Voronoi L-types and Hypermetrics

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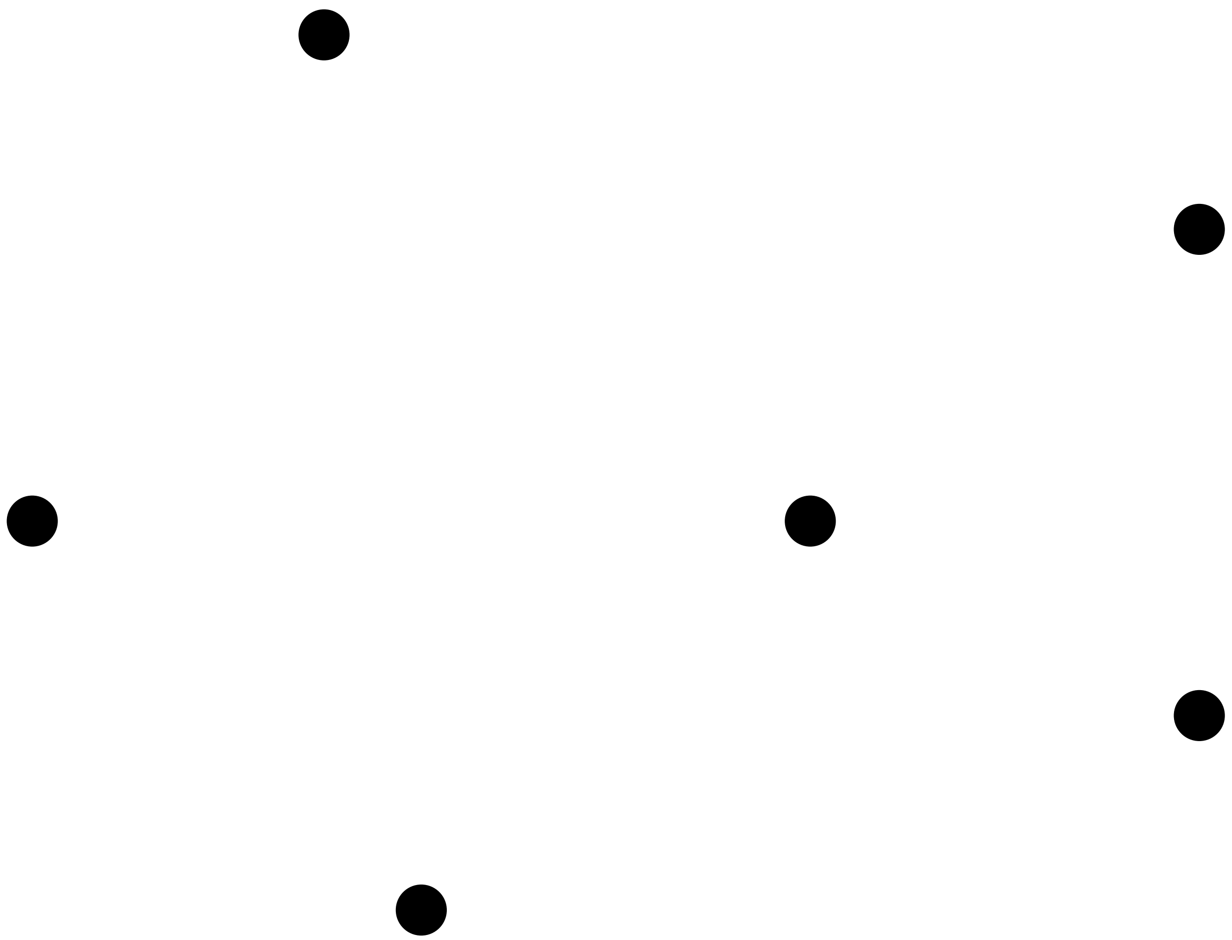
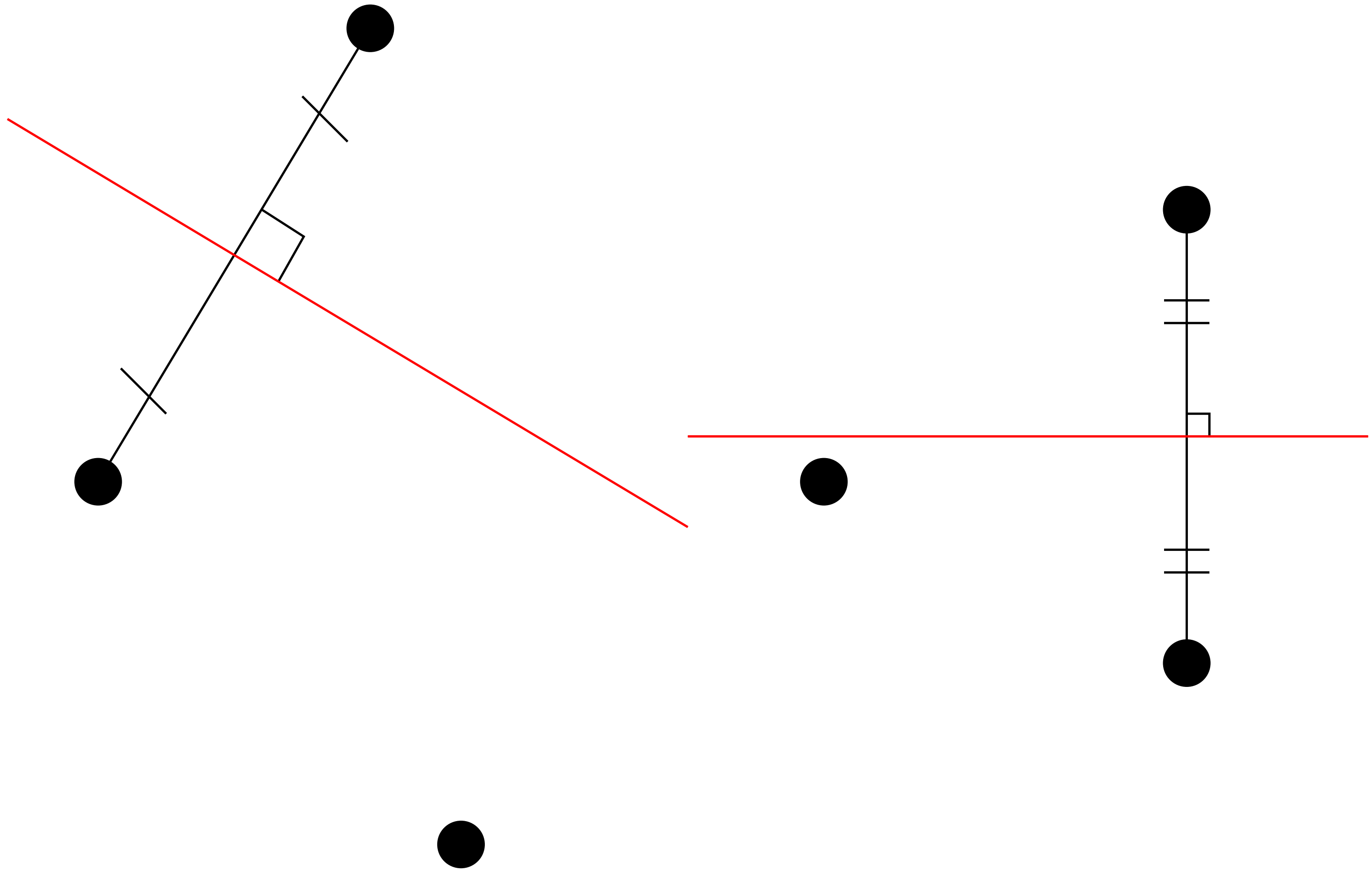
Voronoi and Delaunay polytopes

