Wythoff construction and l_1-embedding

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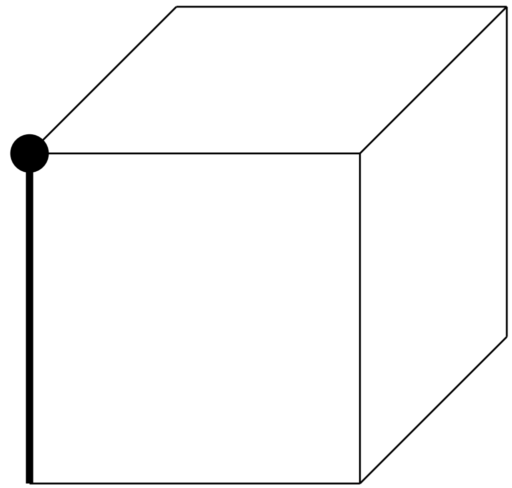
I. Wythoff kaleidoscope construction

W.A. Wythoff (1918) and H.S.M. Coxeter (1935)

Polytopes and their faces

- A polytope of dimension d is defined as the convex hull of a finite set of points in R^d.
- A valid inequality on a polytope P is an inequality of the form $f(x) \geq 0$ on P with f linear. A face of P is the set of points satisfying to $f(x) = 0$ on P.



A face of dimension 0, 1, $d-2$, $d-1$ is called, respectively, vertex, edge, ridge and facet.

Face-lattice

There is a natural inclusion relation between faces, which define a structure of partially ordered set on the set of faces.

- This define a lattice structure, i.e. every face is uniquely defined by the set of vertices, contained in it, or by the set of facets, in which it is contained.
- Given two faces $F_{i-1} \subset F_{i+1}$ of dimension $i-1$ and $i+1$, there are exactly two faces F of dimension i, such that $F_{i-1} \subset F \subset F_{i+1}$.
  This is a particular case of the Eulerian property satisfied by the lattice:

  Nr. faces of even dimension=Nr. faces of odd dimension

Skeleton of polytope

- The skeleton is defined as the graph formed by vertices, with two vertices adjacent if they form an edge.
- The dual skeleton is defined as the graph formed by facets with two facets adjacent if their intersection is a ridge.

In the case of 3-dimensional polytopes, the skeleton is a planar graph and the dual skeleton is its dual, as a plane graph.
Steinitz's theorem: a graph is the skeleton of a 3-polytope if and only if it is planar and 3-connected.

Complexes

We will consider mainly polytopes, but the Wythoff construction depends only on combinatorial information. Also, not all properties of face-lattice of polytopes are needed.
The construction will apply to complexes:

- which are partially ordered sets,
- which have a dimension function associated to its elements.

This concerns, in particular, the tilings of Euclidean d-space.

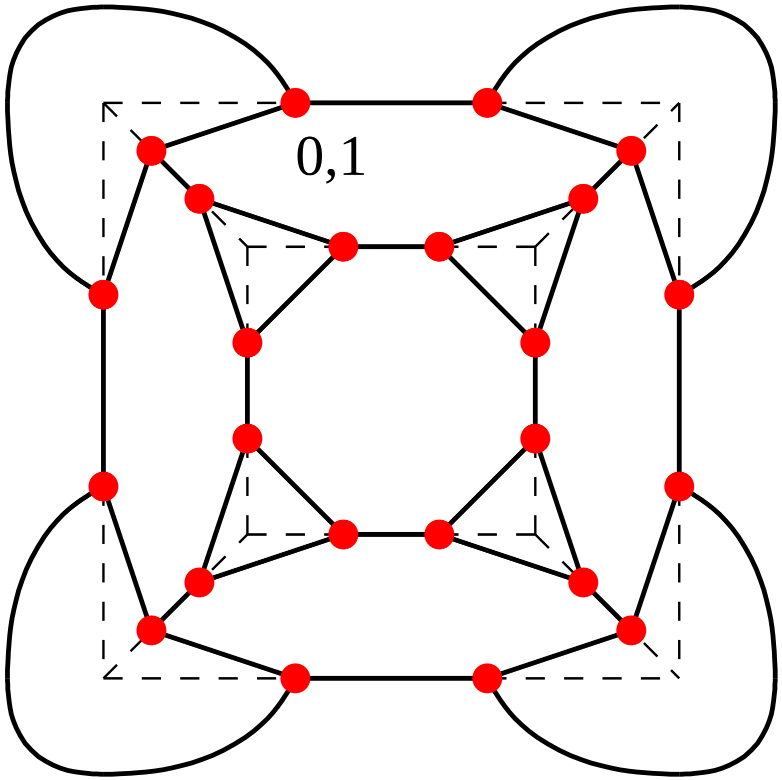
Wythoff construction

- For a $(d-1)$-dimensional complex $\mathcal{K}$, a flag is a sequence (f_i) of faces with $f_0 \subset f_1 \subset \cdots \subset f_u$.
- The type of a flag is the sequence $dim(f_i)$.
- Given a non-empty subset S of $\{0, \ldots, d-1\}$, the Wythoff (kaleidoscope) construction is a complex $P(S)$, whose vertex-set is the set of flags with fixed type S.
- The other faces of $\mathcal{K}(S)$ are expressed in terms of flags of the original complex $\mathcal{K}$.

