientation $Q_{B1}(4)$

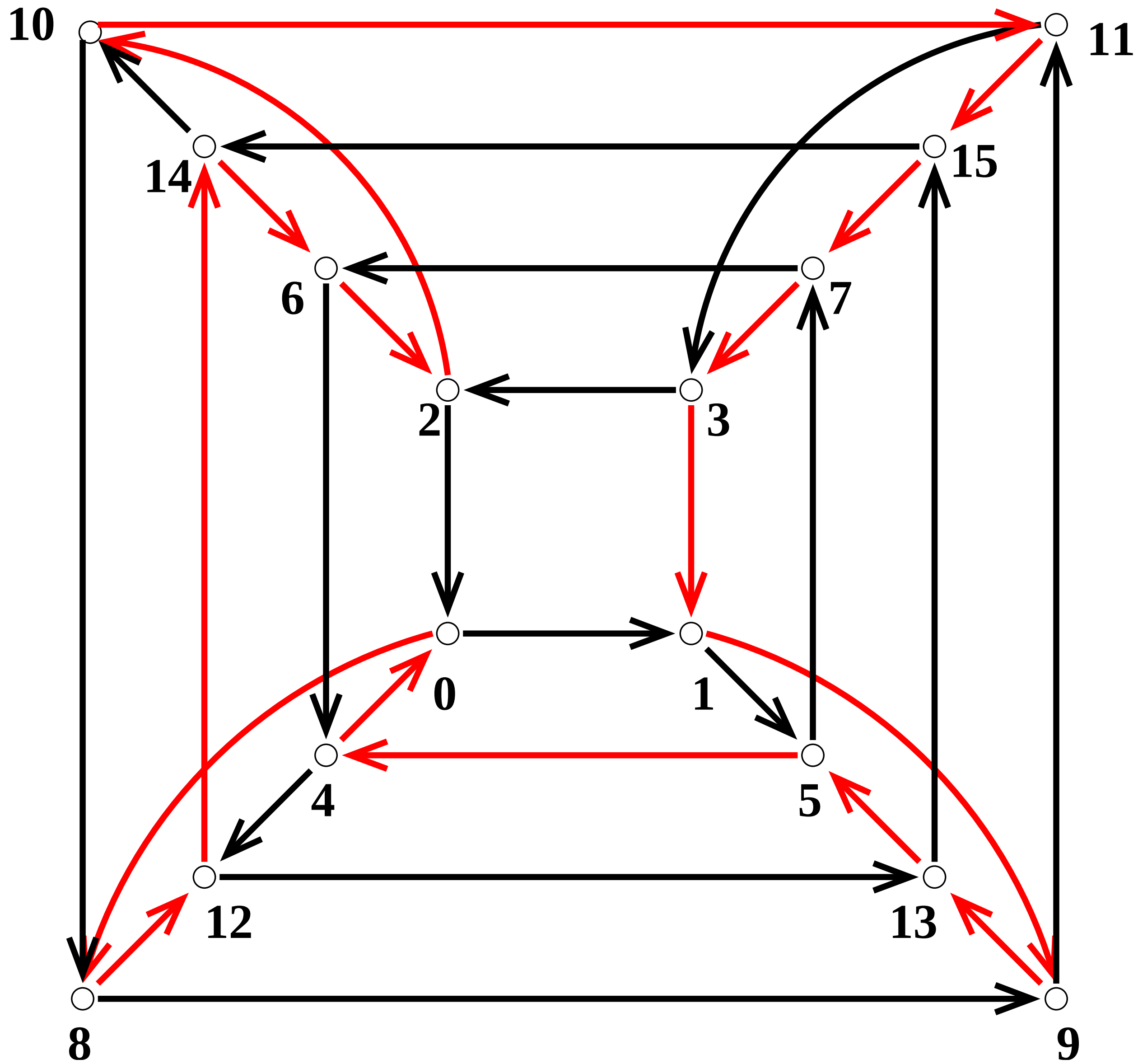


# Hamilton orientation $Q_{B1'}(4)$

The edge-set of $H(4)$ decomposed into two complementary Hamilton cycles with one (so, both) of type B1.
Orientation $Q_{B1'}(4)$ is defined by signature $(1, -1 - 1, 1)$.