A finite set of points



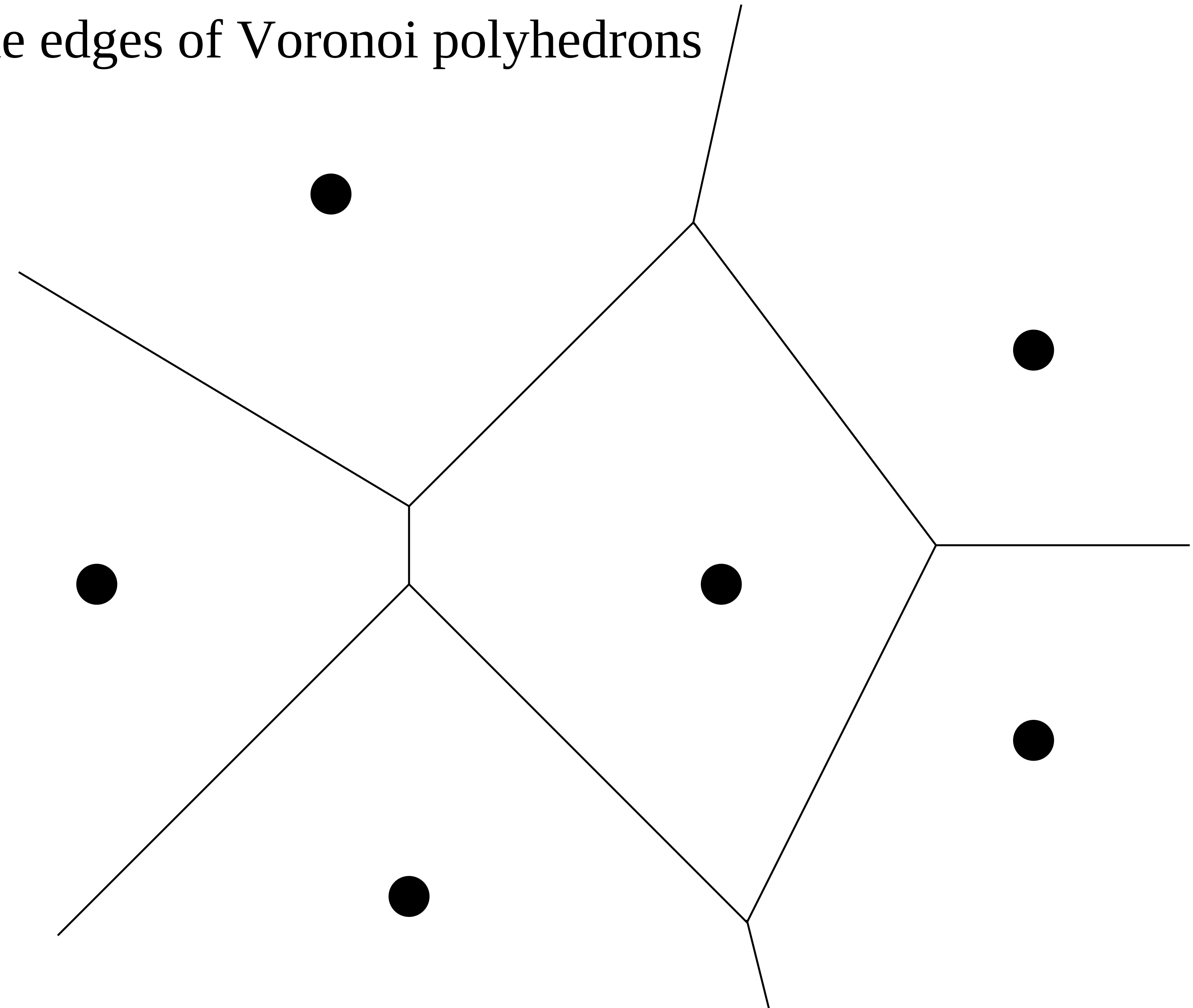
Voronoi and Delaunay polytopes

Some relevant perpendicular bisectors



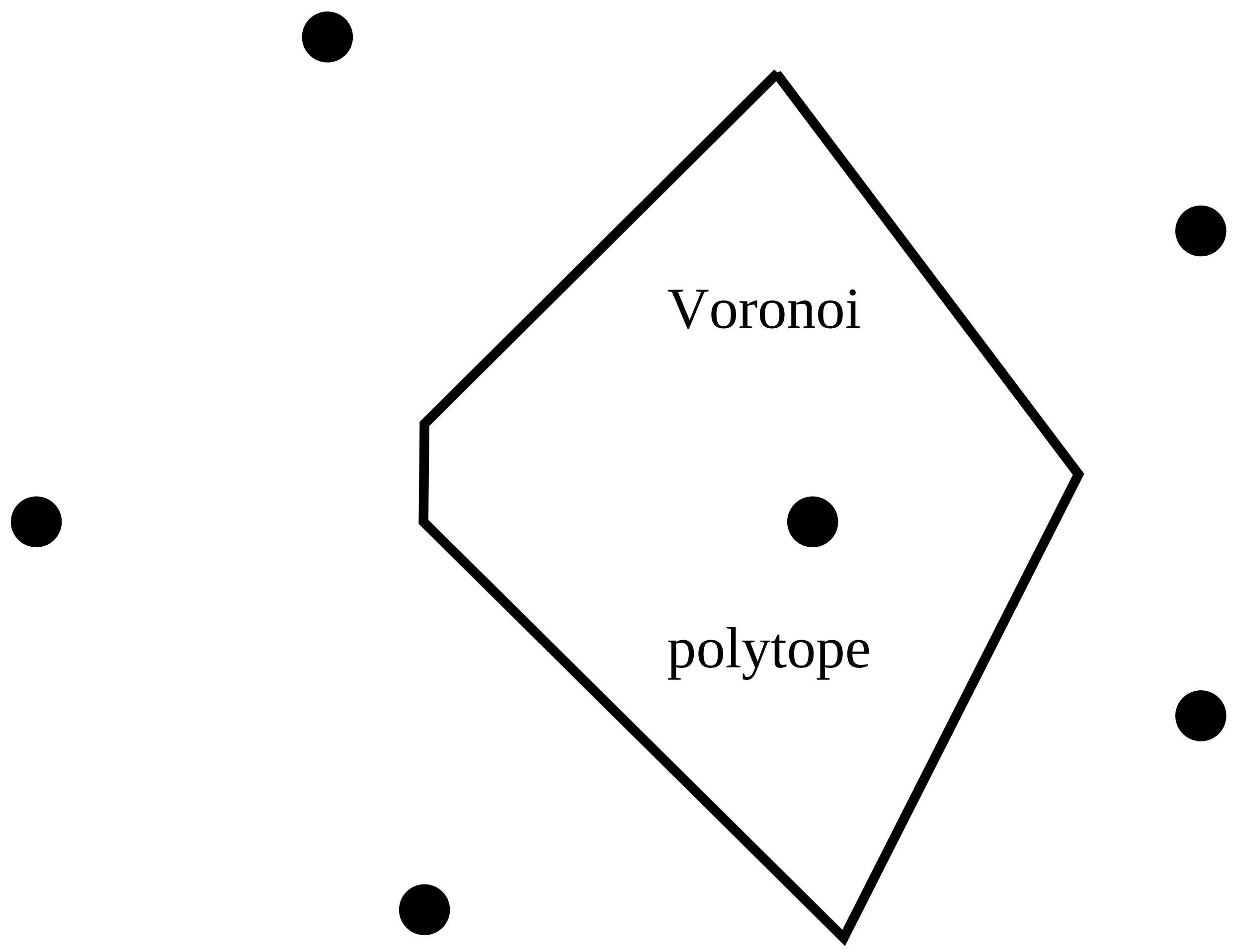
Voronoi and Delaunay polytopes

The edges of Voronoi polyhedrons



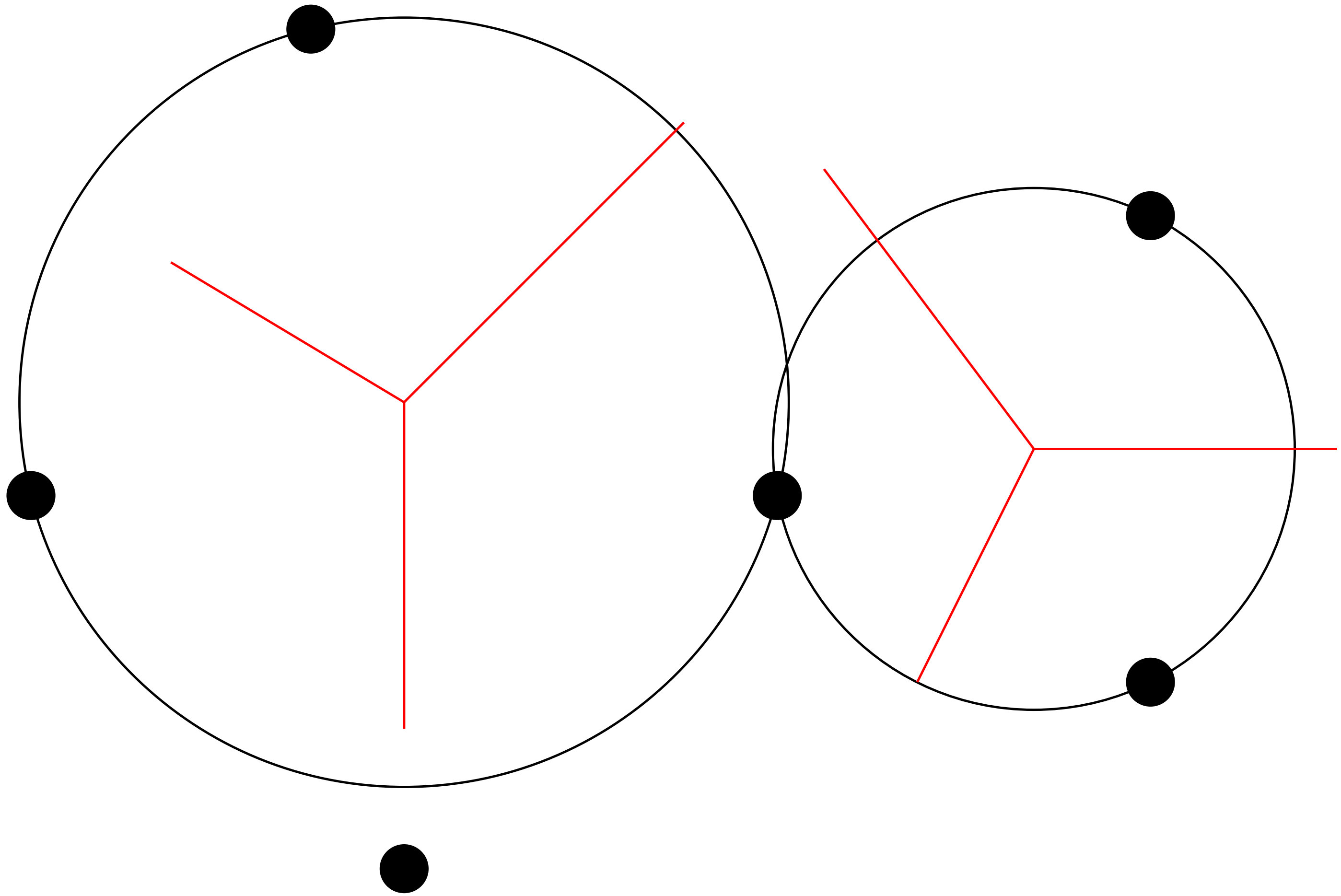
Voronoi and Delaunay polytopes

Voronoi polytope



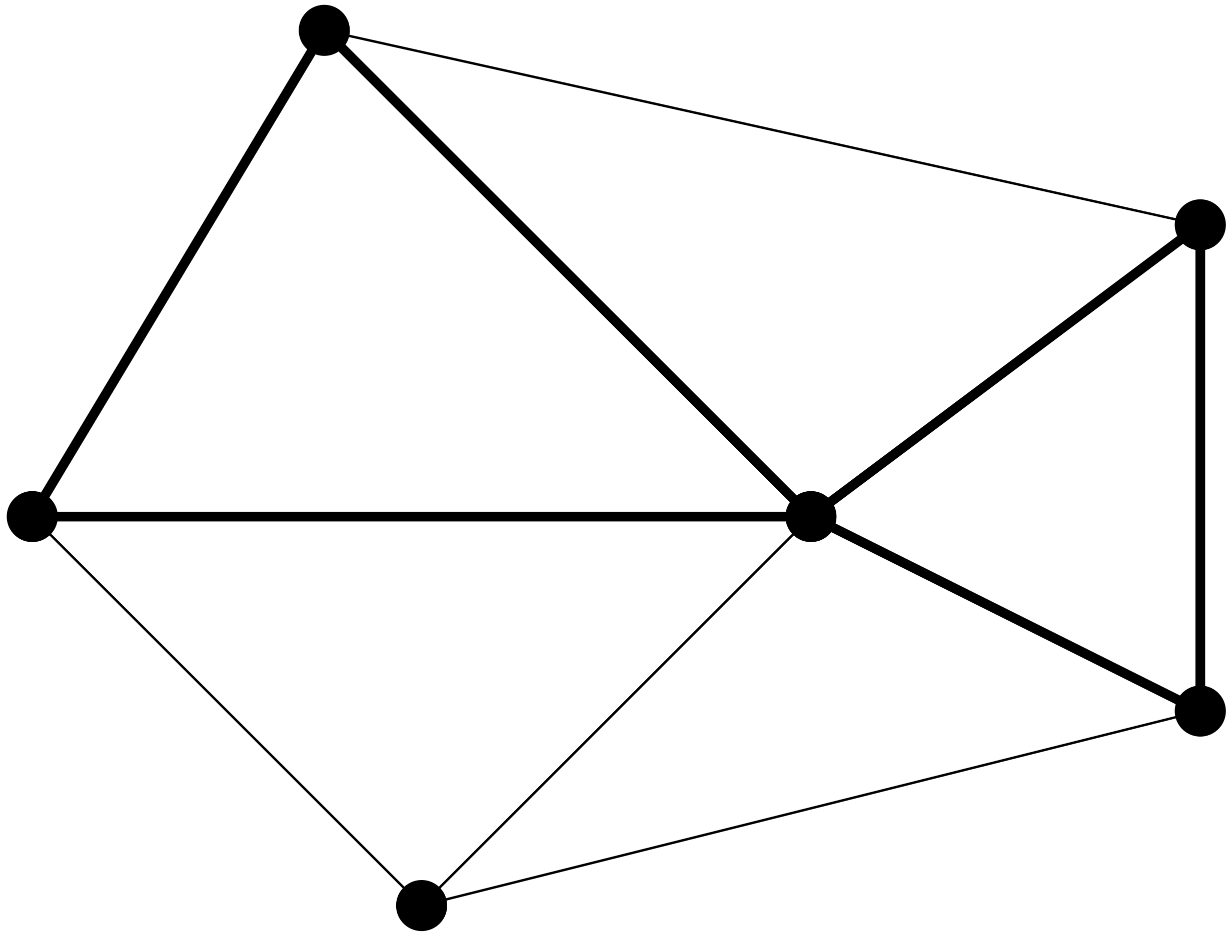
Voronoi and Delaunay polytopes

Empty spheres



Voronoi and Delaunay polytopes

Delaunay polytopes



Synonyms

Voronoi polytope is $\mathcal{V}_x = \{v \in \mathbb{R}^n \ : \ d(v,x) \leq d(v,y)$ for $y \in X - \{x\}\}$, for given locally finite subset X of $\mathbb{R}^n$. Its main synonyms are:

- Dirichlet domain (lattice theory, 2-dimensional case)
- Voronoi polytope (n-dimensional lattices, computational geometry)
- Thiessen polygon (geography), domain of influence (politics)
- Wigner-Seitz cell, first Brillouin zone, Bernal polytope, nearest neighbor region (solid state physics, crystallography)

Delaunay polytope synonyms:

- L-polytope (Voronoi in “Deuxéme mémoire”)
- constellation or, mainly, hole (in Conway-Sloane); hole is deep if of maximal radius and, otherwise, shallow.

Voronoi polytopes $\mathcal{V}_x$ form normal (face-to-face) tiling of $\mathbb{R}^n$.

I. Voronoi and Delaunay polytopes in lattices

The Voronoi polytope of a lattice

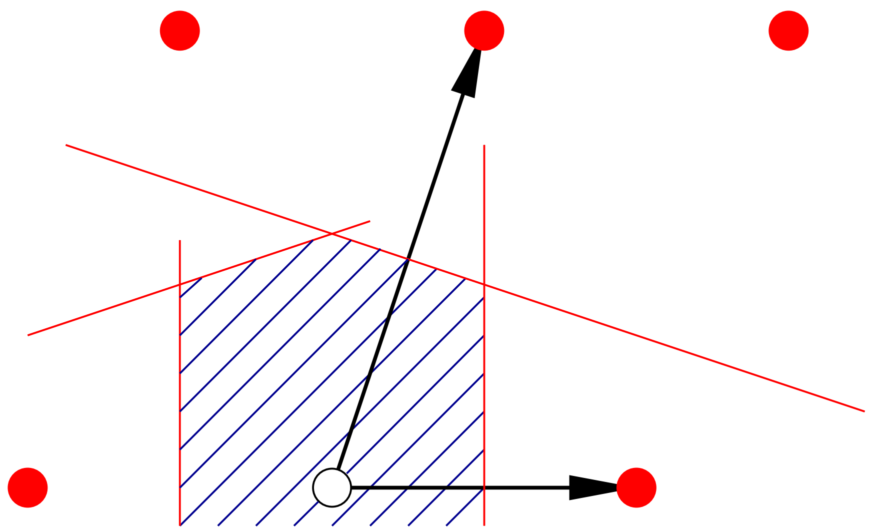
- A lattice L is a rank n subgroup of $\mathbb{R}^n$, i.e.,

$$L = v_1\mathbb{Z} + \cdots + v_n\mathbb{Z} \ .$$

- The Voronoi cell $\mathcal{V}$ of L is defined by inequalities

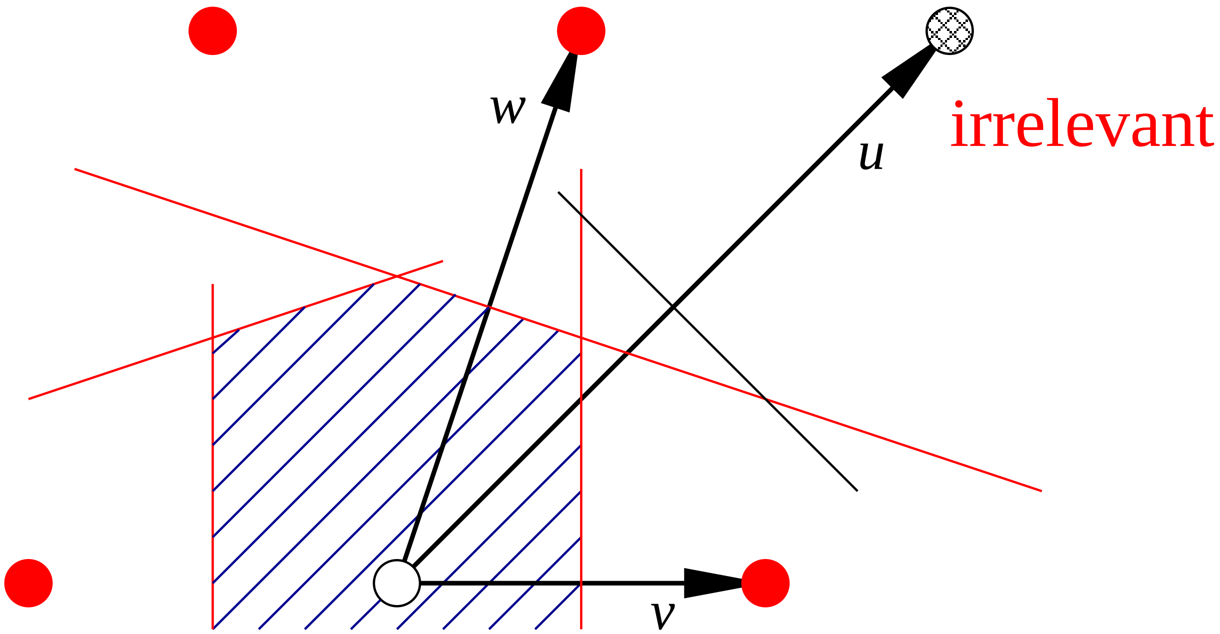
$$\langle x, v\rangle \leq \frac{1}{2}||v||^2 \text{ for } v \in L.$$

- $\mathcal{V}$ is a polytope, i.e., it has a finite number of vertices (of dimension 0), faces and facets (of dimension $n-1$).



The Voronoi polytope of a lattice

- Polytope $\mathcal{V}$ is defined by inequalities $\langle x, v\rangle \leq \frac{1}{2}||v||^2$.
- A vector v_0 is relevant if $\langle x, v_0\rangle = \frac{1}{2}||v_0||^2$ is a facet.
- Voronoi: a vector u is relevant if and only if it can not be written as $u = v + w$ with $\langle v, w\rangle \geq 0$.



The Voronoi polytope of a lattice

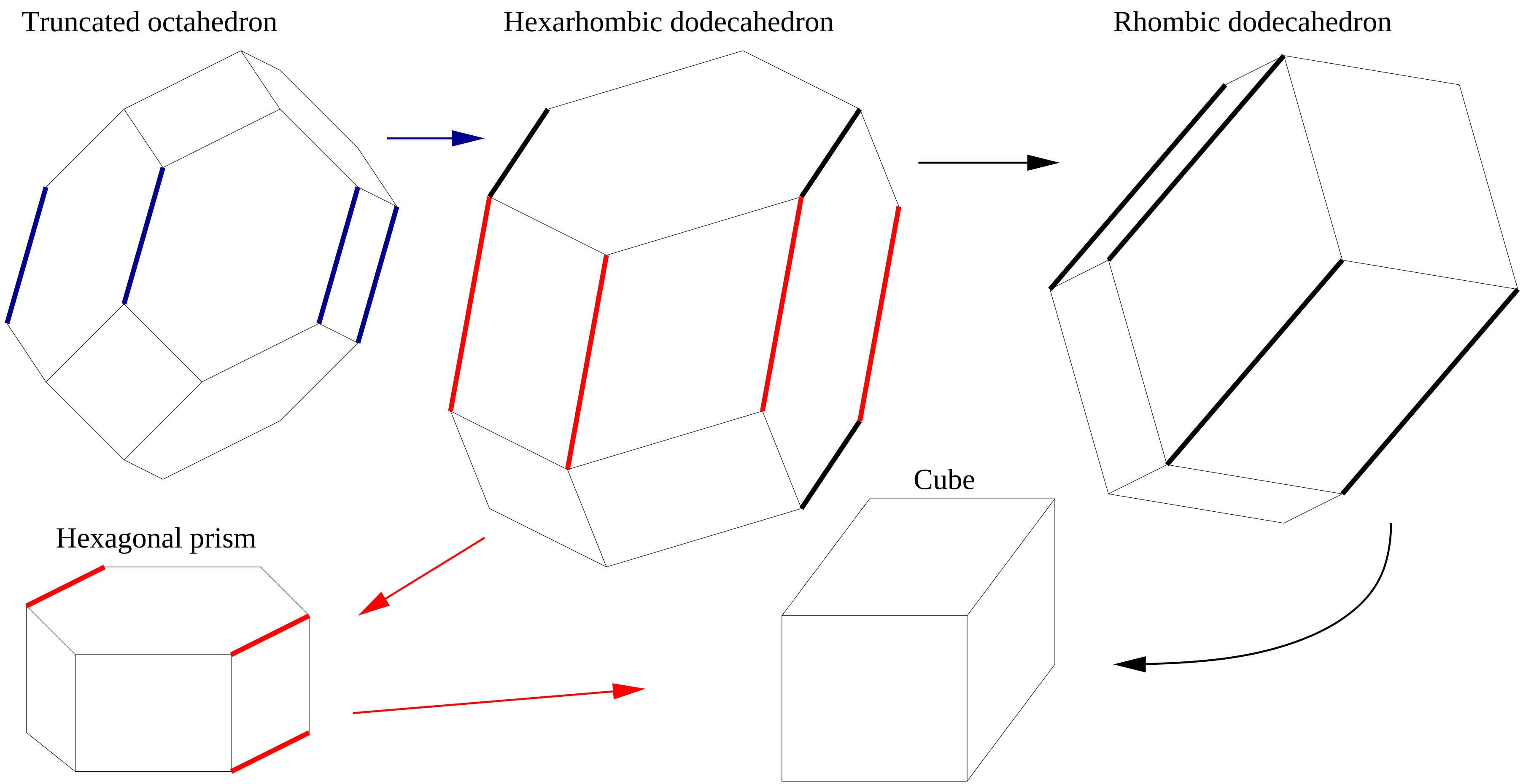
- The translates $v + \mathcal{V}$ with $v \in L$ tile $\mathbb{R}^n$.
- $\mathcal{V}$ has $\leq 2(2^n - 1)$ facets and $\leq (n+1)!$ vertices.
- Shortest vectors in L are relevant vectors.
- L is a root lattice iff all relevant vectors are shortest; they are called roots; their number is number of facets.

Irr. lattice	Nr. facets	Nr. vertices	Nr. orbits
A_n	$n(n+1)$	$2^{n+1} - 2$	$\lfloor \frac{n+1}{2} \rfloor$
D_n	$2n(n-1)$	$2^n + 2n$	2
E_6	72	54	1
E_7	126	632	2
E_8	240	19440	2

- A root lattice is a direct sum of irreducible root lattice.

Comb. types of Voronoi ≤ 3-polytopes

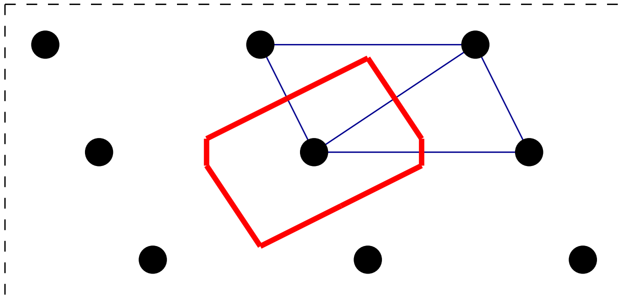
Two combinatorial types of Voronoi 2-polytopes: centrally symmetric hexagons (primitive) and rectangles.



$1+2+5$ with $n \leq 3$ are zonotopes but not 24-cell=$\mathcal{V}(D_4)$.

Voronoi and Delaunay in lattices

- Vertices of Voronoi polytopes are centers of empty spheres which define Delaunay polytopes.
- Voronoi and Delaunay polytopes define dual normal tessellations of the space $\mathbb{R}^n$ by polytopes.
- Every k-dimensional face of a Delaunay polytope is orthogonal to a $(n-k)$-dim. face of a Voronoi polytope.



- Any lattice L has finite number of orbits of Delaunay polytopes under translation; L-star: all with given vertex.

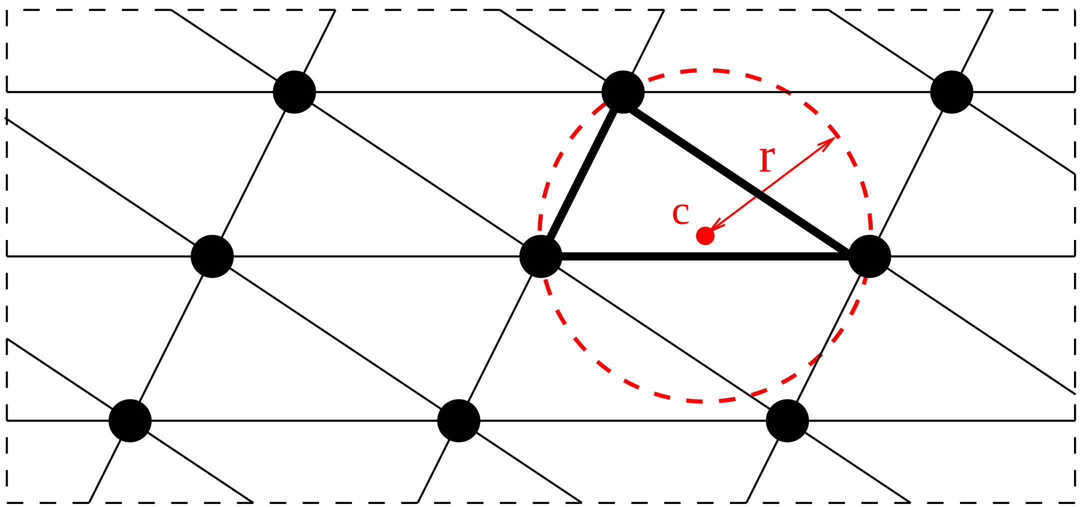
Empty spheres and Delaunay polytopes

A sphere $S(c,r)$ of radius r and center c, in a n-dimensional lattice L, is called an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) $S(c,r) \cap L$ contains $n+1$ affinely independent points.

Delaunay polytope in L is a polytope with vertex-set $L \cap S(c,r)$.



Lattices with ≤ 2 Delaunay polytopes

- $L = \mathbb{Z}^n$; Delaunay polytope is unique:

Polytope	Center	Nr. vertices	Radius
Cube	$(\frac{1}{2})^n$	2^n	$\frac{1}{2}\sqrt{n}$

- D_n=$\{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i$ is even$\}$; Delaunay polytopes:

Polytope	Center	Nr. vertices	Radius
Half-Cube	$(\frac{1}{2})^n$	$\frac{1}{2}2^n$	$\frac{1}{2}\sqrt{n}$
Cross-polytope	$(1, 0^{n-1})$	$2n$	1

- E_8=$\{x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 : \sum_{i=1}^n x_i$ is even$\}$; Delaunays:

Polytope	Center	Nr. vertices	Radius
Simplex	$(\frac{5}{6}, \frac{1}{6}^7)$	9	$\sqrt{\frac{8}{9}}$
Cross-polytope	$(1, 0^7)$	16	1

Digression on the root lattice E_8

- Lattice E_8 is the integral span of its shortest (square length 2) vectors (240 roots): 112 permutations of $(\pm1,\pm1,0,0,0,0,0,0)$ and 128 permutations of $\frac{1}{2}(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,\pm1)$ with even number of $+$. Convex hull of 240 roots is semi-regular 8-polytope 4_{21}.
- E_8 rescaled to minimal sq. length 1: integral octonions.
- Lattice E_8 is self-dual; equiv. E_8 is unimodular (volume of fund. parallelotope is 1). $|Aut(E_8)| = 4! \times 6! \times 8!$.
- E_8 is unique (nontrivial) even (any vector has even square length) self-dual n-dim. lattice with $n < 16$.
- Lie algebra E_8, as a manifold, has dimension $8+240$. In 2007, all its ∞-dim. irr. representations were computed.
- $E_8 \times E_8$ is 1 of 2 even self-dual 16-dim. lattices. Only on them 16 dimensions of heterotic string compactify "well"

II. Voronoi and Delaunay polytopes in lattices

Geometry of numbers by Minkowski

- Denote by PSD_n the convex cone of real symmetric positive definite $n \times n$ matrices.
- Lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$
(spanned by basis $v_1, \ldots, v_n$) corresponds to
the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in PSD_n$.
- Elements of PSD_n can be seen as Gram matrices or as quadratic forms.
It is convenient to think in lattice terms and compute
in terms of quadratic forms.