Formalism of faces of Withoffian $\mathcal{K}(S)$

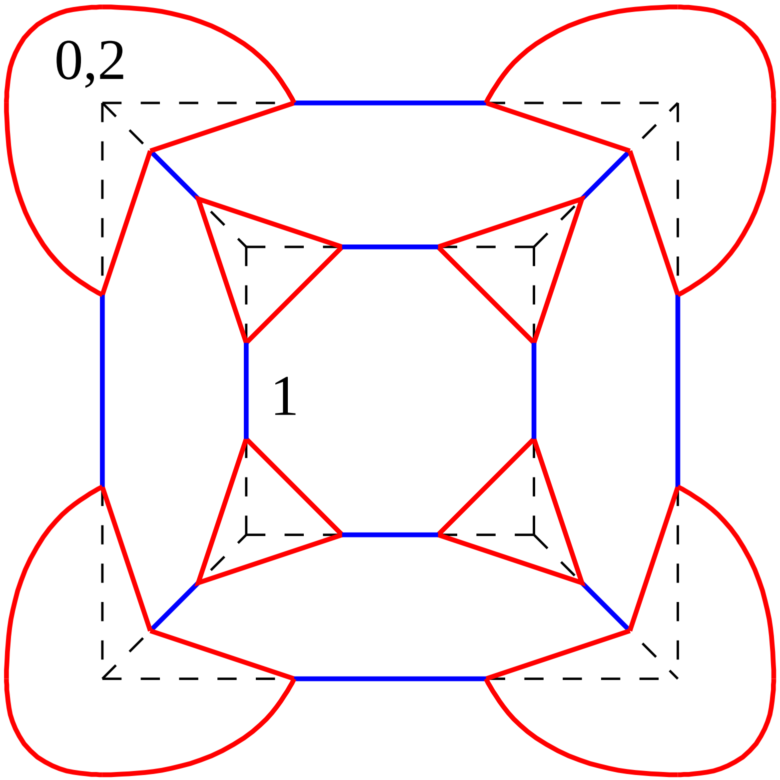
- Set $\Omega = \{\emptyset \neq V \subset \{0, \ldots, d\}\}$ and fix an $S \in \Omega$. For two subsets $U, U' \in \Omega$, we say that U' blocks U (from S) if, for all $u \in U$ and $v \in S$, there is an $u' \in U'$ with $u \leq u' \leq v$ or $u \geq u' \geq v$. This defines a binary relation on Ω (i.e. on subsets of $\{0, \ldots, d\}$), denoted by $U' \leq U$.
- Write $U' \sim U$, if $U' \leq U$ and $U \leq U'$, and write $U' < U$ if $U' \leq U$ and $U \not\leq U'$.
- Clearly, $\sim$ is reflexive and transitive, i.e. an equivalence. $[U]$ is equivalence class containing U.
- Minimal elements of equivalence classes are types of faces of $\mathcal{K}(S)$; vertices correspond to type S, edges to "next closest" type S' with $S < S'$, etc.

Example: the case $S = \{0, 1\}$, vertices



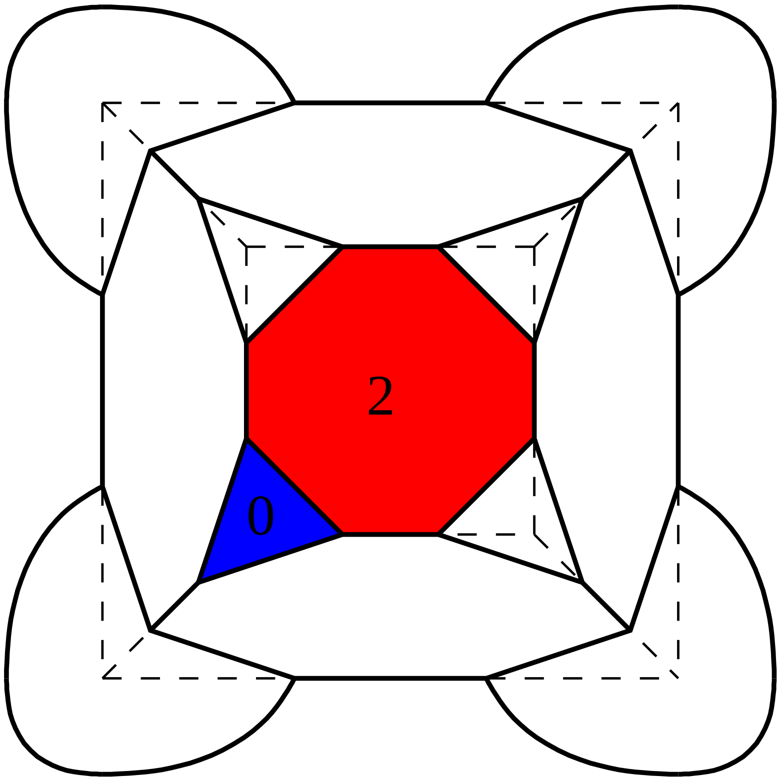
One type of vertices for $Cube(\{0, 1\})$: $\{0, 1\}$ (i.e. type S).