# Hamilton orientation $Q_{B1'}(4)$



# Ten Hamilton orientations of $H(4)$

Edge-complement of Hamilton cycle $h$ of $4$-cube is another Hamilton cycle $h^*$ if and only if $h = B1, C2, C3, C4, D$; moreover, $h^* \sim h$ under $Sym(4)$, shifting and reversals.

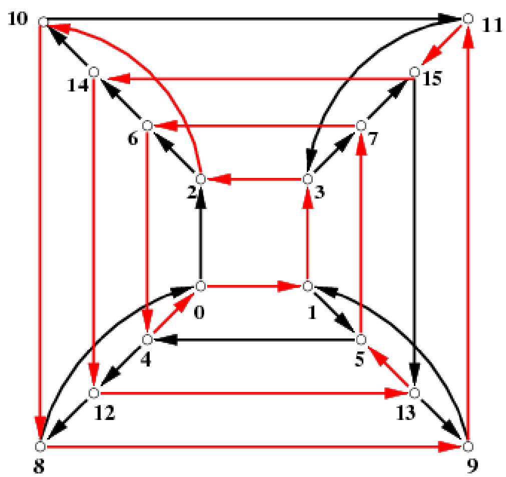
Orient $h$ so to get arc $(0,1)$ on it. Let $O_h$ be orientation of $H(4)$= $h$+$h^*$ with arc $(2,0)$ on $h^*$ and by $O'_h$ one with $(0,2)$. So, signature is $(1,1,-1,-1)$ for all $O_h$, $(1,-1,-1,1)$ for $O'_h$ with $h = B1, C1$ and $(1,-1,1,-1)$ for $O'_h$ with $h = C3, C4, D$.

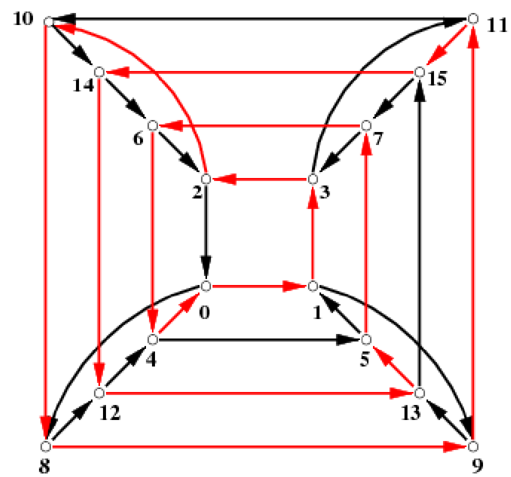
**O-diameter** is $6$ for $Q_{B1}$ and $5$ for other $9$. $Q_{C3}$ has minimal, 4, $|\{(u,v) : q(u,v) = 5\}|$ and **mean** $q(u,v)$ ($\approx 2.5$); cf. $2$ of $H(4)$.

**Conjecture**: for any $m$, there exists a Hamilton orientation of $H(2m)$ with $2^m d(K_4 \times K_4 \times \cdots \times K_4)$ ($m$ times) being the symmetrization of its quasi-metric. It holds for $2$-cube (unique strong orientation) and $4$-cube (orientation $Q_{B1}$). Remind that $H(2m) = C_4 \times C_4 \times \cdots \times C_4)$ ($m$ times).