Isometric lattices

- If $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n$, then $G_{\mathbf{v}} = V^T\, V$ for the following matrix:

$$V = \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}$$

- If $A \in PSD_n$, then there exists V such that $A = V^T\, V$.
- If $A = V_1^T\, V_1 = V_2^T\, V_2$, then $V_1 = OV_2$ with $O^T\, O = I_n$: orthogonal matrix O corresponds to an isometry of $\mathbb{R}^n$.
- Also, if L is a lattice in $\mathbb{R}^n$ with basis $\mathbf{v}$ and u is an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two bases of a lattice L, then $V' = VP$ with $P \in GL_n(\mathbb{Z})$. This implies:

$$G_{\mathbf{v}'} = V'^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

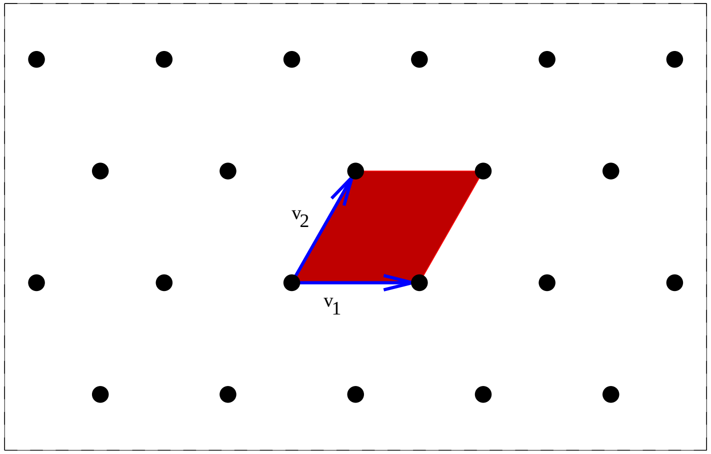
- If $A, B \in PSD_n$, they are called arithmetically equivalent if, for some $P \in GL_n(\mathbb{Z})$, it holds:

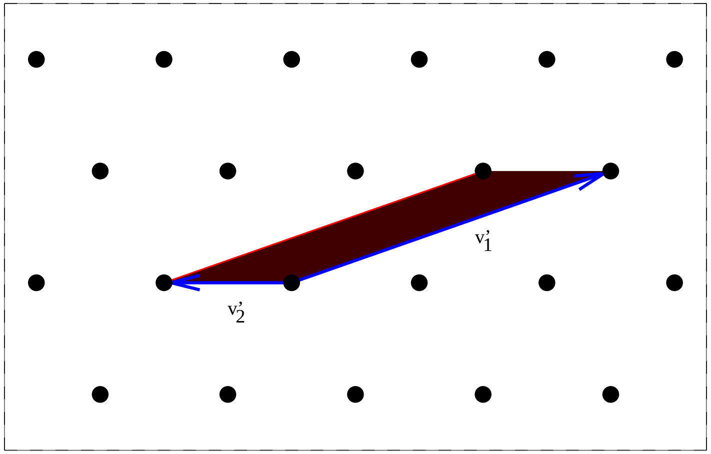
$$A = P^T BP.$$

- There is a bijection between (arithmetic) equivalence classes $PSD_n/GL_n(\mathbb{Z})$ and isometry (equivalence) classes of lattices in $\mathbb{R}^n$.
- $GL_n(F)$ is the general linear group of $n \times n$ invertible matrices over field or ring F with matrix multiplication. Orthogonal group: its subgroup of orthogonal matrices

An example: A_2

- Take the hexagonal lattice A_2 and two bases in it.





$v_1 = (1,0)$ and $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ $\quad v_1' = (\frac{5}{2}, \frac{\sqrt{3}}{2})$ and $v_2' = (-1,0)$

- One has $v_1' = 2v_1 + v_2$, $v_2' = -v_1$ and $P = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

$$G_{\mathbf{v}} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \text{ and } G_{\mathbf{v}'} = \begin{pmatrix} 7 & -\frac{5}{2} \\ -\frac{5}{2} & 1 \end{pmatrix} = P^T G_{\mathbf{v}} P.$$

The enumeration problem

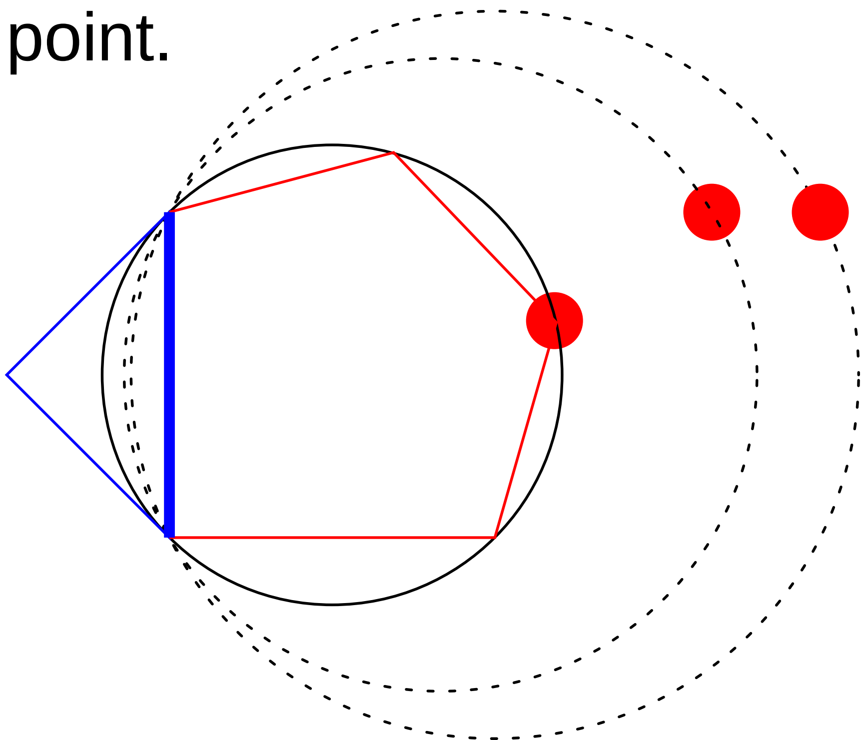
- Given a matrix $A \in PSD_n$, we want to compute the Delaunay polytopes of a lattice corresponding to A.
- There is a finite number of Delaunay polytopes, up to translation, but still of the order of $(n+1)!$.
- If $A \in PSD_n$, then its symmetry group is finite:

$$Aut(A) = \{P \in GL_n(\mathbb{Z}) \quad : \quad A = P^T A P\}.$$

- $Aut(A)$ corresponds to isometries of the corresponding lattice. Those symmetries can be used to accelerate the computation.

Finding Delaunay polytopes

- Given a Delaunay polytope and a facet of it, there exist a unique adjacent Delaunay polytope.
- We use an iterative procedure:
  - Select a point outside the facet.
  - Create the sphere incident to it.
  - If there is no interior point, finish; otherwise, rerun with this point.



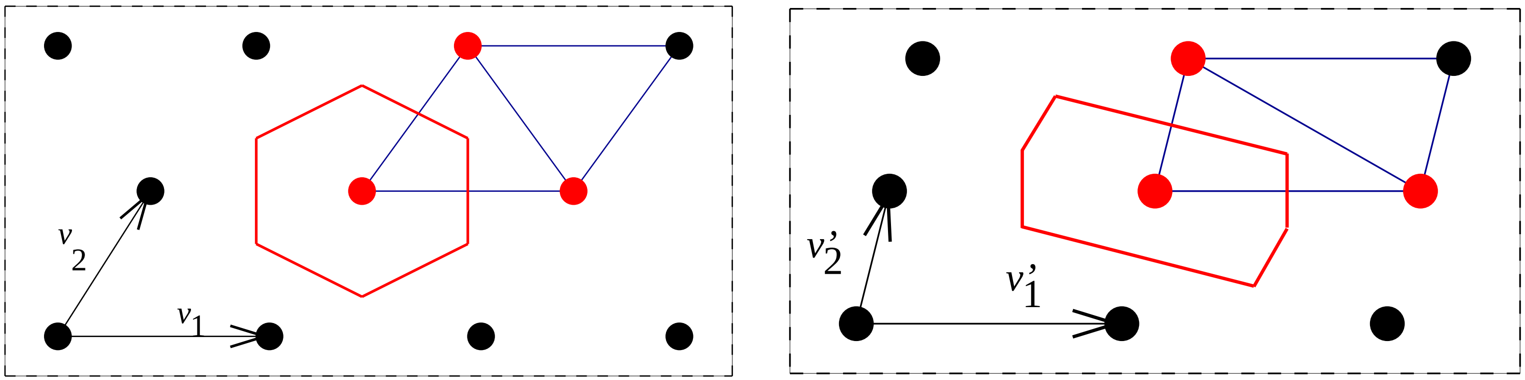
Finding Delaunay decomposition

- Find the isometry group of the lattice (program autom by Plesken & Souvignier).
- Find an initial Delaunay polytope (program finddel by Vallentin) and insert into list of orbits as undone.
- Iterate
  - Find the orbit of facets of undone Delaunay polytopes (GAP + lrs by Avis + Recursive Adjacency Decomposition method by Dutour).
  - For every facet, find the adjacent Delaunay polytope.
  - For every Delaunay polytope, test if it is isomorphic to existing ones.
    If not, insert it into the list as undone.
  - Finish when every orbit is done.

I. L-type domains and L-types

L-type domains by Voronoi

- A L-type domain is the set of matrices $G_{\mathbf{v}} \in PSD_n$ corresponding to the same Delaunay decomposition, i.e., the same combinatorial type of Voronoi polytope.
- Geometrically, for example, Gram matrices of following lattices belong to the same L-type domain:



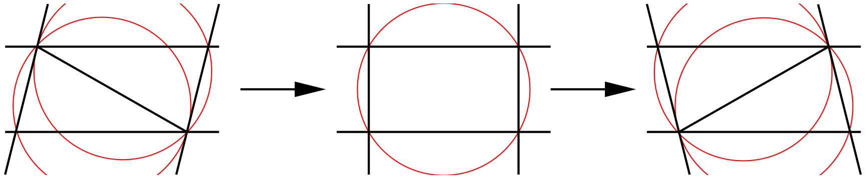
- Specifying Delaunay polytopes means putting some linear equalities and inequalities on the Gram matrix. A priori, infinity of inequalities but finite number suffices

L-types by Voronoi

- An L-type is the union of L-type domains which are isomorphic under linear transformations. The number of L-types is finite, while there is infinity of L-type domains.
- L-type domains are convex polyhedral cones (open but in dim. 1 case - edge forms) face-to-face tiling PSD_n.
- This partition is invariant with respect to $GL_n(\mathbb{Z})$.
- There are finitely many orbits, which correspond to non-isomorphic combinatorial type of Voronoi polytopes
- Two lattices in the same L-type $\mathcal{LT}$ can be continuously deformed without changing the structure.
- If $dim(\mathcal{LT}) = \binom{n+1}{2}$ (i.e., Delaunay partition is simplicial), then $\mathcal{LT}$ is called primitive or non-special.
- If $dim(\mathcal{LT}) = 1$ (i.e. only scaling preserves L-type), then $\mathcal{LT}$ is called rigid and resp. quadratic form is edge form.

Equivalence and enumeration

- Voronoi: the inequalities, obtained by taking adjacent simplices, suffice to describe all inequalities.
- If there is no equalities, i.e., if all Delaunay polytopes are simplices, then the L-type is called primitive.
- The group $GL_n(\mathbb{Z})$ acts on PSD_n by arithmetic equivalence and preserves primitive L-type domains.
- After this action, there is a finite number of them.
- Bistellar flipping creates new triangulation. In dim. 2:

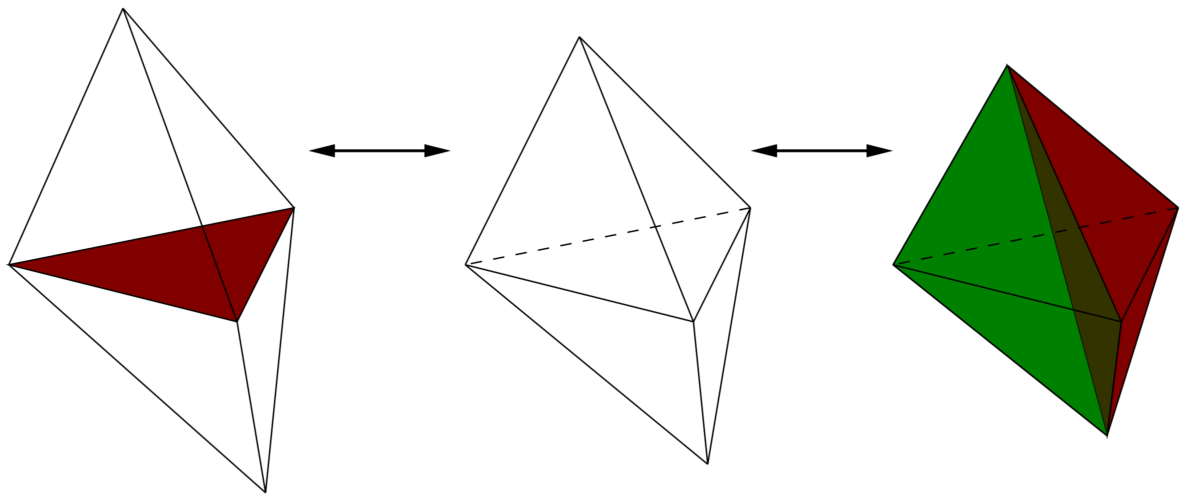


- Enumerating primitive L-types is done classically: find one primitive L-type domain, then adjacent ones and reduce by arithmetic equivalence.

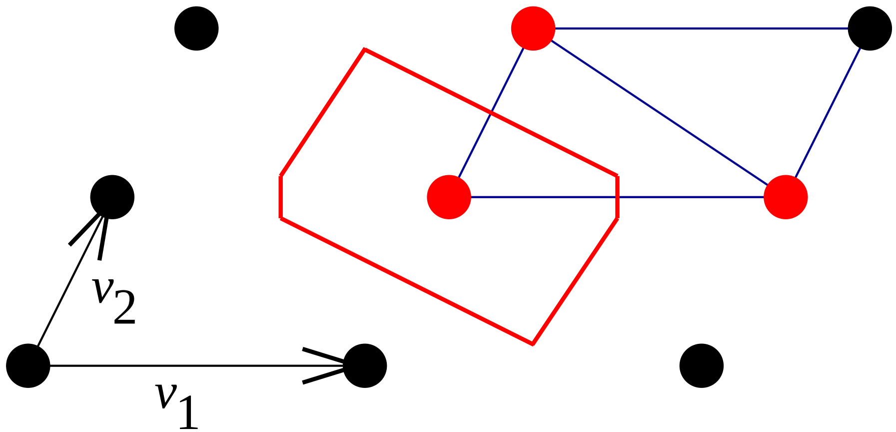
Flipping

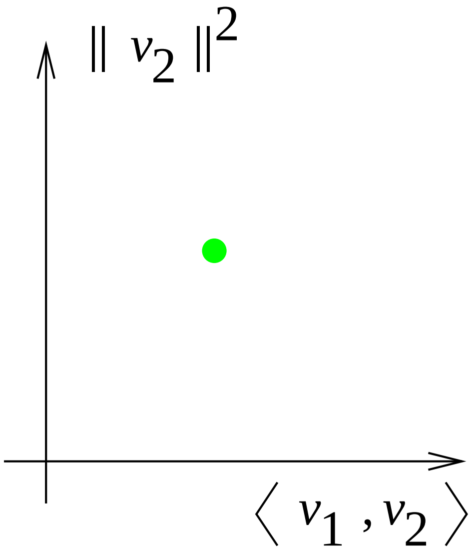
Given a primitive L-type domain, how to describe the structure of adjacent L-type domains?

- Its Delaunay tessellation consists of simplices.
- A facet of the L-type correspond to some Delaunay simplices merging into a Delaunay polytope with $n+2$ vertices, called repartitioning polytope.
- Polytopes with $n+2$ vertices admit exactly two triangulations, the other one yields the adjacent L-type.



Lattices in dimension 2





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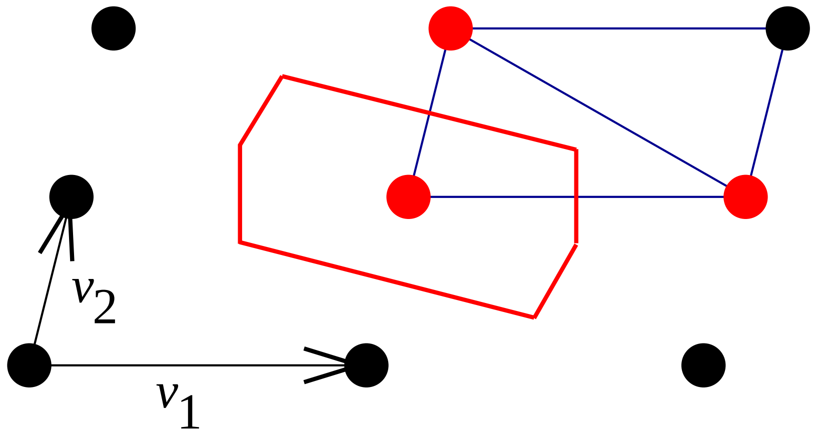
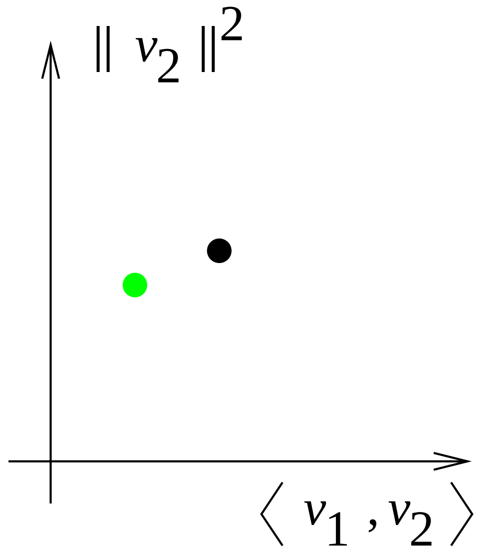
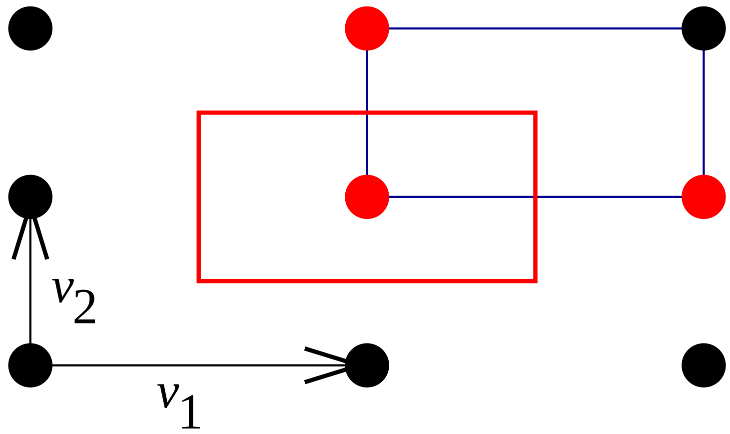
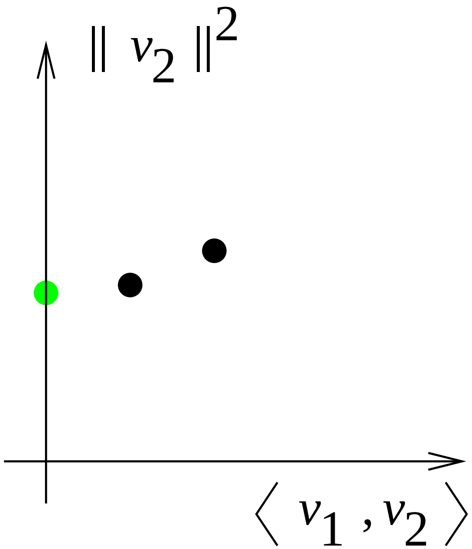