Example: the case $S = \{0, 1\}$, edges



Two types of edges for $Cube(\{0,1\})$: $\{1\}$ and $\{0,2\}$

Example: the case $S = \{0, 1\}$, faces



Two types of faces for $Cube(\{0, 1\})$: $\{0\}$ and $\{2\}$

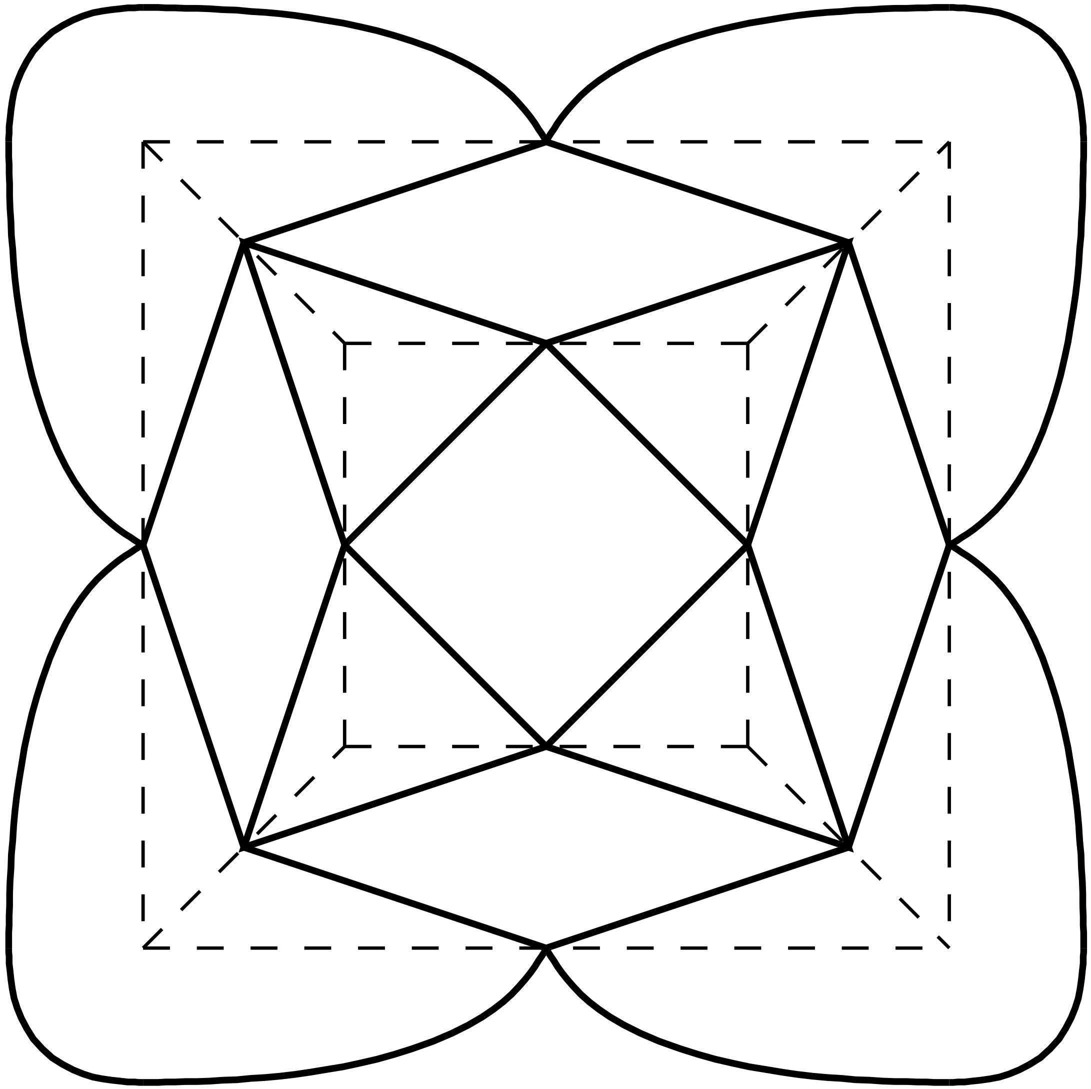
2-dimensional complexes

- 2-dimensional Eulerian complexes are identified with plane graphs.
- If $\mathcal{M}$ is a plane graph

set S	plane graph $\mathcal{M}(S)$
$\{0\}$	original map $\mathcal{M}(S)$
$\{0,1\}$	truncated $\mathcal{M}$
$\{0,1,2\}$	truncated $\mathrm{Med}(\mathcal{M})$
$\{0,2\}$	$\mathrm{Med}(\mathrm{Med}(\mathcal{M}))$
$\{1,2\}$	truncated $\mathcal{M}^*$
$\{1\}$	$\mathrm{Med}(\mathcal{M})$
$\{2\}$	$\mathcal{M}^*$