# Hamilton orientations $O_B(4)$, $O_{B'}(4)$

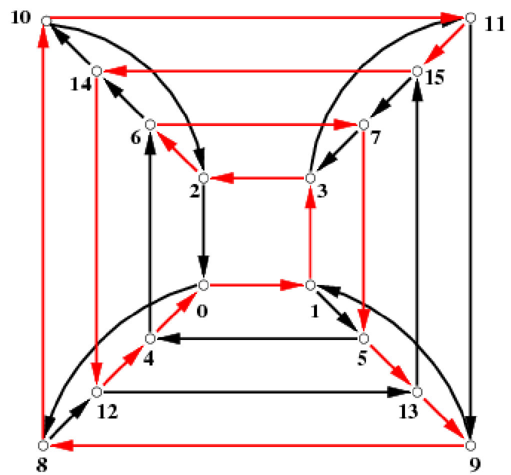
Each Hamilton cycle $V$=$\{v_i\}$, $1\leq i\leq 2^n$, as sequence $t(V)$= $\{1+\lg_2|t_i-t_{i+1}|\}$, $1\leq i\leq 2^n$, where $t_i$ is label of $v_i$, is
**B1** $\{6,6,2,2\}$: 1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4.

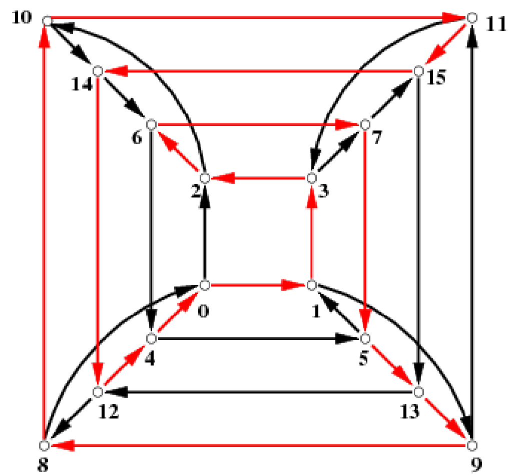


# Hamilton orientations $O_{C2}(4)$, $O_{C2'}(4)$

Each cycle is **C2** $\{6, 4, 4, 2\}$: 1 2 1 3 1 2 4 3 1 2 1 3 1 2 4 3.
**Wrapped grid** $G$ comes from $K_4 \times K_4$ on $((x_{ij}))$ by adding edges of $C_{11,22,33,44}, C_{12,21,43,34}, C_{13,24,42,31}, C_{14,23,41,32}$.
$2d(G)$ is symmetrization of quasi-metric of $O_{C2}(4)$.
This quasi-metric differs from one of Chou-Du $Q_{CD}(4)$ only by permutation $(4,8)(5,9)(6,10)(7,11)$ of vertices.



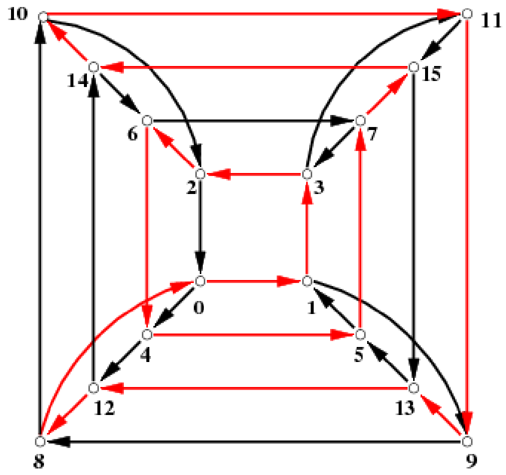



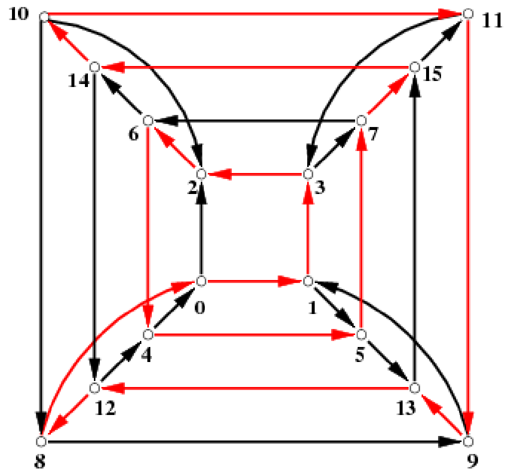
# Hamilton orientations $O_{C3}(4)$, $O_{C3'}(4)$