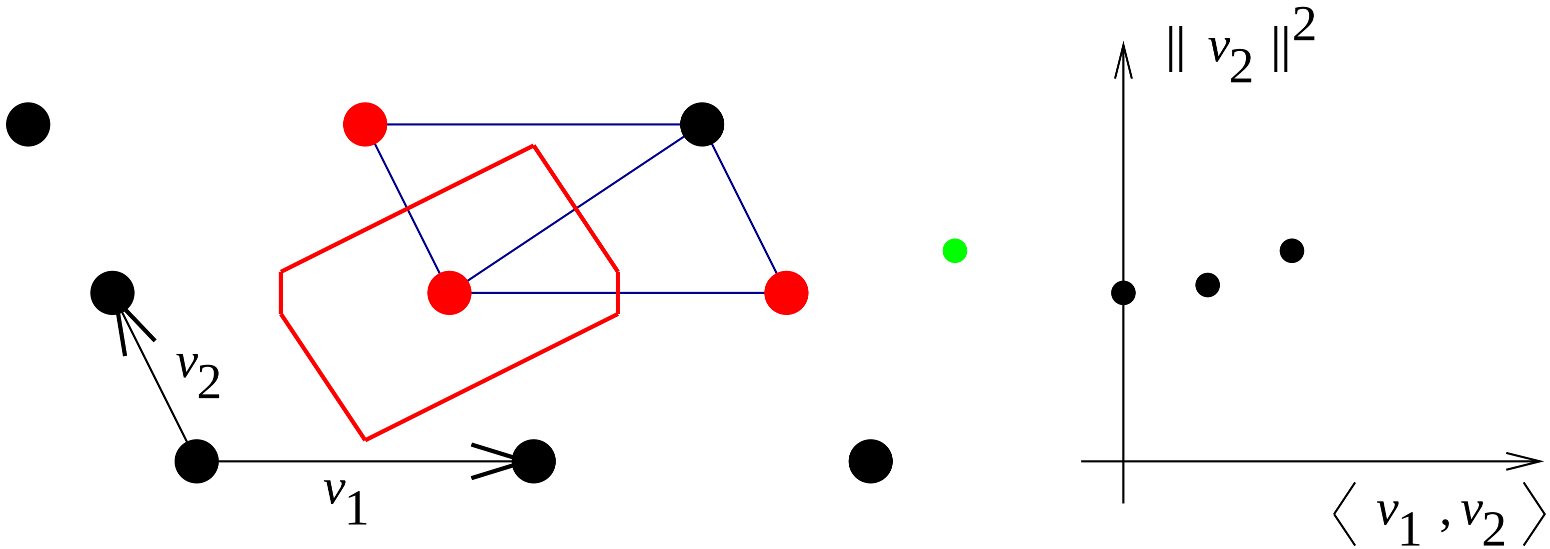


Lattices in dimension 2



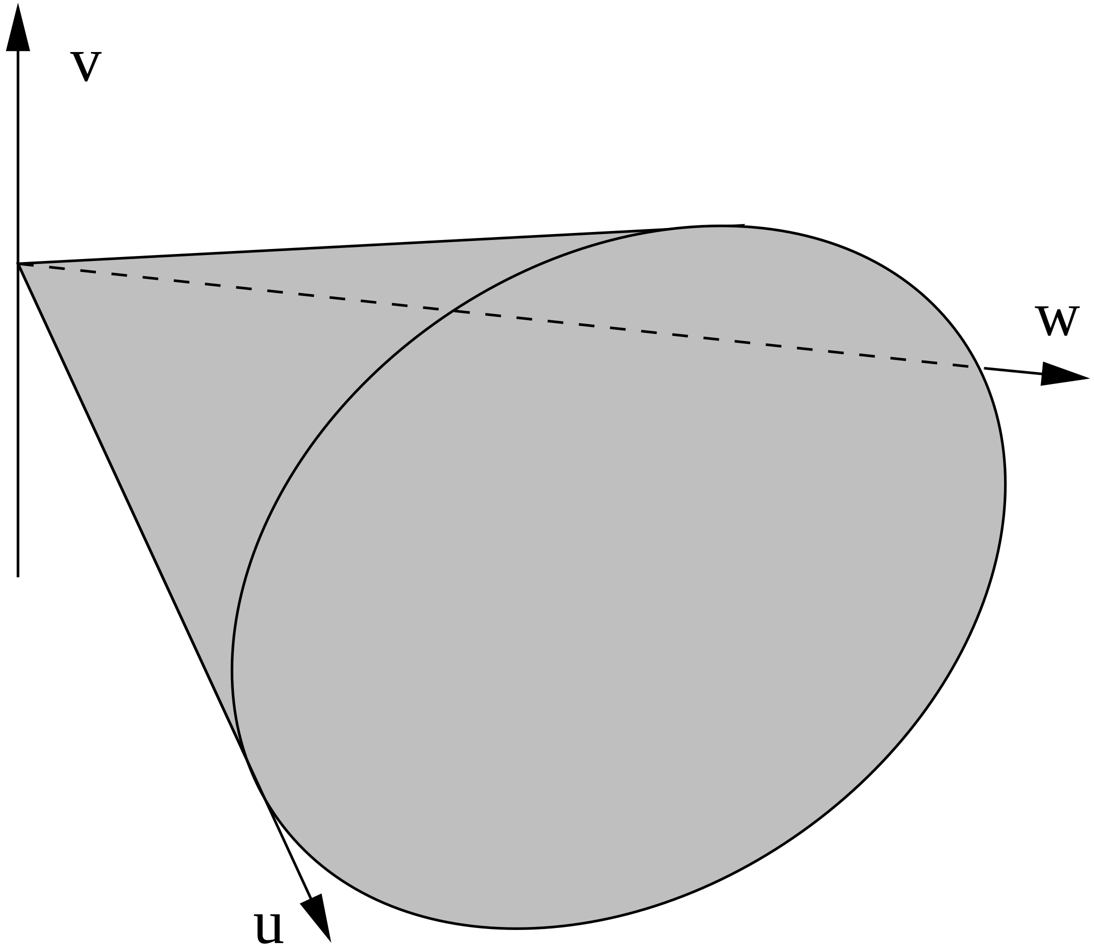


Lattices in dimension 2



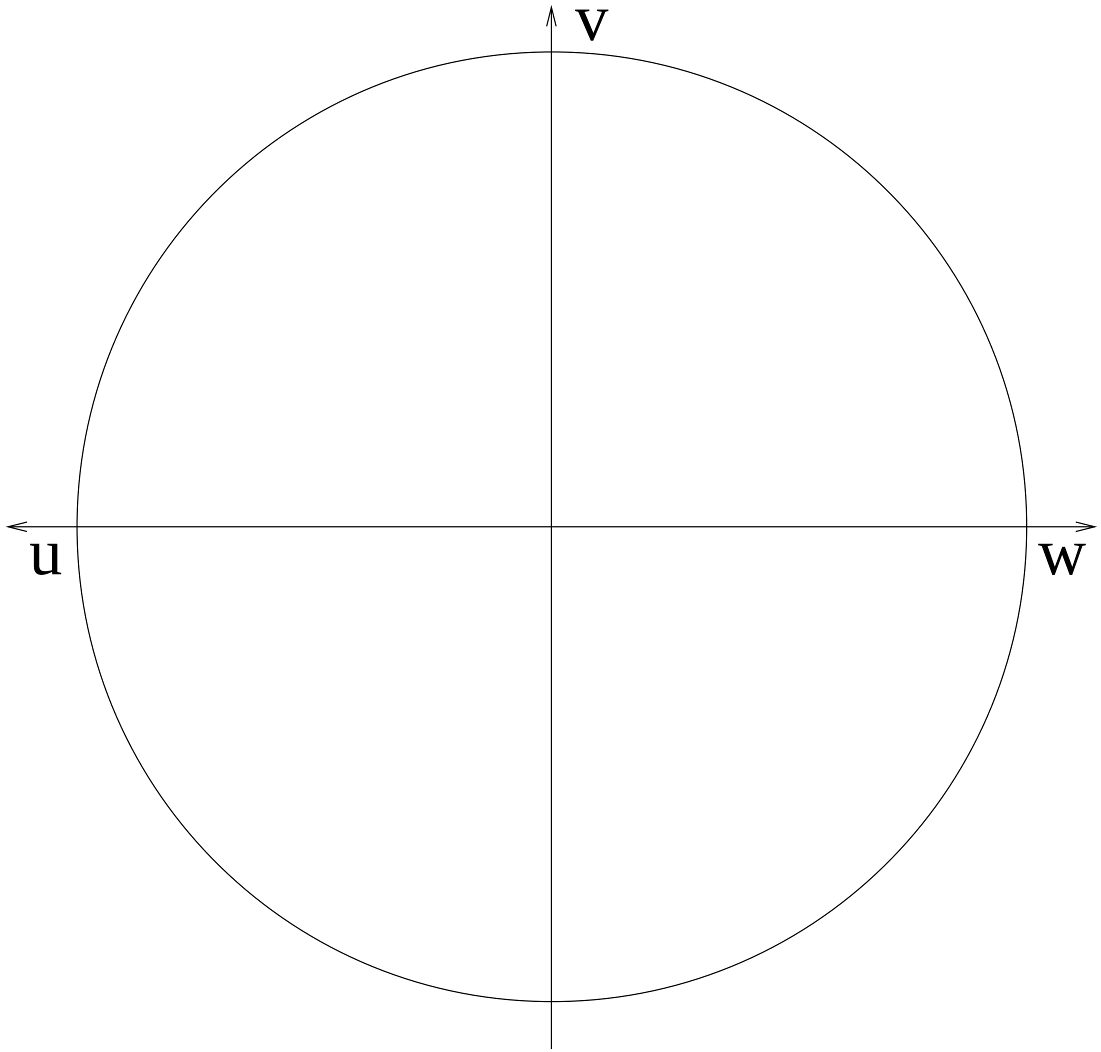
The partition of $PSD_2 \subset \mathbb{R}^3$

$\begin{pmatrix} \mathrm{u} & \mathrm{v} \\ \mathrm{v} & \mathrm{w} \end{pmatrix} \in PSD_2$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



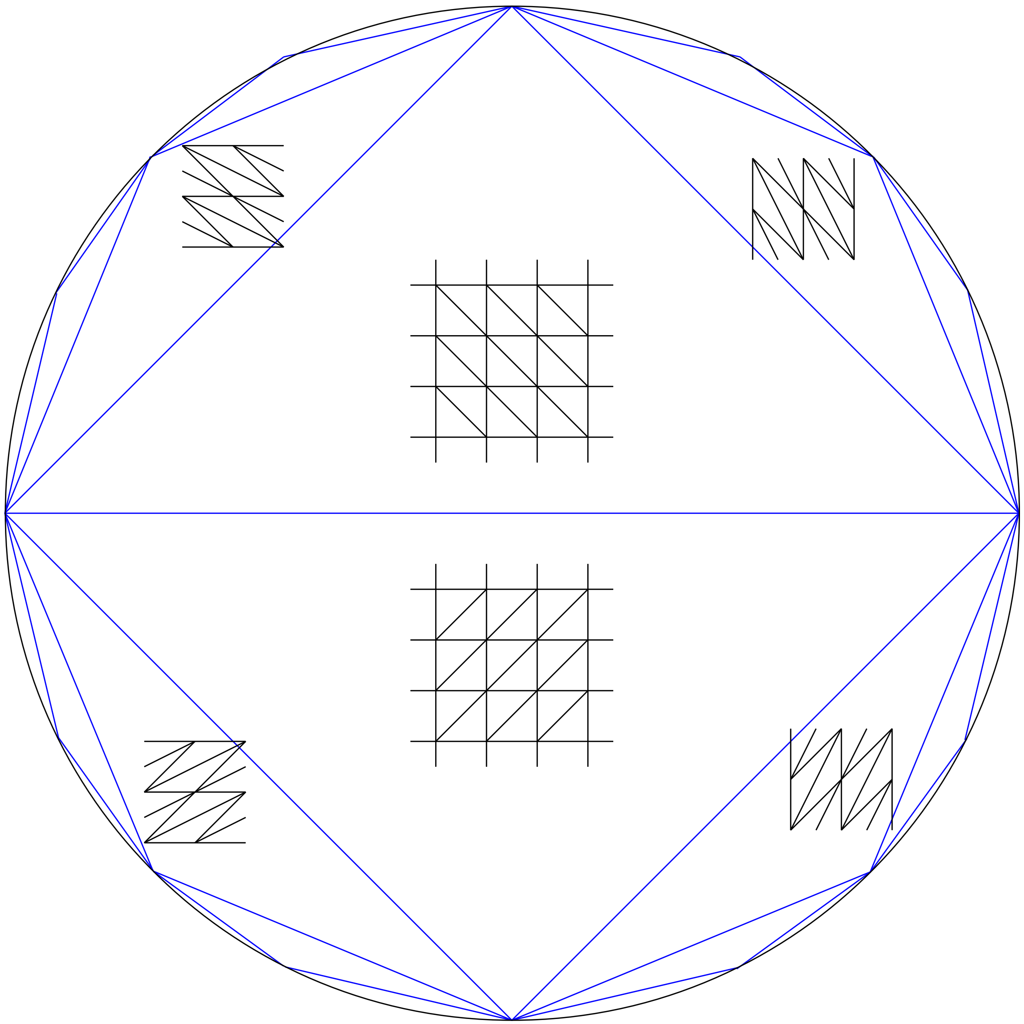
The partition of $PSD_2 \subset \mathbb{R}^3$

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



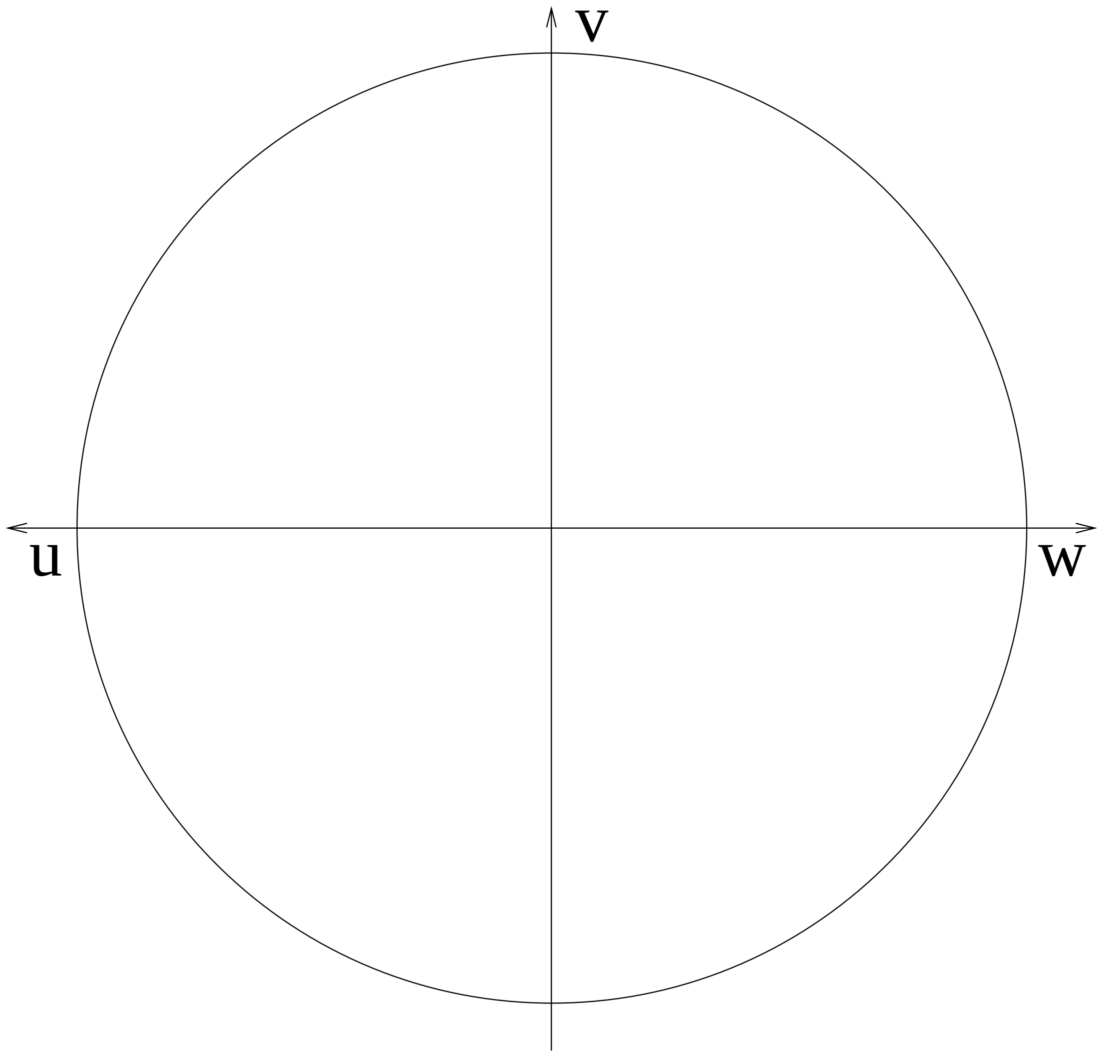
The partition of $PSD_2 \subset \mathbb{R}^3$

Simplicial (primitive) L-types are inside triangles, while on lines, Delaunay partition is the square lattice (special L-type):



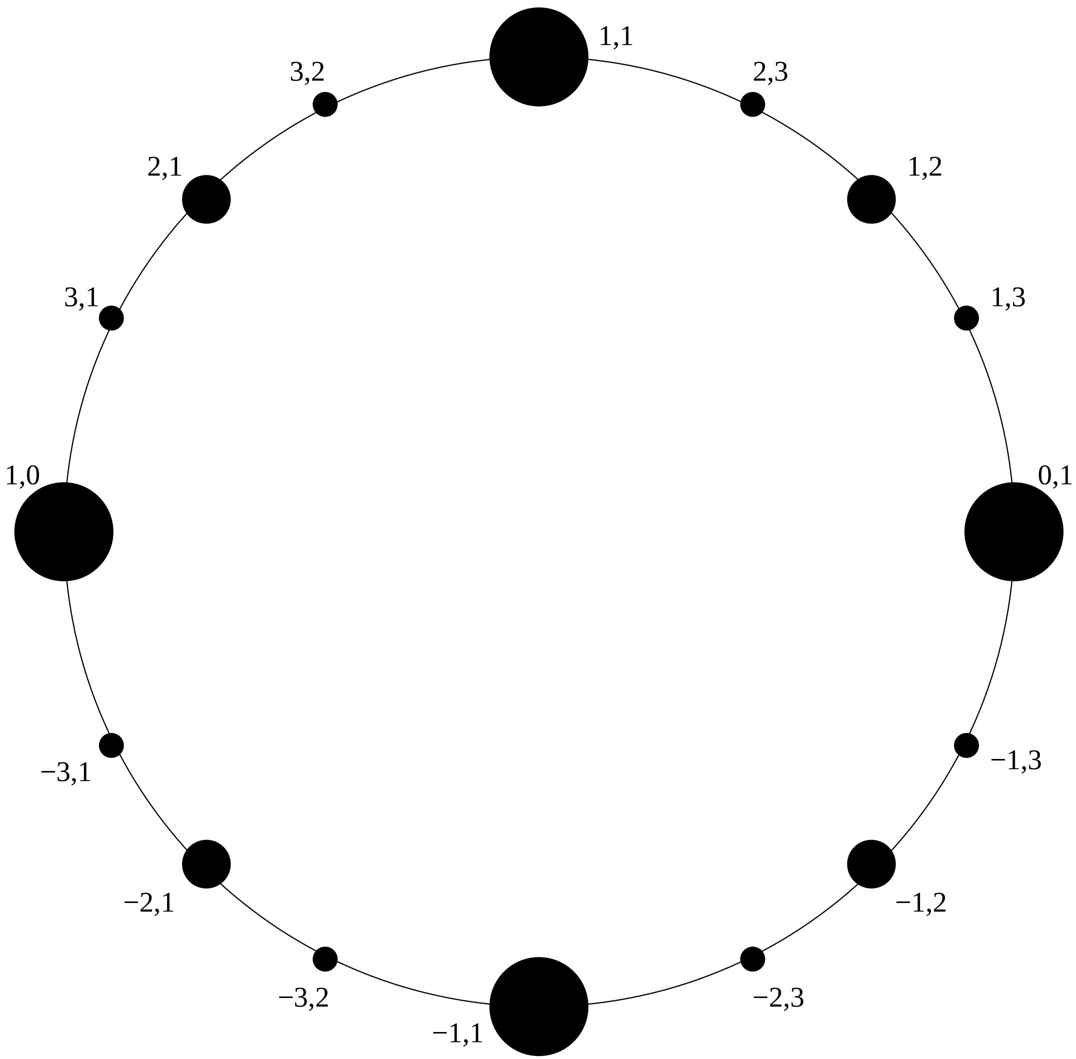
The partition of $PSD_2 \subset \mathbb{R}^3$

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$, then $q \in PSD_2$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$; we cut by the plane $\mathrm{u} + \mathrm{w} = 1$.