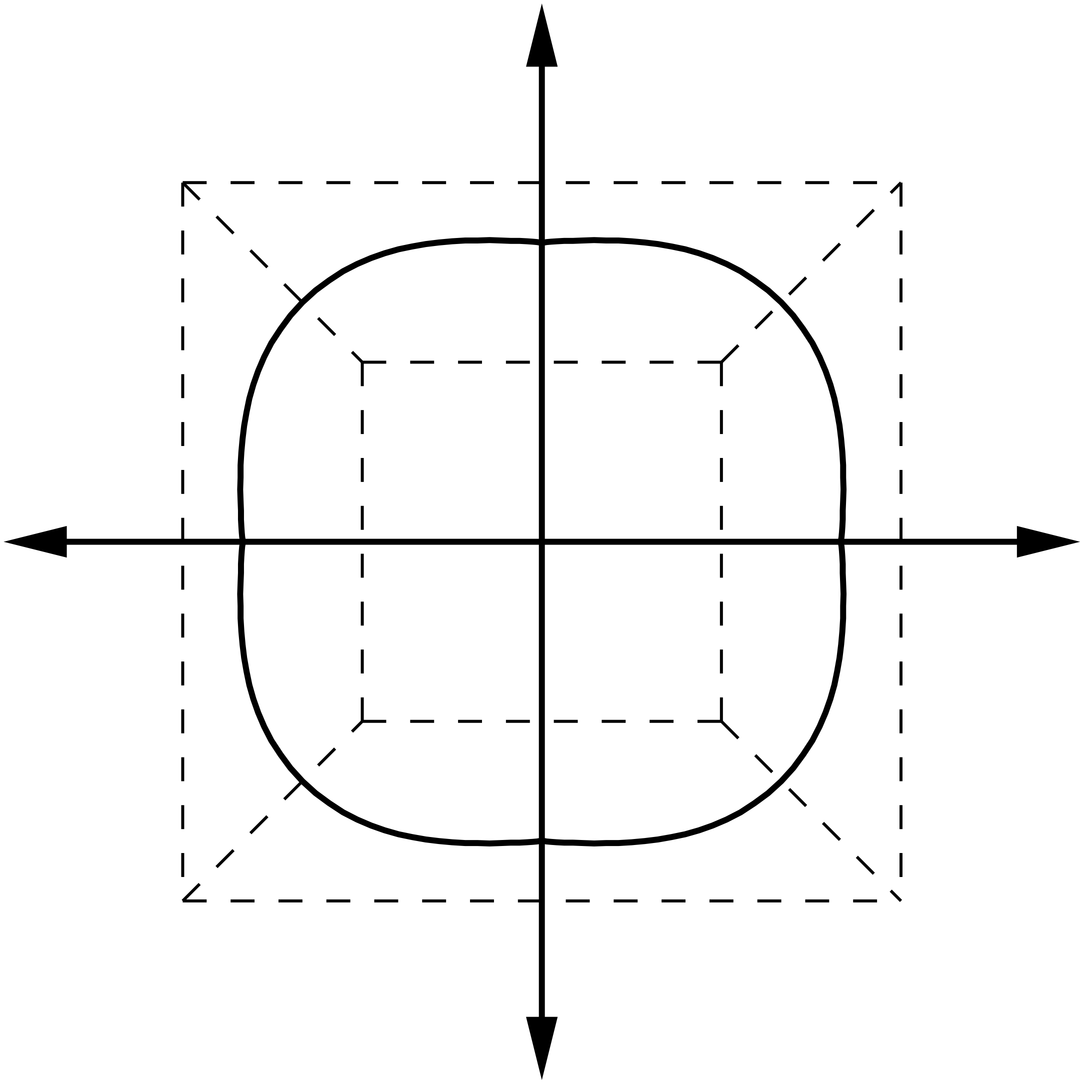
Wythoff on the cube

Cube($\{1\}$)=$Med(Cube) = Cuboctahedron$



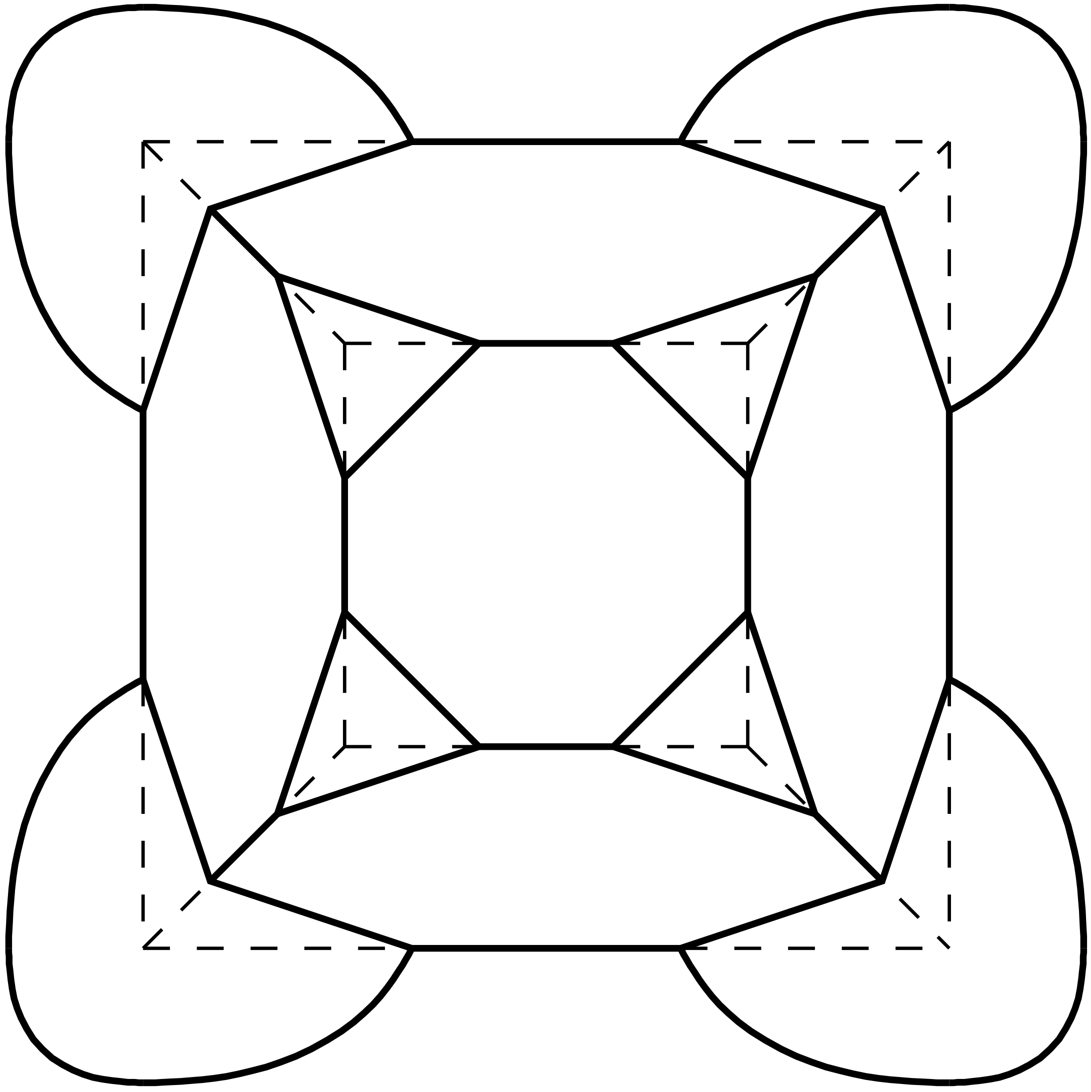
Wythoff on the cube

Cube($\{2\}$)=$Cube^* = Octahedron$



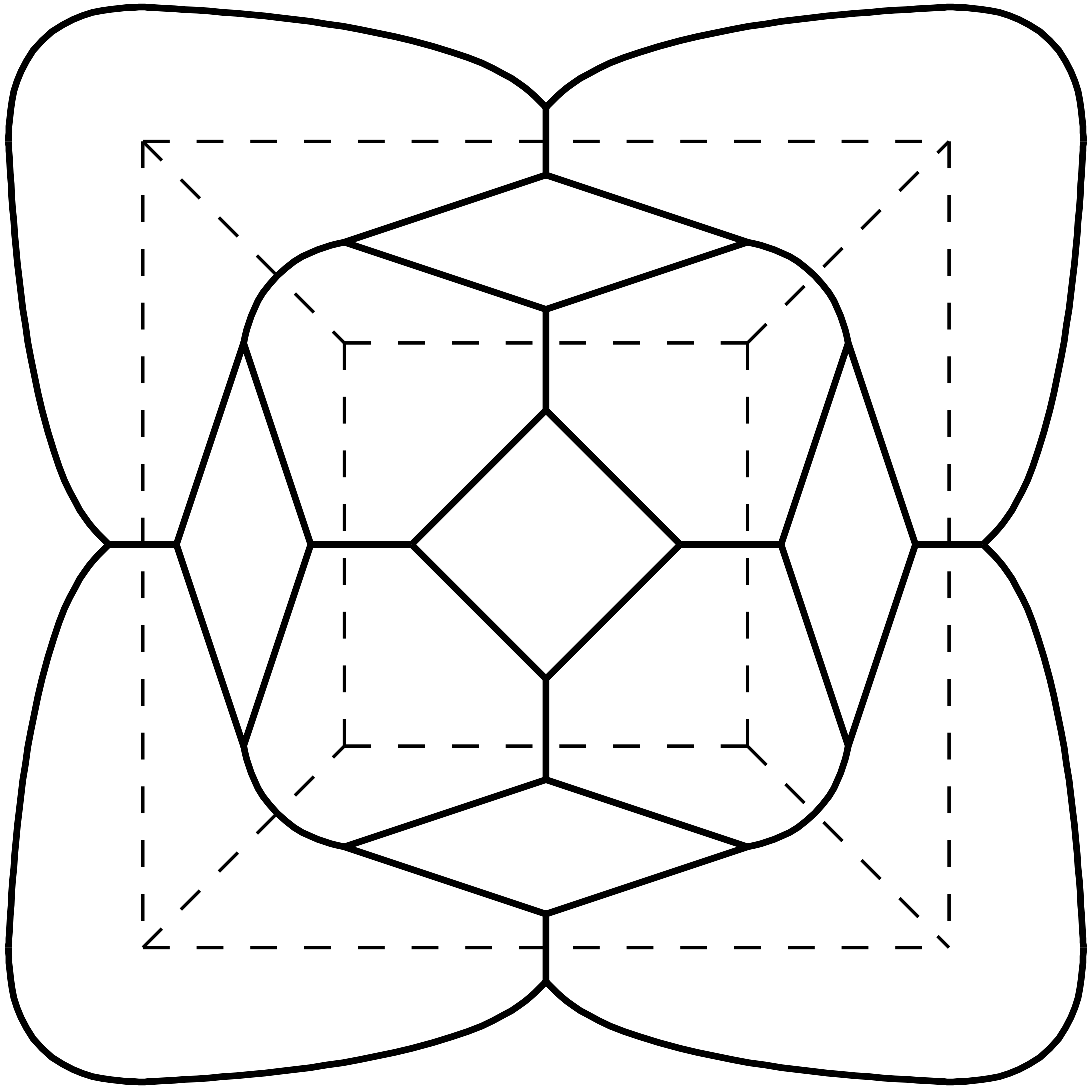
Wythoff on the cube

Cube($\{0, 1\}$)=$truncated\ \ Cube$



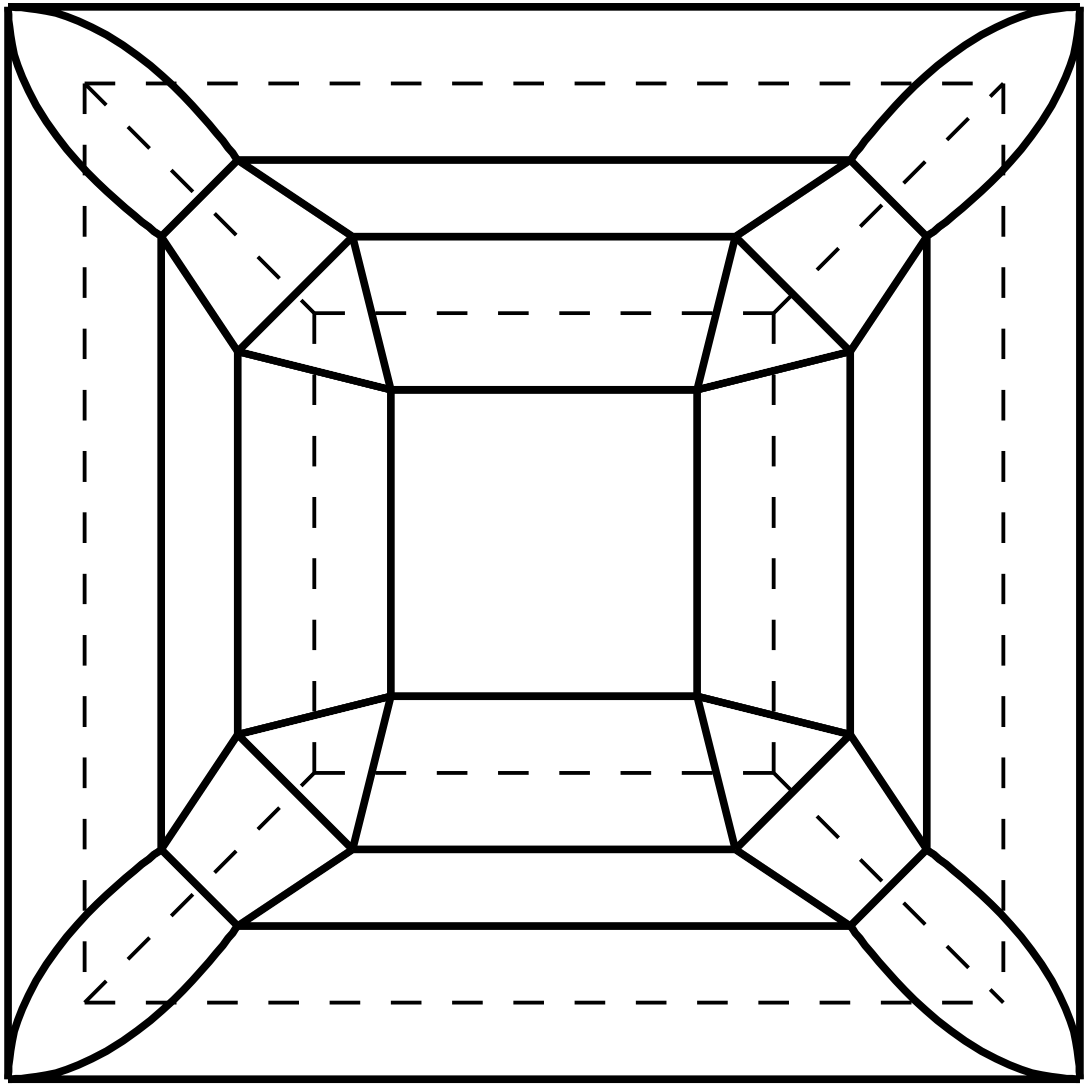
Wythoff on the cube

Cube($\{1,2\}$)=$truncated\ \ Octahedron$



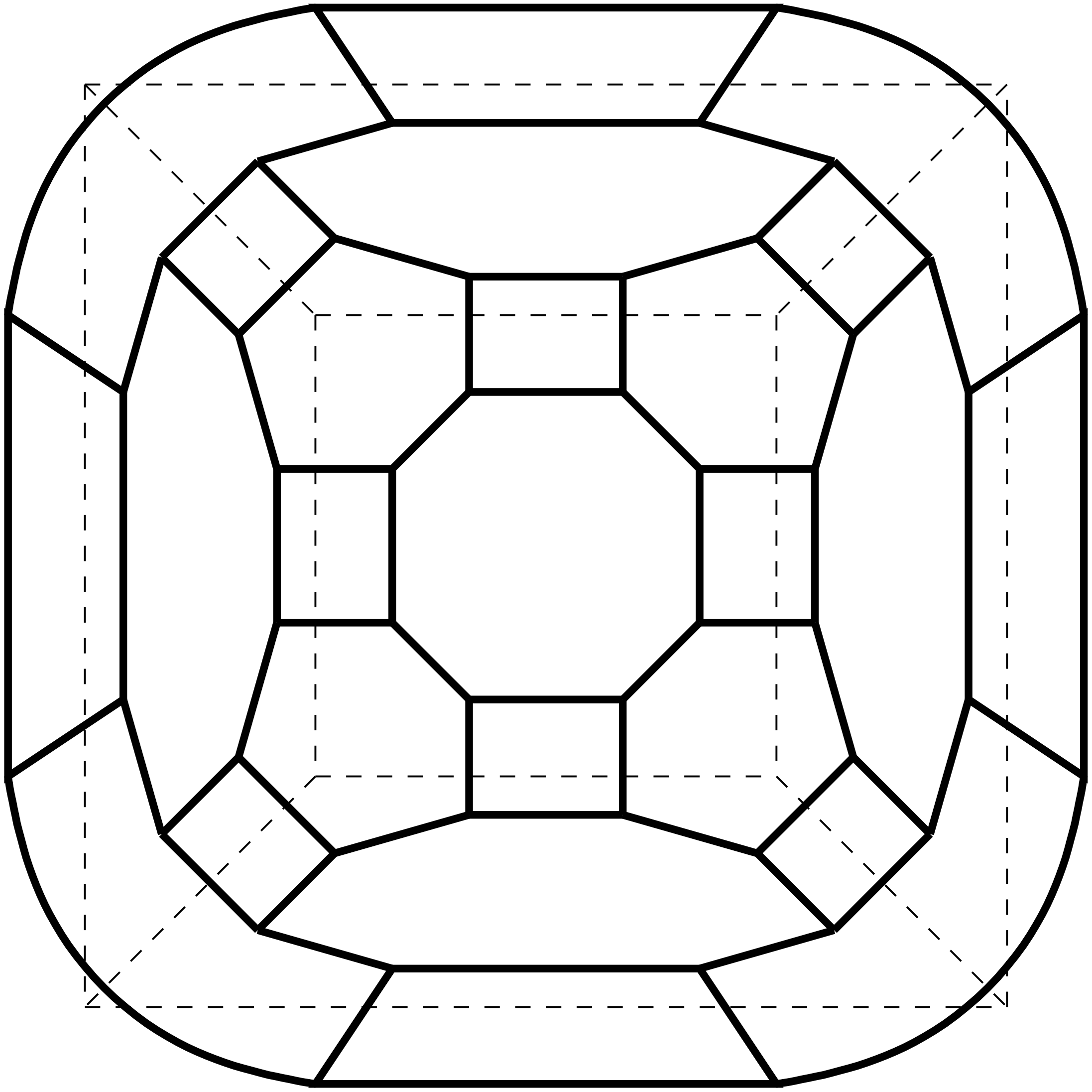
Wythoff on the cube

Cube($\{0, 2\}$)=$Med(Cuboctahedron) = Rhombicuboctahedron$



Wythoff on the cube

Cube($\{0, 1, 2\}$)=$truncated\ \ Cuboctahedron$



Properties of Wythoff construction

If $\mathcal{K}$ is a $(d-1)$-dimensional complex, then:

- $\mathcal{K}(\{0\}) = \mathcal{K}$ and $\mathcal{K}(\{d-1\}) = \mathcal{K}^*$ (dual complex).