Each Hamilton cycle $V$=$\{v_i\}$, $1 \le i \le 2^n$, as sequence $t(V)$= $\{1 + \lg_2 |t_i - t_{i+1}|\}$, $1 \le i \le 2^n$, where $t_i$ is label of $v_i$, is

**C3** $\{6, 4, 4, 2\}$: 1 2 1 3 2 1 2 4 1 3 1 2 3 1 3 4.

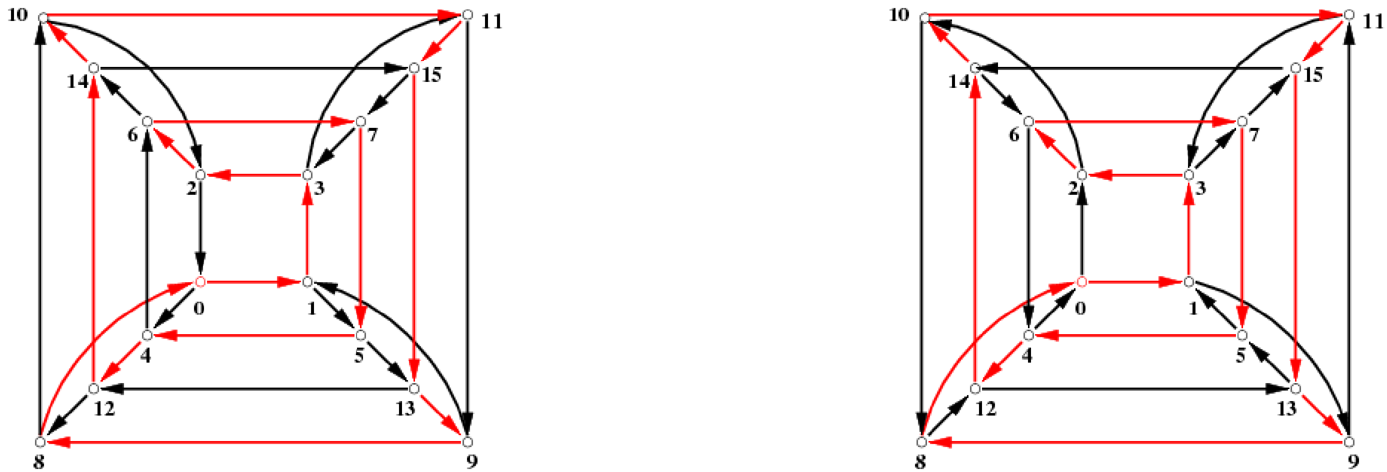
In $O_{C3}(4)$, $q(x, y) < 5$ except $(x, y)$=$(2, 10)$,$(5, 4)$,$(11, 3)$,$(12, 13)$.



# Hamilton orientations $O_{C4}(4)$, $O_{C4'}(4)$

Each Hamilton cycle $V$=$\{v_i\}$, $1{\le}i{\le}2^n$, as sequence $t(V)$= $\{1+\lg_2|t_i - t_{i+1}|\}$, $1{\le}i{\le}2^n$, where $t_i$ is label of $v_i$, is
**C4** $\{6,4,4,2\}$: 1 2 1 3 1 2 1 4 2 3 1 3 2 3 1 4.