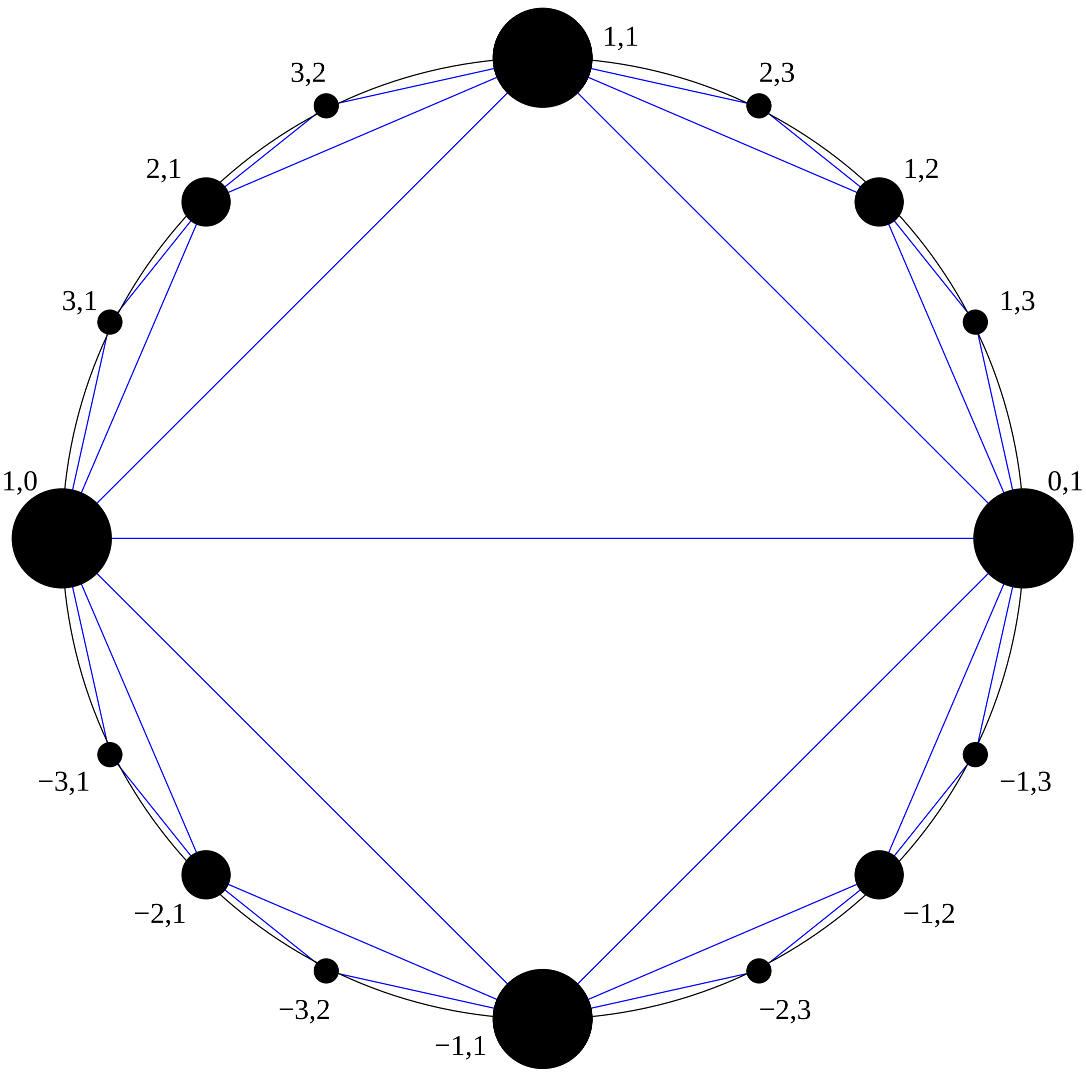
The partition of $PSD_2 \subset \mathbb{R}^3$

The group $GL_2(\mathbb{Z})$ transforms the limit form x^2 into the forms $(ax+by)^2$ with $a, b \in \mathbb{Z}$. Only 1-dim. Voronoi polytopes.



The partition of $PSD_2 \subset \mathbb{R}^3$

Inside triangles: Voronoi polytope is hexagonal (primitive).
On lines: Voronoi polytope is rectangular (special).



Enumeration of primitive, rigid L-types

Dimension	Nr. Voronoi polytopes	Nr. of primitive	Nr of rigid
1	1	1	1
2	2	1	0
3	5 Fedorov	1 Fedorov	0
4	52 Delaunay-Shtogrin	3 Delaunay	1
5	179377 Engel	222 BaRy, Engel	7 $\uparrow$ BaGr
6	?	$\geq 2.5.10^6$ Engel, Va	$\geq 2.10^4$ DuVa
7	?	?	?

Rigid lattices (edge forms)

- All rigid lattices in dimensions $1, 2, 3, 4$: $\mathbb{Z}_1$ and $D_4 = D_4^*$.
- Also rigid: E_6, E_6^*, E_7, E_7^*, $E_8 = E_8^*$, as well as $D_n, n > 4,$ and $D_{2m}^*, m > 2$.
- There are 7 rigid lattices in dimension 5 and ≥ 25263 in dimension 6.
- $\mathbb{Z}$, E_6 and E_7 are first instances of strongly rigid lattices (having extreme one among its Delaunay polytopes).
- A Delaunay is extreme if the lattice, containing it, is unique, i.e., the combinatorics determines the structure.
- Erdahl: a Voronoi polytope $\mathcal{V}$ is a zonotope if and only if all edge forms of the closure of its L-type domain are matrices of rank 1, i.e., of the form $a^T a$; then zonotope $\mathcal{V}$ is the Minkowski sum of such scaled vectors a.

Non-rigidity degree of a lattice

Denote by $nrd(L)$ (non-rigidity degree of lattice L) the dimension of L-type domain to which belongs its quadratic form. It is the number of degrees of freedom, under affine deformation, of (the L-star of) Delaunay partition.

- $1 \leq nrd(L) \leq \binom{n+1}{2}$ with equalities, respectively, iff L is rigid and iff L is primitive (only simplices).
- $nrd(L) \leq \min rank(P)$ over its Delaunay polytopes P.
  $nrd(\mathbb{Z}_n) = n = rank(n\text{-}cube)$ ($n \geq 1$);
  $nrd(A_n) = n + 1 = rank(J(n+1, k), 2 \leq k \leq n)$ ($n \geq 2$);
  $nrd(A_n^*) = \binom{n+1}{2} = rank(n\text{-}simplex\ J(n+1, 1))$ ($n \geq 2$).
- Remaining irreducible root lattices and their duals are either rigid, or $nrd(D^*_{2m+1}) = 2m + 1$ ($m \geq 2$).
- $nrd(L + L')$=$nrdL + nrdL'$; $rank(P \times P')$=$rankP + rankP'$

Non-rigidity degree of a lattice

The lattice vector between any 2 vertices of a Delaunay polytope of L is a minimal vector of a coset of $L/2L$. For an edge, the coset contains, up to sign, unique minimal vector.

- $nrd(L) = \binom{n+1}{2} - rank S(L)$, where $S(L)$ is the system of equations defining the norms of minimal vectors of cosets $L/2L$, i.e., of all minimal affine dependencies of vertices of all non-equiv. Delaunay polytopes in L-tiling.
- rank(P) is the dimension of minimal face of HYP_{n+1} containing generating set of Delaunay polytope P. It is the topological dimension of the set of affine bijections T of $\mathbb{R}^n$ (up to translations and orthogonal transformations) such that $T(P)$ is again Delaunay.
- It holds: $1 \leq nrd(L) \leq \min_P rank(P) \leq \binom{n+1}{2}$.

III. Delaunay polytopes

and

hypermetrics

Hypermetric inequalities

- If $b \in \mathbb{Z}^{n+1}$, $\sum_{i=0}^{n} b_i = 1$, then hypermetric inequality is

$$H(b)d = \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0 \ .$$

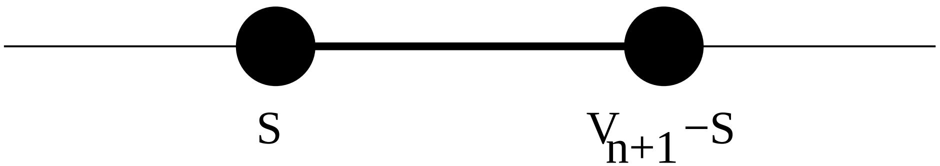
- If $b = (1, 1, -1, 0, \dots, 0)$, then $H(b)$ is triangle inequality.
- The hypermetric cone HYP_{n+1} is the set of all d such that $H(b)d \le 0$ for all b.
- $dim\, HYP_{n+1} = \binom{n+1}{2}$.
- HYP_{n+1} is defined by an infinite set of inequalities, but it is polyhedral (Deza-Grishukhin-Laurent).

Cut cone

The cut semi-metric on $X = \{0, \ldots, n\}$, for any $S \subset X$, is

$$\delta_S(i,j) = \begin{cases} 1 & \text{if } |S \cap \{i,j\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

It can be seen as squared distance on the 1-dim. Delaunay polytope $\alpha_1 = [0,1]$ which is extreme Delaunay of lattice $\mathbb{Z}$.



Denote by CUT_{n+1} the cone generated by all ($2^n - 1$) δ_S.

- $CUT_{n+1} \subset HYP_{n+1}$ for all n and $= HYP_{n+1}$ iff $n \leq 5$. So, other extreme Delaunays appear only from $n \geq 6$.
- $HYP_{n+1} \subset MET_{n+1}$ for all n and $= MET_{n+1}$ iff $n \leq 3$.

Digression on l_p-metrics

$$\{d : ((d_{ij}^2)) \in NEG_n\} \subset CUT_n \subset HYP_n \subset (NEG_n \cap MET_n)$$

iff $d \to l_2^{n-1}$ iff $d \to l_1^m$ iff $d \to l_\infty^{n-2}$

- Given a metric $d = ((d_{ij}))$ on n points, $d \to l_p^m$ means that it is a metric subspace of $\mathbb{R}^m$ with norm l_p, i.e. $d_{ij} = ||\vec{v_i} - \vec{v_j}||_p$ for some $\vec{v_i}, \dots, \vec{v_n} \in \mathbb{R}^m$.
- NEG_n=$\{d : \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0\}$ if $b \in \mathbb{R}^n$, $\sum_0^n b_i = 0$
  PSD_{n-1}=$\{a = ((a_{ij} = \frac{1}{2}(d_{1i} + d_{1j} - d_{ij}))) : d \in NEG_n\}$.
- $d \in HYP_n$ iff $\sqrt{d} \to S^{n-2}$ and it is a generating simplex of a Delaunay polytope of a lattice in $\mathbb{R}^{n-1}$.
- $l_2 \to$ any $l_{p(p \ge 1)} \to l_\infty$ and $l_{p(1 \le p \le 2)} \to l_1$.
- Unit balls of l_p^m is cube γ_m, its dual β_m, S^{m-1}, smooth, for $p = \infty, 1, 2$, any $1 < p < \infty$.

Facets of HYP_7 and CUT_7

HYP_7 has 3773 facets in 14 orbits below.
It has 31170 extreme rays in 29 orbits:
3 of cut semi-metrics and 26 from 7-subsets of 27-vertex-set of extreme Schlafli 6-polytope of the root lattice E_6.

(1 ,1 ,-1 ,0 ,0 ,0 ,0)	(1 ,1 ,1 ,-1 ,-1 ,0 ,0)
(1 ,1 ,1 ,1 ,-1 ,-2 ,0)	(2 ,1 ,1 ,-1 ,-1 ,-1 ,0)
(1 ,1 ,1 ,1 ,-1 ,-1 ,-1)	(2 ,2 ,1 ,-1 ,-1 ,-1 ,-1)
(1 ,1 ,1 ,1 ,1 ,-2 ,-2)	(2 ,1 ,1 ,1 ,-1 ,-1 ,-2)
(3 ,1 ,1 ,-1 ,-1 ,-1 ,-1)	(1 ,1 ,1 ,1 ,1 ,-1 ,-3)
(2 ,2 ,1 ,1 ,-1 ,-1 ,-3)	(3 ,1 ,1 ,1 ,-1 ,-2 ,-2)
(3 ,2 ,1 ,-1 ,-1 ,-1 ,-2)	(2 ,1 ,1 ,1 ,1 ,-2 ,-3)

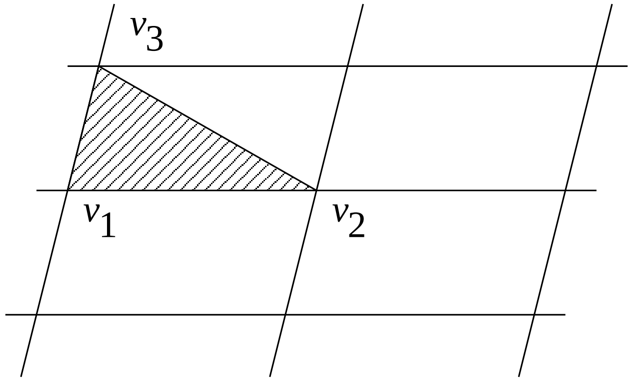
First 10 orbits above are also of facets of CUT_7.
It has 36 orbits of facets, 26 of which are non-hypermetric.

Delaunay polytopes → hypermetrics

If $\mathcal{D}$ is an n-dimensional Delaunay polytope with center c, radius r and vertices $\{v_0, \dots, v_N\}$, then $d(i,j) = \|v_i - v_j\|^2$ satisfies, for any $b \in \mathbb{Z}^{N+1}$ with $\sum_{i=0}^{N} b_i = 1$,

$$H(b)d := \sum_{i,j} b_i b_j d(i,j) = 2(r^2 - \|\sum_i b_i v_i - c\|^2) \leq 0,$$

i.e., distance $d(i,j)$ is a hypermetric on $X = \{0, 1, \dots, N\}$.
Moreover, $\sum_i b_i v_i$ is a vertex of $\mathcal{D}$ if and only if $H(b)d = 0$.



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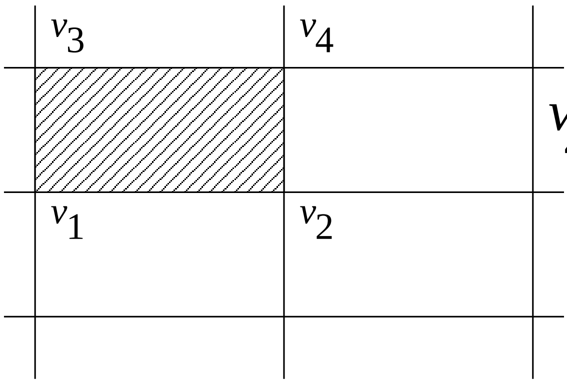


$v_4 = v_2 + v_3 - v_1$

Delaunay polytopes $\Leftrightarrow$ hypermetrics

- Let $||v_i - c|| = ||c||$ for $c, v_0 = 0, v_1, \dots, v_N \in \mathbb{R}^n$, and $||\sum_i b_i v_i - c|| \geq ||c||$ for all $b \in \mathbb{Z}^N$. Then the set $L := \mathbb{Z}(v_1, \dots, v_N)$ is a lattice.
- A distance d on $X = \{0, 1, \dots, N\}$ is a hypermetric if and only if (X, d) has a representation $i \in X \to v_i \in \mathbb{R}^k$, $k \leq N$, on sphere S, which is empty (not containing its elements in the interior) for the set $L_{af} := \{\sum_i b_i v_i : b \in \mathbb{Z}^X, \sum_{i=0}^N b_i = 1\}$. The elements of L_{af} on this sphere, generate a lattice (root lattice iff hypermetric d is graphic, i.e., $d_{path}(G)$). They form an affine basis of its Delaunay polytope.
- D.-Terwilliger: $d_{path}(G) \in HYP_n$ iff $2d$ is an isometric subspace of a direct product $\frac{1}{2}H_m \times K_{m \times 2} \times G_{56}$ for m>6. Shpectorov; D., Grishukhin: $d_{path}(G) \in CUT_n$ iff no G_{56}.

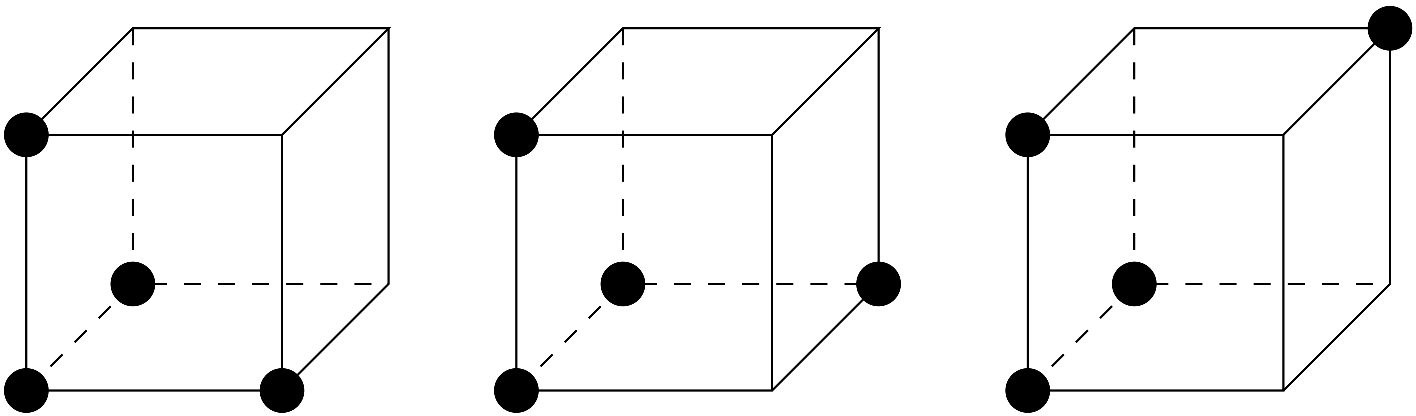
Radius of Delaunay polytopes

- D.-Grishukhin, 1993: Let (X, d) be a hypermetric space and P_d be its associated Delaunay polytope; let r be the radius of the sphere circumscribing P_d. If $\sum_{i \in X} d(i, j)$ does not depend on $j \in X$, then $r^2 = \frac{1}{2|X|} \sum_{j \in X} d(i, j)$.
- D.-Grishukhin, 1996:
  Let L be a n-dimensional lattice in $\mathbb{R}^n$ with covering radius (maximum radius of a Delaunay polytope) $\rho(L)$.
  Let R denote the maximum radius of a symmetric Delaunay polytope of L (setting $R = 0$ if none exists).
  Let r denote the maximum radius of a proper symmetric face of a Delaunay polytope of L.
  Then $\rho(L) = R$ if $R \geq \frac{2r}{\sqrt{3}}$ and, otherwise, $R \leq \rho(L) \leq \frac{2r}{\sqrt{3}}$.