- In general, $\mathcal{K}(S) = \mathcal{K}^*(\{d-1-s : s \in S\})$.
- $\mathcal{K}(\{1\})$ is median complex and $\mathcal{K}(\{0,1\})$ is (vertex) truncated complex.
- $\mathcal{K}$ admits at most $2^d - 1$ different Wythoff constructions. most different constructions.
- $\mathcal{K}(\{0,\dots,d-1\}) = \mathcal{K}^*(\{0,\dots,d-1\})$ is order complex. Its skeleton is bipartite and the vertices are full flags. Edges are full (maximal) flags minus some face. In general, flags with i faces correspond to faces of dimension $d-i$.

II. l_1-embedding

Hypercube and Half-cube

- The Hamming distance $d(x,y)$ between two points $x,y \in \{0,1\}^m$ is $d(x,y) = |\{1 \le i \le m : x_i \ne y_i\}|$ $=|N_x \Delta N_y|$ (where N_x denotes $\{1 \le i \le m : x_i = 1\}$), i.e. the size of symmetric difference of N_x and N_y.
- The hypercube H_m is the graph with vertex-set $\{0,1\}^m$ and with two vertices adjacent if $d(x,y) = 1$. The distance d is the path-distance on H_m.
- The half-cube $\frac{1}{2}H_m$ is the graph with vertex-set

$$\{x \in \{0,1\}^m : \sum_i x_i \text{ is even}\}$$

and with two vertices adjacent if $d(x,y) = 2$. The distance d is twice the path-distance on $\frac{1}{2}H_m$.

Scale embedding into hypercubes

- A scale λ embedding of a graph G into hypercube H_m is a vertex mapping $\phi : G \to \{0,1\}^m$, such that

$$d(\phi(x), \phi(y)) = \lambda d_G(x,y)$$

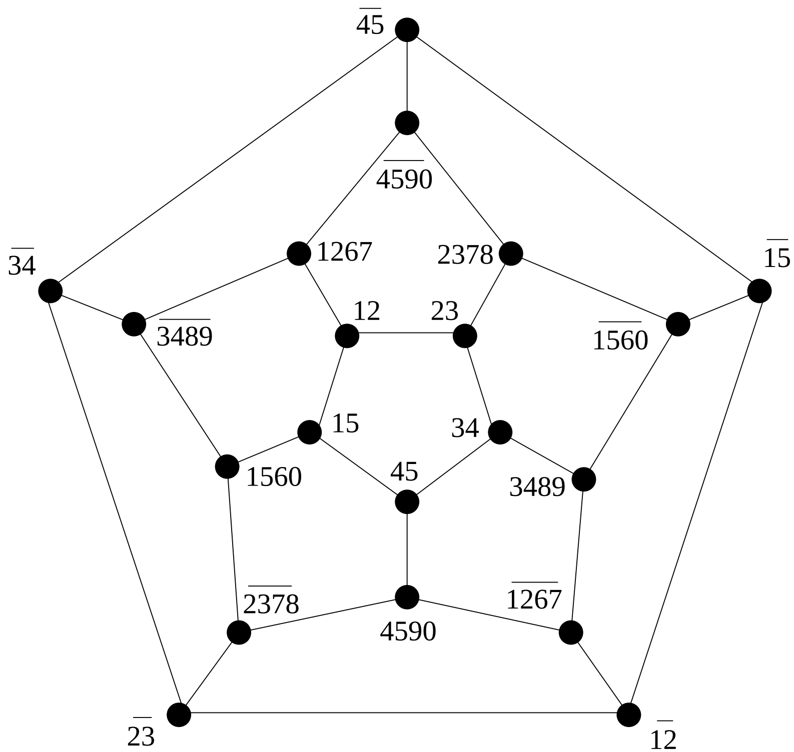
with d_G being the path-distance between x and y.