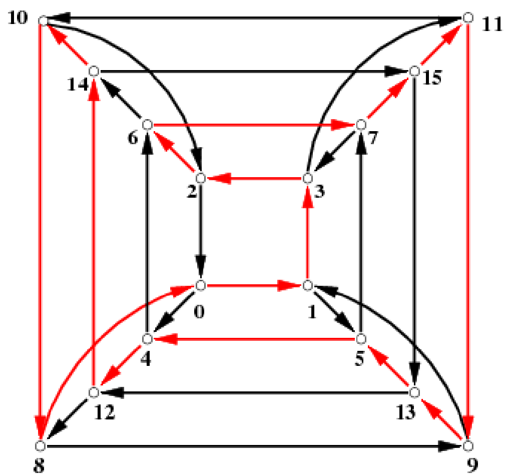


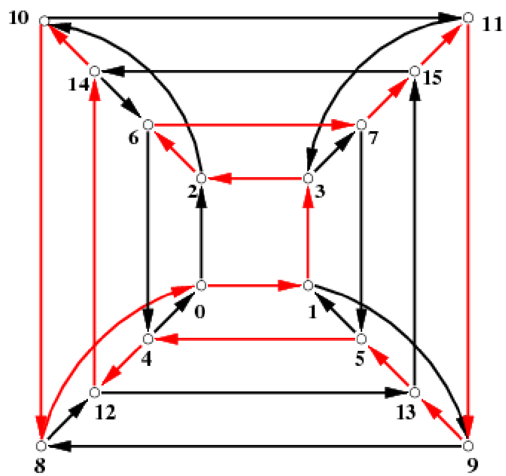
# Hamilton orientations $O_D(4)$, $O_{D'}(4)$

Each Hamilton cycle $V$={$v_i$}, $1\leq i\leq 2^n$, as sequence $t(V)$, is

**D** $\{4,4,4,4\}$: 1 2 1 3 1 4 3 2 3 4 1 4 2 3 2 4.

In $O_D(4)$, $q(x,y)<5$ except $(x,y)$=$(0,14)$,$(6,8)$,$(10,4)$,$(12,2)$ and $(3,13)$,$(5,11)$,$(9,7)$,$(15,1)$. In $O_{D'}(4)$, $q(x,y)=5$ 10 times.

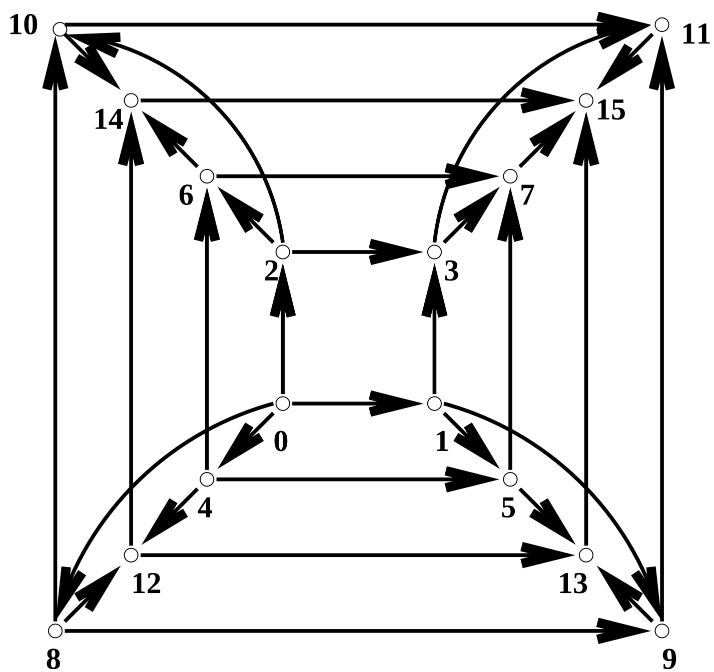


# VII. Unique-sink orientations of hypercubes

# Inclusion (or Boolean) orientation $Q_I(n)$

Label vertices $0 \le x \le 2^n - 1$ of $n$-cube by subsets $A_x = \{1 \le i \le n : x_i = 1\}$ of $[n] = \{1, \dots, n\}$.
**Inclusion orientation** $Q_I(n)$: do arc $AB$ if $A \subset B$ and $|B \backslash A|$=1.
Its **path quasi-semi-metric** is $|B \backslash A|$ if $A \subset B$ and =$\infty$, else,
while **measure q-s-metric** on $(\Omega = [n], \mathcal{A} = 2^{[n]}, \mu)$ is $\mu(B \backslash A)$.




Become strongly connected if add sink-souce arc $(2^n - 1, 0)$.