Affine basis

Any Delaunay has $n+1$ affinely independent vertices. $\{v_0, \dots, v_n\}$ is affine basis of an n-dimensional polytope P if, for every vertex v of P, there is $\{b_i\} \in \mathbb{Z}^{n+1}$ with

$$b_0 + \dots + b_n = 1 \text{ and } b_0 v_0 + b_1 v_1 + \dots + b_n v_n = v \; .$$



Baranovski & Ryshkov: every Delaunay polytope of dimension ≤ 6 has an affine basis. Dutour-Grishukhin: contre-example (12-dimensional 14-vertex polytope).

Polyhedrality of HYP_n

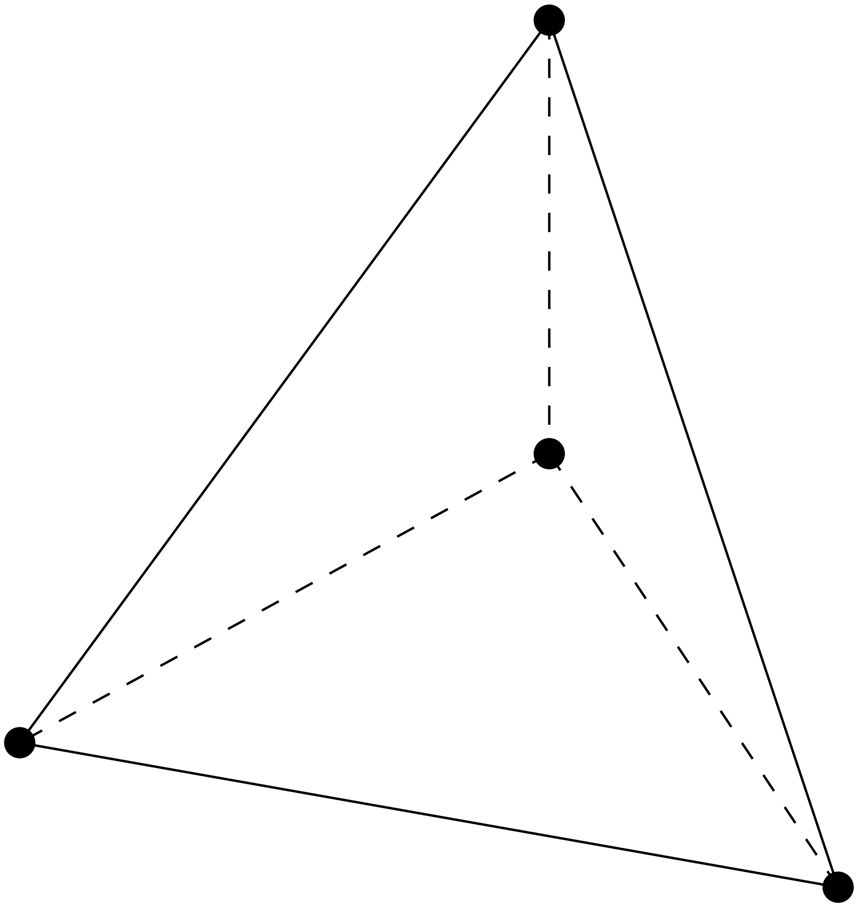
- DGL: HYP_n is polyhedral as union of L-type domains.
- HYP_8 has ≥ 294.056 (84 orbits) of facets.
  It has $\geq 7.126.560$ (374 orbits) of extreme rays (all generating simplexes of the Gosset polytope in E_7) including 55 orbits of graphic hypermetrics.
- Lovasz: if $H(b)$ defines a facet, then $|b_i| \leq \frac{2^n}{\binom{2n}{n}} n!$.
- To find all faces of HYP_n implies to find all Delaunay polytopes of dimension $\leq n-1$.

Rank of Delaunay polytope

The $rank(P)$ is dimension of the face F with $P \in F$; it is the number of degrees of freedom (parameters) of affine deformation preserving it as Delaunay polytope.

- if $rank\,(P) = \binom{n+1}{2}$, then P is a simplex α_n.
- if $rank\,(P) = \binom{n+1}{2} - 1$, then $P = Pyr(\alpha_p \cup \alpha_q)$ (*repartitioning polytope* in Voronoi terms).
- if $rank\,(P) = 1$, then P is an *extreme Delaunay polytope* (one degree of freedom: homotheties and rotations).
- Example: rank of α_n, β_n, γ_n, $\frac{1}{2}\gamma_{n>4}$, $J(n+1, k>1)$ is, respectively, $\binom{n+1}{2}$, $\binom{n}{2} + 1$, n, n, $n+1$.

3-dimensional case

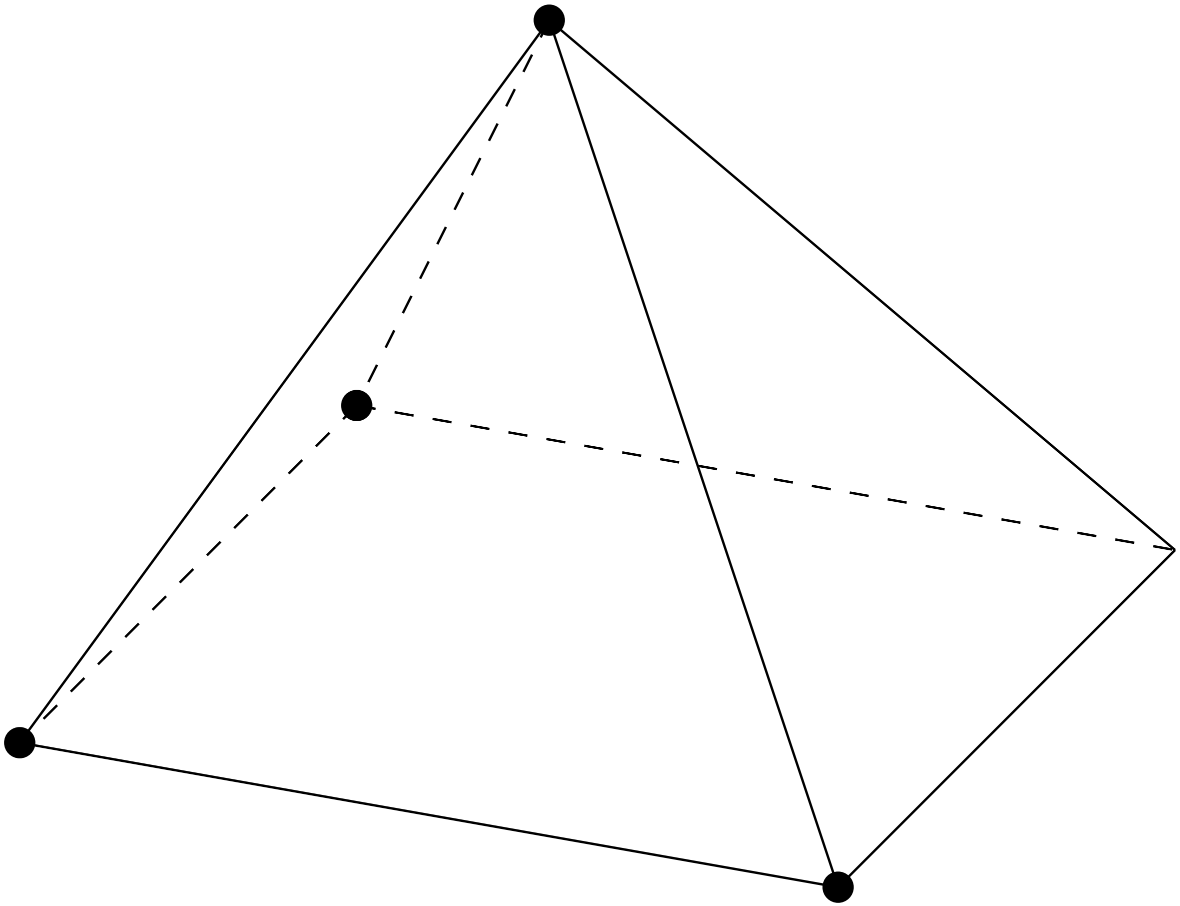


3−simplex

Hypermetric Vectors

Rank: 6

3-dimensional case



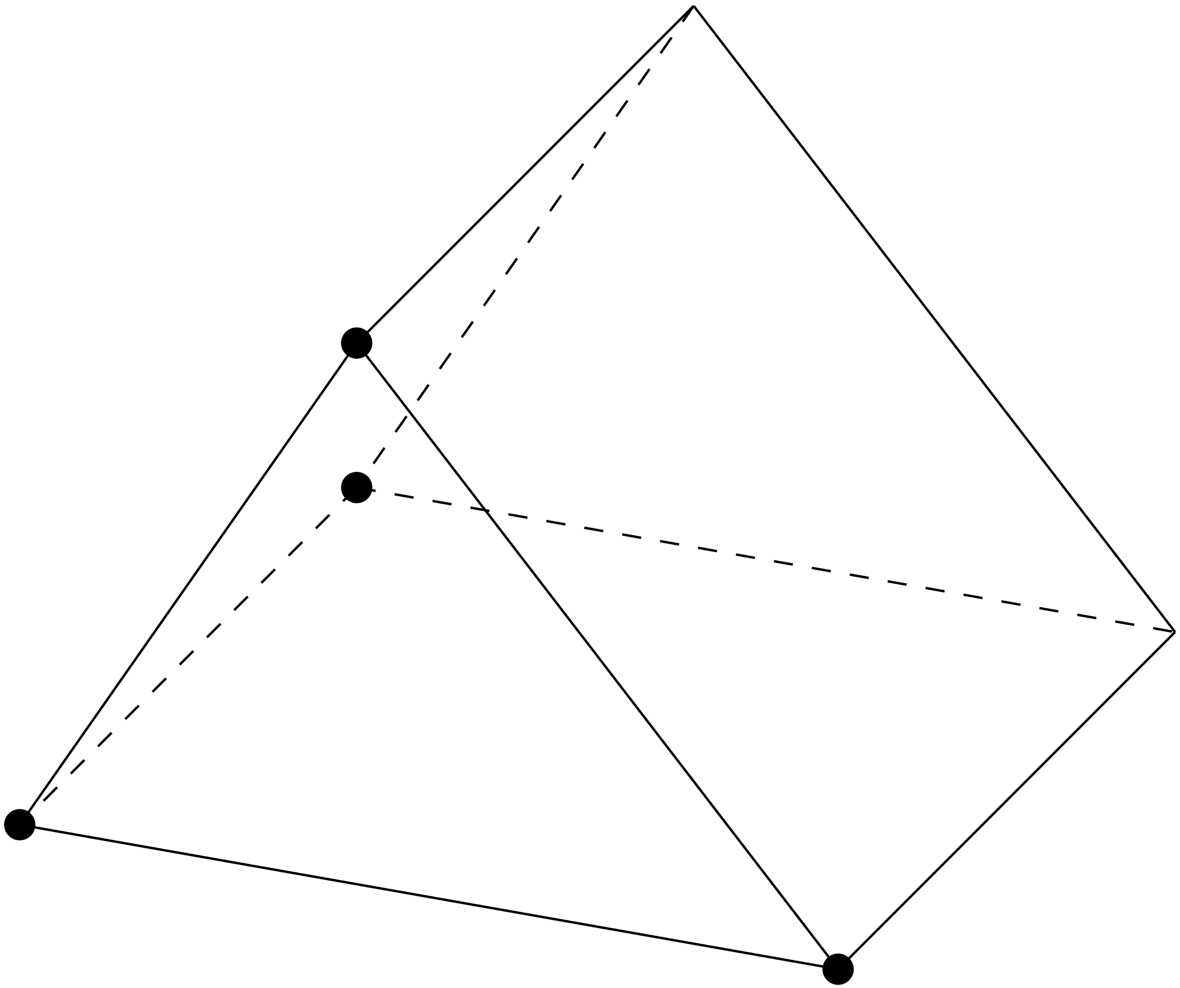
Pyramid

Hypermetric Vectors

$(-1\,,\,0\,,\,1\,,\,1)$

Rank: 5

3-dimensional case



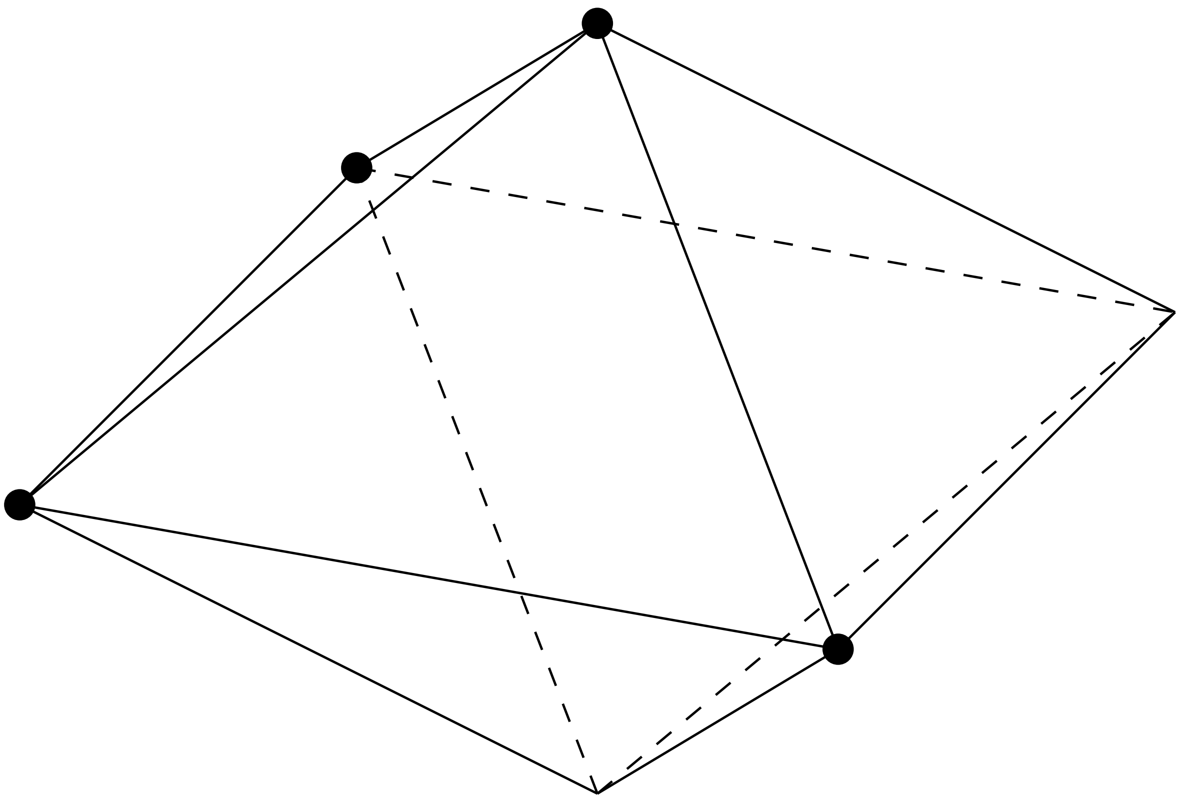
3−Prism

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(-1\ ,\ 1\ ,\ 0\ ,\ 1)$

Rank: 4

3-dimensional case



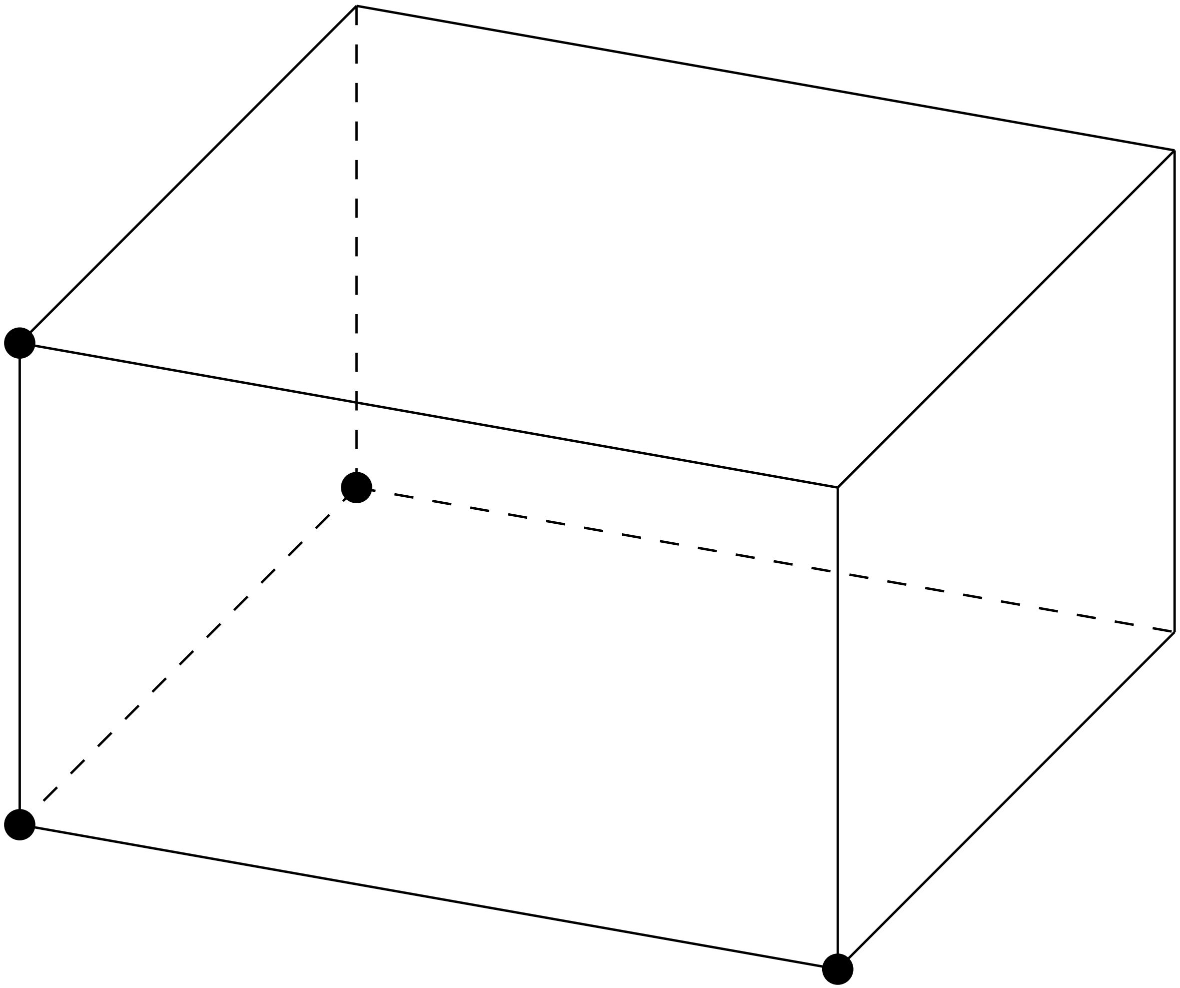
Octahedron

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(\ 0\ ,-1\ ,\ 1,\ 1)$

Rank: 4

3-dimensional case



Cube

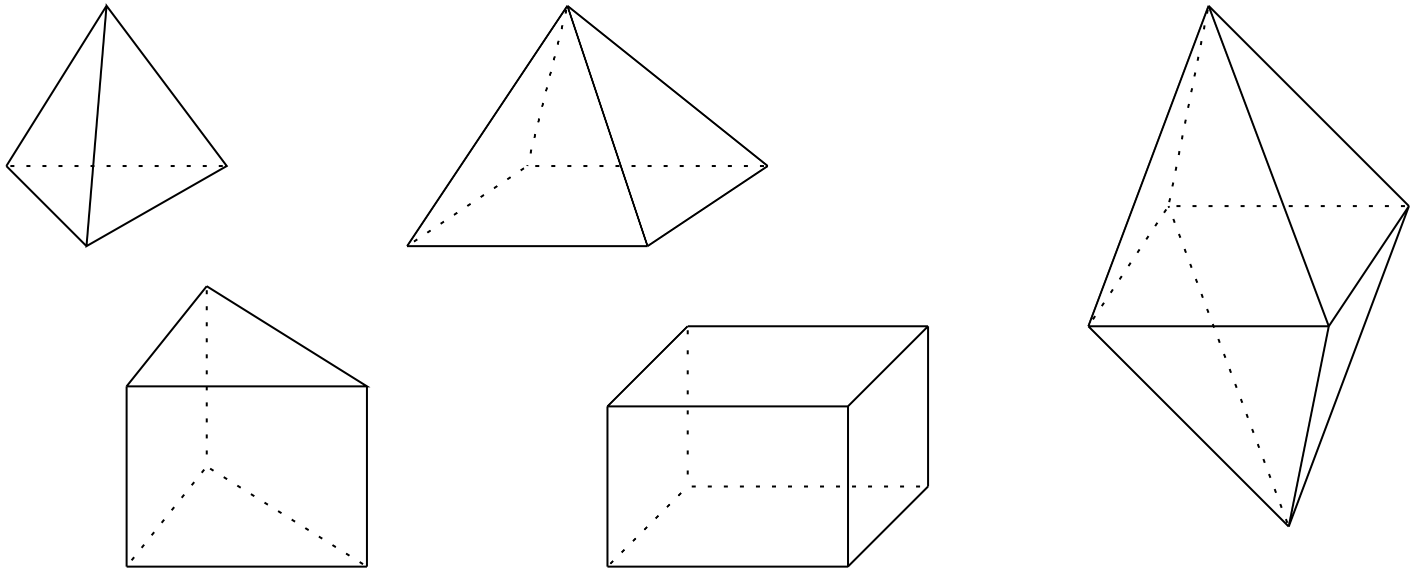
Hypermetric Vectors

(−1 , 0 , 1 , 1)
(−1 , 1 , 0 , 1)
(−1 , 1 , 1 , 0)
(−2 , 1 , 1 , 1)