- An isometric embedding of a graph G into a graph G' is a mapping $\phi : G \to G'$, such that

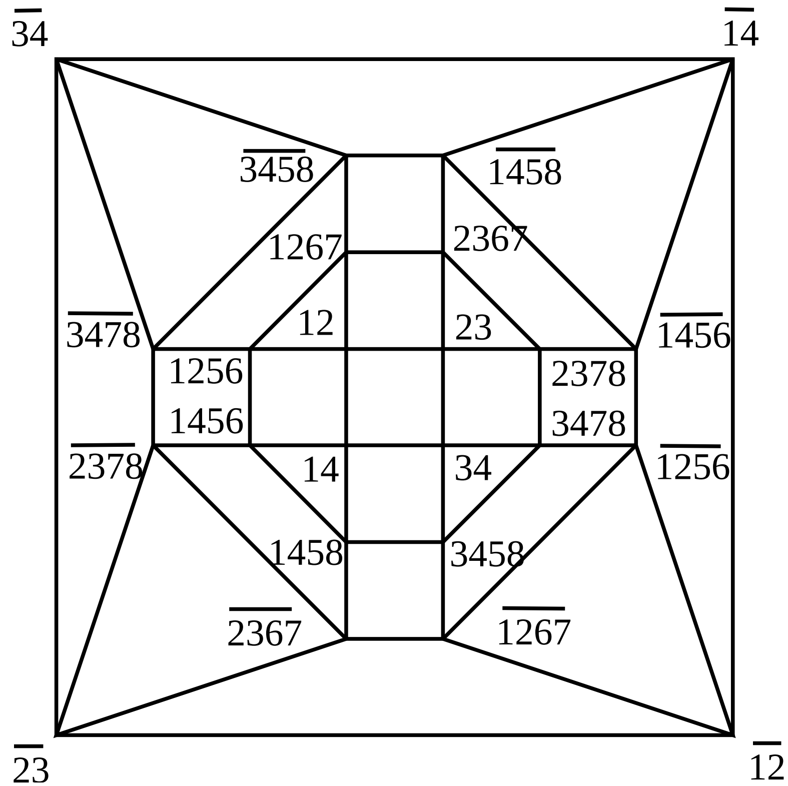
$$d_{G'}(\phi(x), \phi(y)) = d_G(x,y) \ .$$

- Scale 1 embedding is hypercube embedding, scale 2 embedding is half-cube embedding.

Examples of half-cube embeddings



Dodecahedron embeds into $\frac{1}{2}H_{10}$

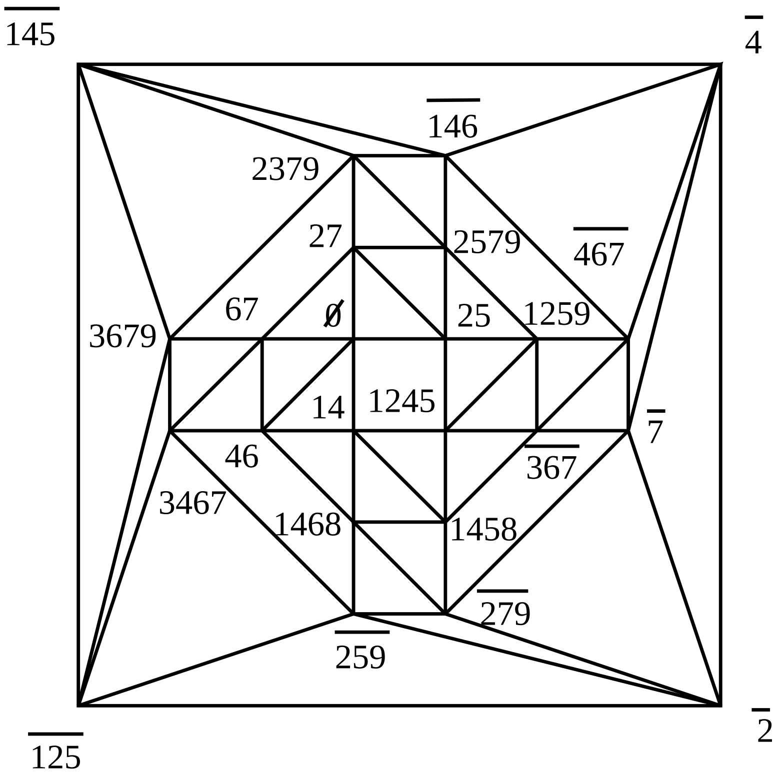


Rhombicuboctahedron embeds into $\frac{1}{2}H_{10}$ (moreover, into $J(10, 5)$: add 9 to vertex-addresses)