# Unique-sink orientations

An orientation of n-cube is called **unique-sink orientation** if every face has unique sink.

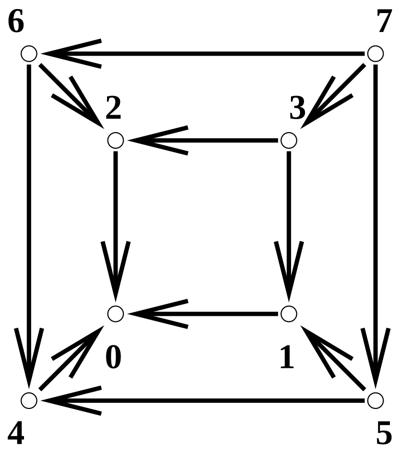
**Examples**:

1) the inclusion orientation $Q_I(n)$ and the arc-reversal of it on any fixed **matching** (set of disjoint edges) $M$ of n-cube;

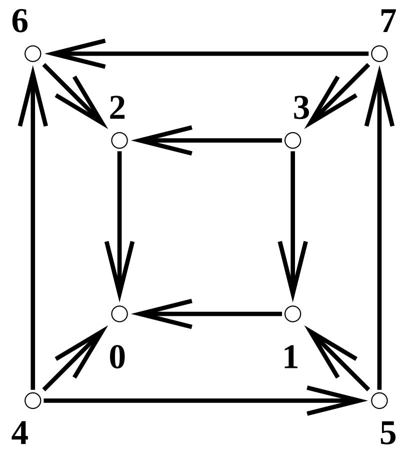
2) every acyclic orientation with unique-sink **on each 2-face**;

3) the **Klee-Minty orientation** $Q_{KM}(n)$: if the binary expansions of vertices $x, x' \in H(n)$ differ only in i-th position, then do arc $(xx')$ if $\sum_{i \le j \le n} x_j$ is odd and arc $(x'x)$, otherwise.

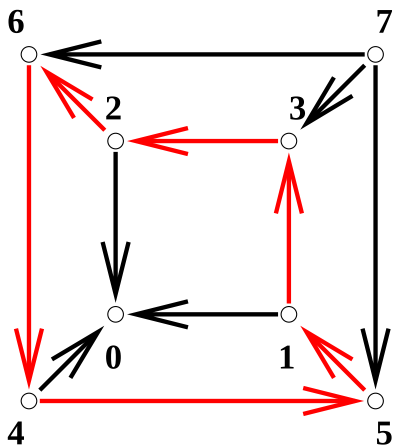
# $3$-cube: some unique-sink orientations



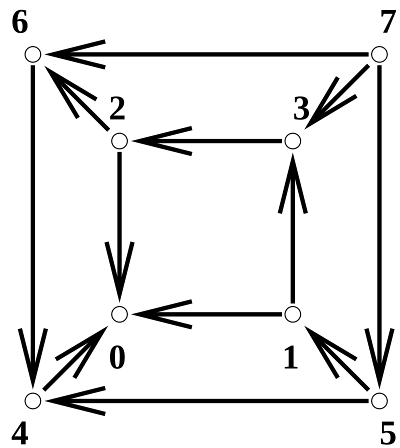

Inclusion orientation $Q_I(3)$




Klee-Minty orientation $Q_{KM}(3)$




(62,31,54)-reversed $Q_I(3)$




(62,31)-reversed $Q_I(3)$

# Digression: Klee-Minty orientation

**Klee-Minty orientation**: if the binary expansions of vertices $x, x' \in H(n)$ differ only in i-th position, then do arc $(xx')$ if $\sum_{i\leq j\leq n} x_j$ is odd and arc $(x'x)$, otherwise.

It is acyclic unique-sink orientation; moreover, each face has unique source.

It comes from combinatorial model (Avis-Chvatal, 1978) of **Klee-Minty cubes**, 1972, i.e., linear programs whose polytopes are deformed n-cubes (with skeleton of $H(n)$) but for which some pivot rules follow path through all $2^n$ vertices and hence, need exponential number of steps.

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