Rank: 3

H(−2, 1, 1, 1)=H(−1, 0, 1, 1)+H(−1, 1, 0, 1)+H(−1, 1, 1, 0)

Comb. types of Delaunay 3-polytopes



Dim.	Nr. of types	Authors	Computing time
2	2	Fedorov (1885)	
3	5	Fedorov (1885)	23s
4	19	Erdahl & Ryshkov (1987)	52s
5	138	Kononenko (1997)	5m
6	6241	Dutour (2002)	50h

Maximal volume of Delaunay polytope

- $det(L)$ is the volume of parallelepiped on basis of L. $\frac{det(L)}{n!}$ is the fundamental volume (of simplex on basis).
- The volume of any Delaunay n-polytope is its integral multiple, say, α. For any α, there is a lattice in $\mathbb{R}^{2\alpha+1}$ with Delaunay simplex of relative volume α.
- Any Delaunay n-simplex with $\alpha > 1$ does not contain a basis of L; so, it generates only proper sublattice of L.
- Conjecture: $\max \alpha = n - 3$ for Delaunay n-simplex; proved for $n = 4, 5, 6$ by Voronoi, Baranovski, Ryshkov.
- Santos, Schürmann & Vallentin: $\max \alpha \geq 1.5^n$ if $24 \mid n$. Lovasz: $\max \alpha \leq \frac{2^n}{\binom{2n}{n}} n!$; remind his bound $|b_i| \leq \frac{2^n}{\binom{2n}{n}} n!$
- Sph. packing density $\leq \frac{v_n}{\kappa_n}$, where κ_n, v_n are volume of unit ball and minimal volume of Voronoi polytope.

Number of vertices of Delaunay polytope

- Every incidence $H(b)d = 0$ corresponds to a vertex $b_0v_0 + \cdots + b_nv_n$ of a Delaunay n-polytope P.
- The number N of vertices satisfies $n+1 \le N \le 2^n$ (with equalities for n-simplex and n-cube) and:

$$rank(P) \ge \binom{n+2}{2} - N \quad for \quad any \quad Delaunay \quad and$$
$$rank(P) \ge \binom{n+1}{2} - \frac{N}{2} + 1 \quad for \quad centr. \quad symmetric \quad ones.$$

- So, $N \ge \binom{n+2}{2} - 1$ for extreme Delaunay polytopes and $N \ge 2\binom{n+1}{2}$ if, moreover, polytope is centrally symmetric
- If equality in above bounds (as for Erdahl-Rybnikov polytopes), then the adjacency computation for the corresponding extreme ray of HYP_{n+1} is easy.

IV. Extreme Delaunay polytopes

Extreme Delaunay polytopes

- The interval $[0,1]$ is the only extreme Delaunay polytope in dimension $n \leq 5$, since $HYP_n = CUT_n$ if $n \leq 6$.
- Deza & Dutour: there is an unique extreme Delaunay polytope in dimension 6 (the Schläfli polytope 2_{21}).
- Deza, Grishukhin & Laurent found 6 extreme polytopes:

Name	Dimension	Nr. vertices	Equality	Section of
Schläfli	6	27	yes	E_8
Gosset	7	56	no	E_8
	16	512	no	Barnes-Wall
B_{15}	15	135	yes	Barnes-Wall
	22	275	yes	Leech
	23	552	no	Leech

The Schläfli polytope

Root lattices E_6 and E_8:

$$E_6 = \{x \in E_8 : x_1 + x_2 = x_3 + \cdots + x_8 = 0\},$$
$$E_8 = \{x \in \mathbb{Z}^8 \cup (\tfrac{1}{2} + \mathbb{Z})^8 \text{ and } \textstyle\sum_i x_i \in 2\mathbb{Z}\}.$$

E_6 has unique Delaunay polytope called Schläfli polytope; its skeleton is the (strongly regular) Schläfli graph.

- Schläfli polytope has 27 vertices.
- Symmetry group has size 51840, transitive on vertices.
- Schläfli polytope is extreme Delaunay polytope.
- Deza-Grishukhin-Laurent: it has 26 orbits of affine bases, which gives 26 orbits of extreme rays in HYP_7.

Computing methods

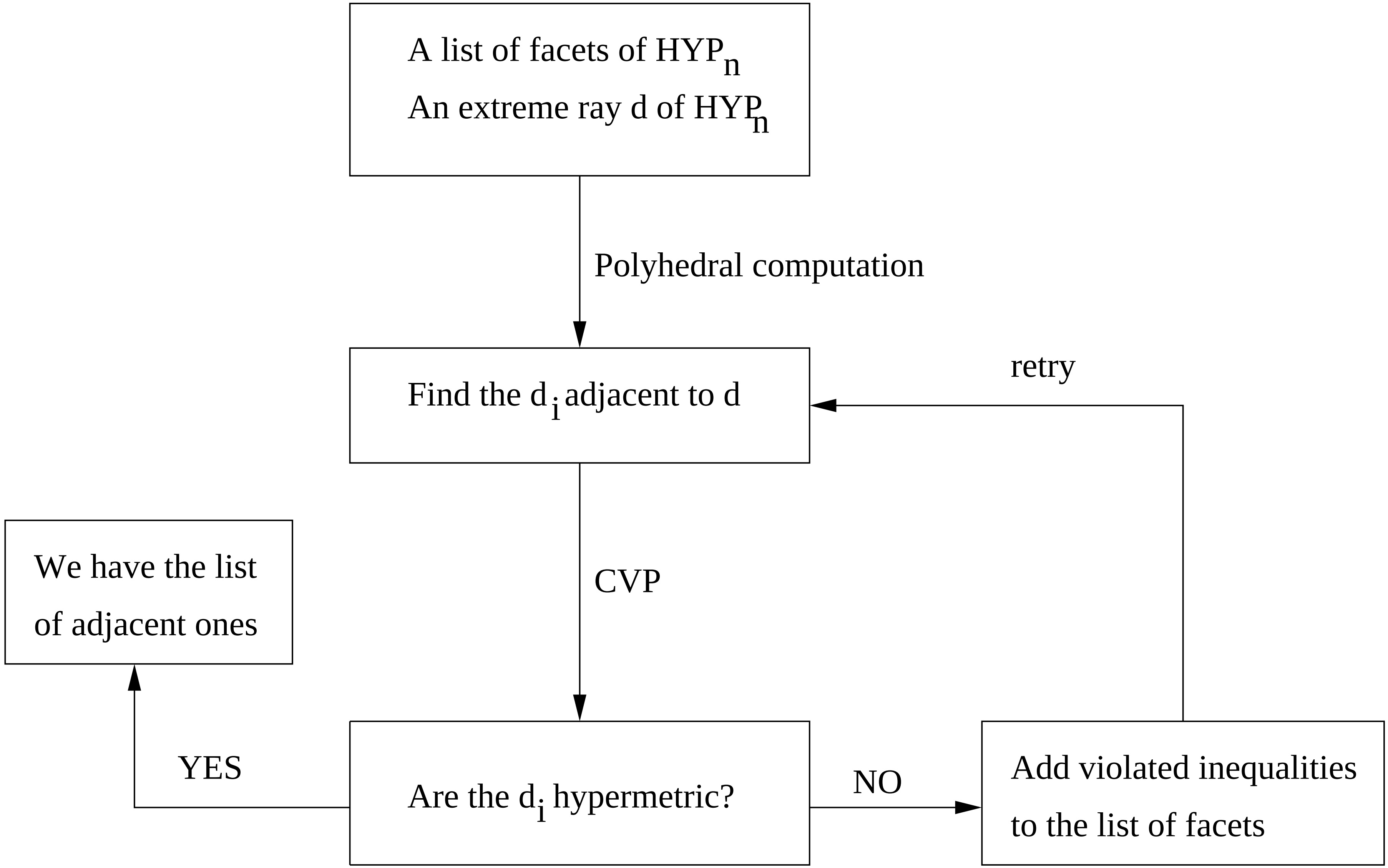
Given a distance vector $d_{ij} = \|v_i - v_j\|^2$,

- one can compute the Gram matrix $\langle (v_i - v_0), (v_j - v_0) \rangle$,
- test if d is non-degenerate,
- compute the sphere $S(c, r)$ incident to v_i.
- $d \in HYP_{n+1}$ if and only if there is no b with

$$\|b_0 v_0 + \cdots + b_n v_n - c\| < r$$

  (i.e., Closest Vector Problem).
- Find b, such that $H(b)d = 0$, is also a CVP.

Bounding method



8-dimensional extreme Delaunay

Delaunay B_{15} satisfies equality in $N \geq \binom{n+2}{2} - 1 = 135$.
Dutour: its adjacent extreme rays correspond to an extreme Delaunay 8-polytope Du_8 with f-vector:

$$(79, 1268, 7896, 23520, 36456, 29876, 11364, 1131)$$

Its symmetry group has size 322560, not vertex-transitive.
There are three orbits of vertices:

- a vertex
- 64-vertices: the 7-half-cube
- 14-vertices: the 7-cross-polytope

Dutour series of extreme Delaunays

- If n even, $n \geq 6$, there is an asymmetric extreme Delaunay Du_n formed with 3 layers of lattice D_{n-1}:
  - a vertex
  - the $(n-1)$-half-cube
  - the $(n-1)$-cross-polytope

  $n = 6$: Schläfli polytope; $n = 8$: Du_8
- If n odd, $n \geq 7$, there is a centrally symmetric extreme Delaunay Du_n formed with 4 layers of lattice D_{n-1}:
  - a vertex
  - the Du_{n-1} extreme Delaunay
  - the Du_{n-1} extreme Delaunay
  - a vertex

  $n = 7$: Gosset polytope 3_{21}.

Erdahl-Rybnikov infinite series

- Du_n have exponential number N of vertices:
  $2^{n-2} + 2n - 1$ for even $n \geq 6$,
  $2^{n-2} + 4n - 4$ for odd $n \geq 7$.
  Conjecture: this N is maximal amongst all extreme Delaunay n-polytopes.
  For all other known ones, N is a polynomial of n.
- Erdahl & Rybnikov, 2002 constructed series of extreme Delaunay polytopes ER_n with minimal N:
  asymmetric ones, for $n \geq 6$, with $N = \binom{n+2}{2} - 1$
  starting with Schläfli polytope, and
  centrally symmetric ones, for $n \geq 7$, with $N = 2\binom{n+1}{2}$
  starting also with Gosset polytope.
- For both series, $N = 27$ for $n = 6$ and $N = 56$ for $n = 7$.

Grishukhin infinite series

- Asymmetric Du_n and ER_n (with even n) generalize partitions $\alpha_0 + \frac{1}{2}\gamma_5 + \beta_5$ and $\alpha_5 + J(6,2) + \alpha_5$ of Schläfli polytope into D_5- and A_5-layers, resp. No extreme Delaunay decomposes into < 3 layers (lamina); all known ones have a decomposition with exactly 3 layers.
- Grishukhin, 2006 constructed series $Gr_n(t)$ depending on a second parameter $1 \leq t < \frac{n-3}{2}$; the case $t = 1$ gives ER_n. $Gr_n(t)$ has $N = 2n + \binom{n}{t+1}$ vertices.
  He constructed also series $Gr_n^a(t)$ of assymmetric ones (for the same n, t) by adding $\binom{n}{t} + 2$ new vertices to $Gr_n(t)$; so, it has same N as $Gr_{n+1}(t)$ but it is different.
  He also gave, for each $Gr_{n=2t+4}(t)$, a symmetric one having it as a section.
- Erdahl, Ordine, Rybnikov, 2007: 3-parametric series.

Extreme Delaunays with $n \leq 9$

Idea is to apply the bounding method to the extreme Delaunay Du_8, obtain new extreme Delaunay polytopes, reapply the method, test by isomorphy, etc.

- Only segment and Schlafli in dimensions ≤ 6.
- Conjectured list in dimension 7:

$Nr.vertices$	$Nr.facets$	$\vert Sym\vert$
$35\, asymm.$	228	1440 (Erdahl & Rybnikov)
$56\, symm.$	702	2903040 (Gosset)

- 27 extreme Delaunay polytopes in dimension 8; all? $\binom{8+2}{2} - 1 = 44 \leq N \leq 79$ with equality for Du_8.
- ≥ 1500 in dimension 9. Perhaps, not so big growth.

27 extreme Delaunays in dimension 8

$Nr.vertices$	$Nr.facets$	$\|Sym\|$
79	1131	322560 ED_8
72	1798	80640
72	354	80640
58	664	1440
55	355	288
54	375	864
54	539	384
52	634	192
49	546	288
49	535	960
47	534	48
47	474	24
47	395	48

46	523	36
46	476	288
45	571	192
45	559	48
45	582	144
45	414	1296
44	559	48
44	559	240
44	504	2880
44	599	144
44	529	10080
44	538	72
44	480	288
44	559	72

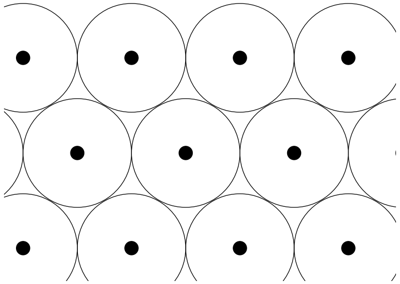
V. Lattice packing

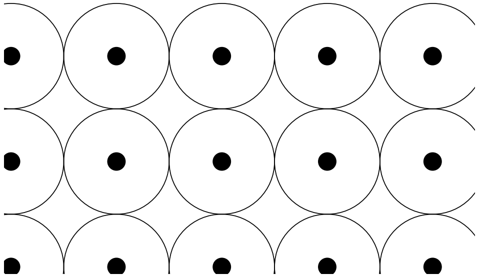
and

lattice covering

Lattice packing

- We consider packing by n-dimensional balls of the same radius, whose center belong to a lattice L.





- Objective is to maximize the packing density:

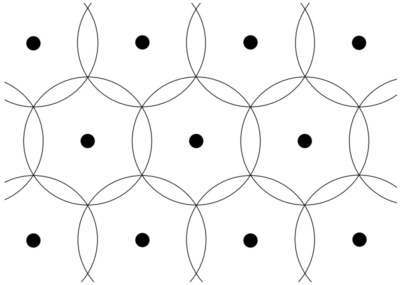
$$\alpha(L) = \frac{\lambda(L)^n \kappa_n}{det(L)} \leq 1,$$

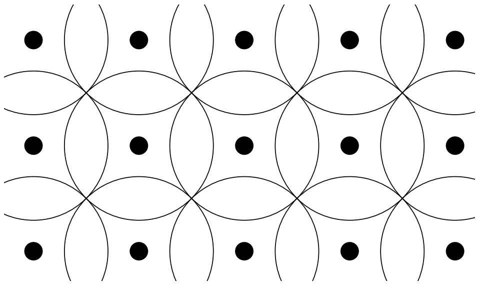
where $\kappa_n, det(L)$ are volumes of the unit ball, unit cell, and packing radius (inradius of Voronoi polytope) is:

$$\lambda(L) = \frac{1}{2} \min_{v \in L - \{0\}} ||v||.$$

Lattice covering

- We consider covering of $\mathbb{R}^n$ by n-dimensional balls of the same radius, whose centers belong to a lattice L.





- Objective is to minimize the covering density:

$$\Theta(L) = \frac{\mu(L)^n \kappa_n}{det(L)} \geq 1$$

with covering radius $\mu(L)$ being the maximum distance of points of $\mathbb{R}^n$ to a closest lattice vector.
$\mu(L)$ is largest radius of Delaunay polytopes of L.

- Best (i.e., sparest) lattice covering for $n \leq 5$: A_n^*.

Lattice packing-covering

- We want a lattice L in $\mathbb{R}^n$, such that the sphere packing (resp, covering) obtained by taking spheres centered in L with maximal (resp, minimal) radius, are both good.
- The quantity of interest is:

$$\frac{\Theta(L)}{\alpha(L)} = (\frac{\mu(L)}{\lambda(L)})^n \geq 1.$$

- Lattice packing-covering problem: minimize $\frac{\Theta(L)}{\alpha(L)}$.

Dim.	Solution	Dimension	Solution
2	A_2^*	4	H_4 (Horvath lattice)
3	A_3^*	5	H_5 (Horvath lattice)

Minimal volume of Voronoi polytope

- For spherical packing: let v_n be minimal volume v_n of Voronoi polytope over subsets X of $\mathbb{R}^n$ with $d(x,y) \geq 2$.
- For $n = 2, 3$ and $4, 8, 24$, it is hexagon, Dodecahedron (regular ones) and, conjecturally, Voronoi polytopes of D_4, E_8, Leech lattice, respectively.
- Density of any sphere packing of $\mathbb{R}^n$ is at most $\frac{v_n}{\kappa_n}$ with equality if and only if a minimal $\mathcal{V}_x$ face-to-face tiles $\mathbb{R}^n$.
- So, equality $n = 4, 8, 24$, but not for $n = 3$: Dodecahedron gives local, but not global min.
- Best (i.e., densest) packing is $\frac{\pi}{\sqrt{12}} \approx 0.9069$ (Lagrange, for $n = 2$) by A_2. For $n = 3$, it is $\frac{\pi}{\sqrt{18}} \approx 0.74$ by A_3 (Hales-Ferguson, 1998, proved Kepler Conjecture).

Perfect forms

Optimal lattice packings come from the theory of perfect forms and perfect domains; see “Premier mémoire” by Voronoi (1908) and book by Martinet.

For a form $A \in PSD_n$, define $\min(A) = \min_{v \in \mathbb{Z}^n \neq 0} vA^tv$ and $Min(A)$ the set of shortest (realizing $\min(A)$) vectors $v \in \mathbb{Z}^n$.

- A form A is called perfect if is defined by $Min(A)$, i.e., $vB^tv = \min(A)$ with $v \in Min(A)$ implies $B = A$.
- A lattice is perfect if it has a basis $\mathbf{v}$ with perfect $G_{\mathbf{v}}$.
- Korkine-Zolotarev: if a form has local maximum of packing density, then it is perfect.
- Perfect forms have rational coefficients.
- The number $|Min(A)|$ of shortest vectors is $\geq n(n+1)$, since cone dimension is $\binom{n+1}{2}$ and they come as $\{v, -v\}$

Perfect domains

- If A is perfect, its perfect domain (P-domain) is cone:

$$\{\sum_{v \in Min(A)} \lambda_v v^T v : \lambda_v \geq 0\}.$$

- Voronoi: all P-domains form a polyhedral normal tiling of PSD_n, and there is a finite number of perfect forms A_i, up to $GL_n(\mathbb{Z})$-equivalence $A = P^T A_i P$.
- L-tiling coincides with P-tiling for $n = 2, 3$ and refines it for $n = 4, 5$ but not (Erdahl-Rybnikov) for $n = 6$.
- Dickson: P-domain = L-domain only for A_n.
- L- and P-tilings are two polyhedral reduction tilings of PSD_n into open polyhedral cones (domains): both tilings are invariant with respect to $GL(n, \mathbb{Z})$ and have finitely many non-equivalent domains.

Enumeration of perfect forms

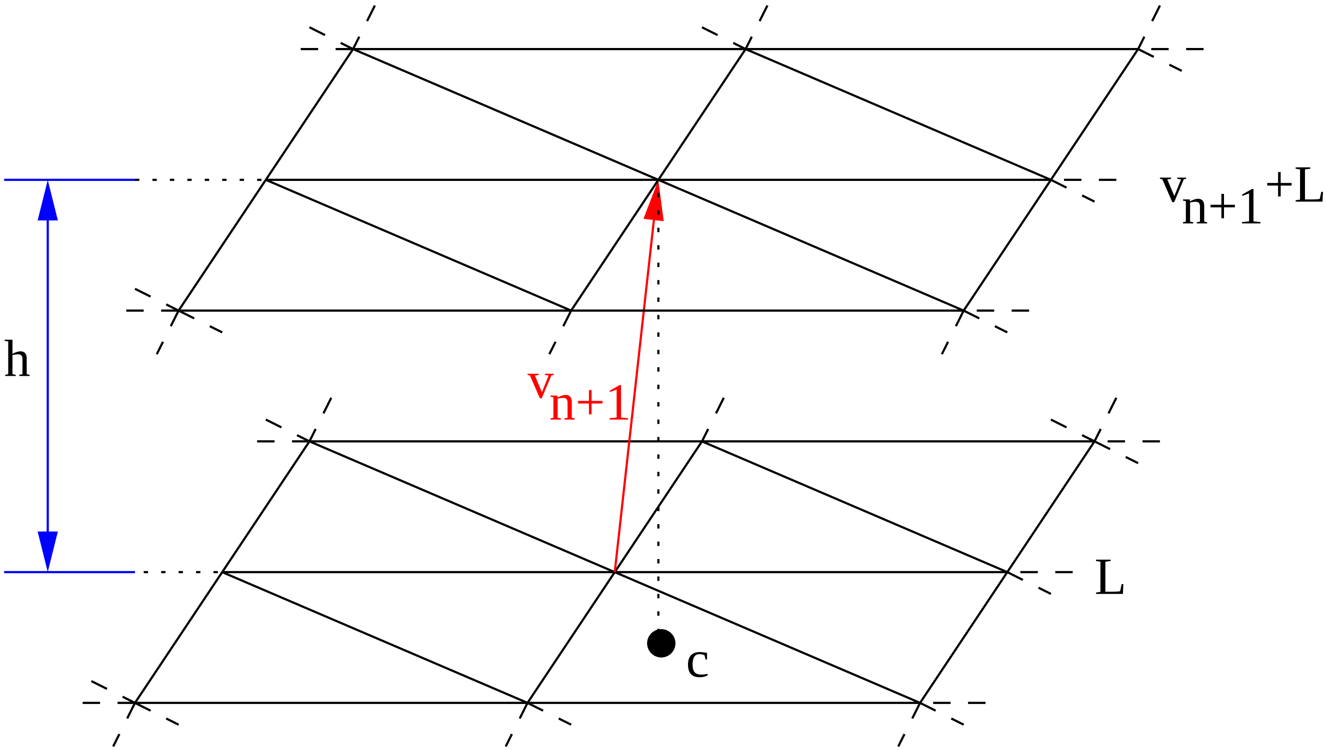
Dim.	Nr. of forms	Best form	Authors
1	1	A_1	
2	1	A_2	Lagrange
3	1	A_3	Gauss
4	2	D_4	Korkine-Zolotareff
5	3	D_5	Korkine-Zolotareff
6	7	E_6	Barnes
7	33	E_7	Jaquet
8	10916	E_8	Dutour, Schurmann, Vallentin

Best packings (all by lattices)

Dimension	Best lattice packing	Best packing
2	A_{hex} (Lagrange)	A_{hex} (Lagrange)
3	A_3 (Gauss)	A_3 (Hales & Ferguson)
4	D_4 (Korkine & Zolotarev)	?
5	D_5 (Korkine & Zolotarev)	?
6	E_6 (Blichfeldt)	?
7	E_7 (Blichfeldt)	?
8	E_8 (Blichfeldt)	?
9,11	Λ_9, Λ_{11} (laminated lattices)?	?
10,12	K'_{10}, K_{12} (Coxeter-Todd lattices)?	?
16	BW_{16} (Barnes-Wall lattice)?	?
24	$Leech$ (Cohn & Kumar)	?

Lamination

- Given a n-dim. lattice L, create a $(n+1)$-dim. lattice L':



- The point c is fixed orthogonal projection of v_{n+1} on L. We vary the value of h to get a PSD_n-space.

Lamination

- In terms of Gram matrices,

$$Gram(L) = A \text{ and } Gram(L') = \begin{pmatrix} A & A^t c \\ cA & \alpha \end{pmatrix}$$

  c is the projection of the vector $(0, \dots, 0, 1)$ on lattice L.
- The symmetries of L' are the symmetries of L preserving the center c and, if $2c \in \mathbb{Z}^n$, the othogonal symmetry

$$\begin{pmatrix} I_n & 0 \\ 2c & -1 \end{pmatrix}$$

- c can be chosen as the center of a Delaunay polytope.

Lamination

- Conway & Sloane: for the packing problem, one finds that the best lattice, containing L as a section, is defined by taking c to be a deep hole (i.e., Delaunay polytope of maximal radius). They obtain a family Λ_n of lattices.
- For the covering problem, things are not so simple: one cannot solve the general problem with c unspecified, since it has no symmetry and too much parameters.
- By doing lamination over Coxeter-Barnes lattices A_9^5 and A_{11}^4, one gets a record covering (Coxeter-Todd lattices K_{10}, K_{12}) in dimension 10 and 12.

Best known coverings (all by lattices)

n	Lattice	Covering density Θ			
1	$\mathbb{Z}^1$	1	13	L_{13}^c	7.762108
2	A_2^*	1.209199	14	L_{14}^c	8.825210
3	A_3^*	1.463505	15	L_{15}^c	11.004951
4	A_4^*	1.765529	16	A_{16}^*	15.310927
5	A_5^*	2.124286	17	A_{17}^9	12.357468
6	L_6^c	2.464801	18	A_{18}^*	21.840949
7	L_7^c	2.900024	19	A_{19}^{10}	21.229200
8	L_8^c	3.142202	20	A_{20}^7	20.366828
9	L_9^c	4.268575	21	A_{21}^{11}	27.773140
10	L_{10}^c	5.154463	22	Λ_{22}^*	≤ 27.8839
11	L_{11}^c	5.505591	23	Λ_{23}^*	≤ 15.3218
12	L_{12}^c	7.465518	24	*Leech*	7.903536

Bambah, Ryshkov-Baranovskii: A_n^* is best for $2 \leq n \leq 5$.

Coxeter-Barnes lattices

- The lattice A_n is defined as

$$A_n = \{x \in \mathbb{Z}^{n+1}, \text{ such that } \sum x_i = 0\}.$$

- If r divides $n+1$, then define the lattice A_n^r by

$$A_n^r = A_n \cup v_{n,r} + A_n \cup \ldots \cup (r-1)v_{n,r} + A_n,$$

$$\text{where } v_{n,r} = \frac{1}{r}\sum_{i=1}^{n+1} e_i - \sum_{i=1}^{q} e_i \text{ and } q = \frac{n+1}{r},$$

i.e., A_n^r is the union of r translates of A_n.

- The dual $(A_n^r)^*$ of A_n^r is A_n^q.
- One has $A_n^1 = A_n \subset A_n^r \subset A_n^* = A_n^{n+1}$.

Coxeter-Barnes lattices

- $A_8^3 = E_8$, $A_7^2 = E_7$; so, their automorphism groups are $W(E_7), W(E_8)$.
- For other (n,r), $Aut(A_n^r) = Aut(A_n) = \mathbb{Z}_n \times Sym(n+1)$. So, the symmetry group of A_n^r is relatively large, much larger than one of Leech or Thompson lattices. Those lattices are interesting since their symmetry groups are large and related to sporadic simple groups.
- Their Delaunay polytopes can also be big: one of 21 orbits of A_{21}^2 consists of 40698-vertex polytopes.
- Strong algebraic structures are expected because stabilizers of their known Delaunay polytopes are large.
- Is there a general analytical description of the Delaunay partition of A_n^r?

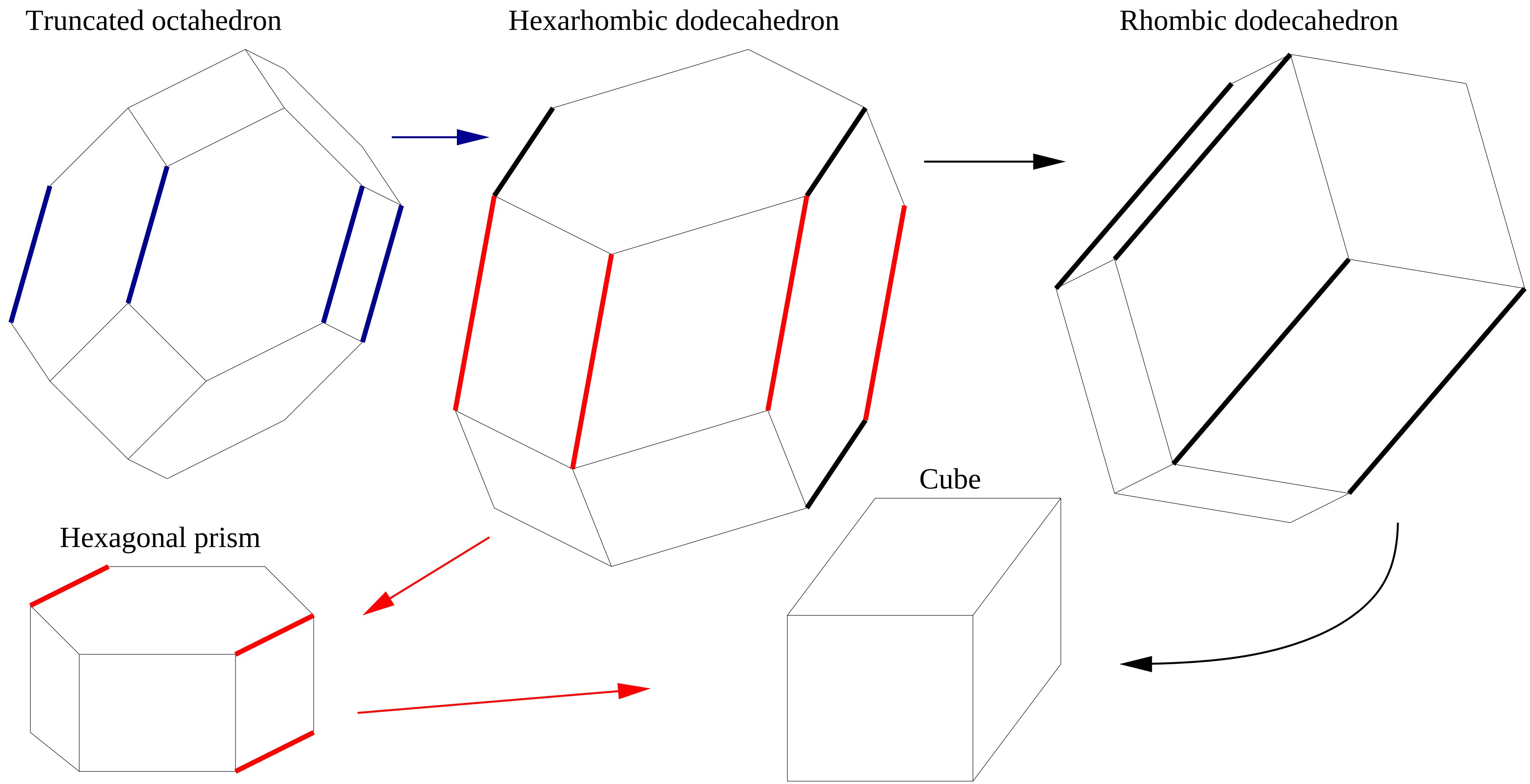
VI. Voronoi Conjecture

Tilings

- A tiling (or tesselation) is a set of polytopes in $\mathbb{R}^n$ which do not overlap (a packing) and cover $\mathbb{R}^n$ without gaps (a covering). Those polytopes are called tiles (or cells).
- A tiling is face-face if any 2 tiles intersect in a face or $\emptyset$.
- A tiling is monohedral if all its tiles are congruent.
A tile of monohedral tiling is called spacefiller.
- A tiling is isohedral if its symmetry group (all rigid motions of $\mathbb{R}^n$ preserving tiling) is transitive on tiles.
A tile of isohedral tiling is called stereotope.
- A parallelotope is a stereotope which can be moved to any other tile by pure translation (with no rotation).
- Voronoi polytope of any lattice is a parallelotope;
Voronoi conjecture: and vice versa.

Comb. types of ≤ 3-paralelotopes

All 2-parallelotopes are combinatorially equiv. to Voronoi 2-polytopes: centrally symmetric hexagons and rectangles.



$1 + 2 + 5$ with $n \leq 3$ are zonotopes but not 24-cell=$\mathcal{V}(D_4)$.

The Voronoi conjecture, 1908

A parallelotope is a polytope whose translation copies form a normal (face-to-face) partition of $\mathbb{R}^n$.
Voronoi conjecture: every parallelotope is a linear image of the Dirichlet domain (i.e., Voronoi polytope) of a lattice.

- Fedorov, 1885: $1+2+5$ parallelotopes with $n \leq 3$ are Voronoi and zonotopes (projections of n-cube).
- Delone, 1929, 51 and Shtogrin, 1975, 52-nd, found all 52 4-dimensional parallelotopes; all are Voronoi.
  Deza and Grishukhin: 17 are zonotopes, one is the regular 24-cell $\{3,4,3\}$ and remaining 34 are sums of 24-cell with non-zero zonotopal parallelotopes.
- Engel, 1998 by computer: all 179397 5-dimensional ones; all are Voronoi including 222 primitive ones.
- McMullen–Erdahl: OK for zonotopes: comb, arith. eqv.

The Voronoi conjecture: facet vectors

- Venkov, 1954, and McMullen, 1984: a convex polytope P is a parallelotope if and only if it holds:
(i) P and each its facet are centrally symmetric and
(ii) the projection of P along any $(n-2)$-face is either parallelogram, or a centrally symmetric hexagon.
- Let I be index-set of facets up to central symmetry.
For $i \in I$, facet vector q_i is the one orthogonal to facet i.
- If 0 be the center of P, then it holds:
$P(0) = \{x \in \mathbb{R}^n : -\frac{1}{2}q_i^T t_i \le q_i^T x \le \frac{1}{2}q_i^T t_i, i \in I\}$,
where t_i is lattice vector from 0 to the center of the parallelotope sharing with P the facet i.
- $t_i, i \in I$, generate a lattice L (as translation group).
A parallelotope $P(0)$ is the Voronoi polytope $\mathcal{V}(0)$ of L if and only if $t_i = q_i, i \in I$ (the relevant vectors).

The Voronoi conjecture: primitivity

- An n-parallelotope and its tiling are k-primitive if each k-face of P (and of the tiling) belongs exactly to $n-k+1$ parallelotopes of the tiling.
- Primitivity is 0-primitivity. Voronoi, 1908, proved his conjecture for primitive parallelotopes.
- The number of primitive n-parallelotopes is 1 for $1 \leq n \leq 3$ and $3,\ 222,\ \geq 2 \times 10^6$ for $n = 4,\ 5,\ 6$.
- k-primitivity implies $(k+1)$-primitivity.
Zhitomirskii, 1929, proved Voronoi conjecture for k-primitive parallelotopes with $k \leq n-2$.
- Any parallelotope is $(n-1)$- and n-primitive.

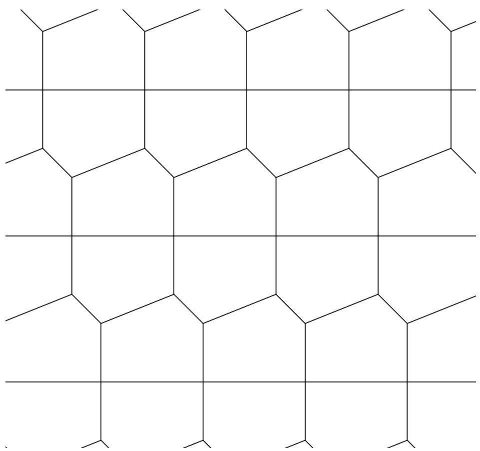
VII. Crystallographic structures

A generalization

- Given a lattice L, a crystallographic structure is a subset of $\mathbb{R}^n$ (periodic structure) of the form

$$\mathcal{CS} = \cup_{i=0}^{m}(a_i + L).$$

- For example, if G is a Bravais group of $\mathbb{R}^n$ and x a point, then the orbit Gx is a crystallographic structure.
- A stereotope is a polytope whose orbit under a space group form a normal (face-to-face) tiling of the space.



L-type theory, linear

- Fixing vectors a_i and varying lattice L (or quadratic form), one gets linear theory (L-types, perfect domains etc.) for such periodic structures.
- One can consider the Delaunay decomposition of $\mathcal{CS}$ and compute it with almost the same algorithms.
- Every Delaunay is circumscribed by an empty sphere. The condition that no points are inside the sphere, translates into linear inequalities.
- Many conjectures says that the best packings and coverings are not lattice ones. So, the hope is to choose well the a_i, do our classical polyhedral computation and get better packings or coverings than the lattice ones.

L-type theory, nonlinear

- Fix a lattice L (or quadratic form on $\mathbb{R}^n$) and vary a_i.
- L-types exist again in that context.
  They again partition the parameter space.
- But the obtained theory is no longer linear.
- The key for a combinatorial attack is to have the solution of the problem: among polynomial inequalities

  $$f_i(x) \geq 0 \text{ for } 1 \leq i \leq, p$$

  find the non-redundant ones.
- Project: use real algebraic geometry software for solving above problem and realize enumeration of L-types of stereotopes.