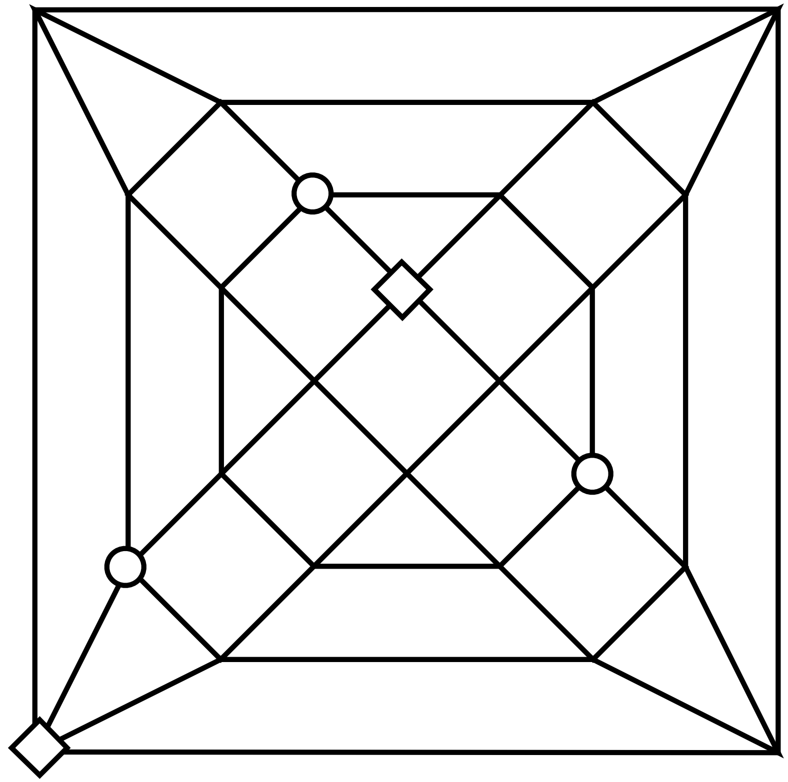
Johnson and l_1-embedding

- the Johnson graph $J(m,s)$ is the graph formed by all subsets of size s of $\{1,\ldots,m\}$ with two subsets S and T adjacent if $|S\Delta T|=2$.
- H_m embeds in $J(2m,m)$, which embeds in $\frac{1}{2}H_{2m}$.
- A metric d is l_1-embeddable if it embeds isometrically into the metric space l_1^k for some dimension k.
- A graph is l_1-embeddable if and only if it is scale embeddable (Assouad-Deza). The scale is 1 or even.

Further examples



snub Cube embeds into $\frac{1}{2}H_9$, but not in any Johnson graph



twisted Rhombicuboctahedron is not 5-gonal

Hypermetric inequality

- If $b \in \mathbb{Z}^{n+1}$ and $\sum_{i=0}^{n} b_i = 1$, then the hypermetric inequality is

$$H(b)d = \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0 \; .$$

- If a metric admits a scale λ embedding, then the hypermetric inequality is always satisfied (Deza).
- If $b = (1, 1, -1, 0, \ldots, 0)$, then $H(b)$ is triangular inequality

$$d(x,y) \le d(x,z) + d(z,y) \; .$$

- If $b = (1, 1, 1, -1, -1, 0, \ldots, 0)$, then $H(b)$ is called the 5-gonal inequality.

Embedding of graphs

- The problem of testing scale λ embedding for general metric spaces is NP-hard (Karzanov).
- Theorem(Jukovic-Avis): a graph G embeds into H_m if and only if:
  - G is bipartite and
  - d_G satisfies the 5-gonal inequality.
- In particular, testing embedding of a graph G into H_m is polynomial.
- The problem of testing scale 2 embedding of graphs into $\frac{1}{2}H_m$ is also polynomial problem (Deza-Shpectorov).

III. l_1-embedding of Wythoff construction

Regular (convex) polytopes

A regular polytope is a polytope, whose symmetry group acts transitively on its set of flags. The list consists of:

regular polytope	group
regular polygon P_n	$I_2(n)$
Icosahedron and Dodecahedron	H_3
120-cell and 600-cell	H_4
24-cell	F_4
γ_n(hypercube) and β_n(cross-polytope)	B_n
α_n(simplex)	A_n=$Sym(n+1)$

There are 3 regular tilings of Euclidean plane (36, 63, $44 = \delta_2 = Z^2$) and infinity of (p^q) on hyperbolic plane $\mathbb{H}^2$.
All non-polytopal regular tilings of dimension $d \geq 3$, are:
3 Euclidean ($\delta_d = Z^d$ and 2 sporadic tilings of $\mathbb{R}^4$)
and $15, 7, 5$ tilings of $\mathbb{H}^d$ with $d = 3, 4, 5$, respectively.

2-dim. regular tilings and honeycombs

Columns and rows indicate vertex figures and facets, resp.
Blue are elliptic (spheric), red are parabolic (Euclidean).

	2	3	4	5	6	7	m	∞
2	22	23	24	25	26	27	2m	2∞
3	32	α_3	β_3	Ico	36	37	3m	3∞
4	42	γ_3	δ_2	45	46	47	4m	4∞
5	52	Do	54	55	56	57	5m	5∞
6	62	63	64	65	66	67	6m	6∞
7	72	73	74	75	76	77	7m	7∞
m	m2	m3	m4	m5	m6	m7	mm	$m\infty$
∞	$\infty 2$	$\infty 3$	$\infty 4$	$\infty 5$	$\infty 6$	$\infty 7$	∞m	$\infty\infty$

All above tilings embed, since it holds:

- Hyperbolic tiling pq (i.e. $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$) embeds (for $q \leq \infty$) into $\frac{1}{2}Z^{\infty}$ if p is odd and into Z^{∞} if p is even or ∞.
- Euclidean (parabolic, i.e. $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$) 2∞ and $\infty 2$ embed into H_1 and Z^1, resp. Spheric (elliptic, i.e. $\frac{1}{p}+\frac{1}{q}>\frac{1}{2}$) $2m$ embeds into H_1 for any m, spheric $m2$ embeds into $H_{\frac{m}{2}}$ and $\frac{1}{2}H_m$ for m even and odd, respectively.
- $\delta_2=Z^2$, $\gamma_3=H_3$, $\beta_3=J(4,2)$, $\alpha_2=J(4,1)$;
  Icosahedron 35 and Dodecahedron 53 embed into $\frac{1}{2}H_6$, $\frac{1}{2}H_{10}$, respectively.
  63 and 36 embed into Z^3 and $\frac{1}{2}Z^3$, respectively.

3-dim. regular tilings and honeycombs

	α_3	γ_3	β_3	Do	Ico	δ_2	63	36
α_3	α_4*		β_4*		600-			336
β_3		24-				344		
γ_3	γ_4*		δ_3*		435*			436*
Ico				353				
Do	120-		534		535			536
δ_2		443*				444*		
36							363	
63	633*		634*		635*			636*

All emb. ones with $d \geq 3$ are, besides α_{d+1} and β_{d+1}: all bipartite ones (i.e. with cell γ_d, δ_{d-1} or 63): γ_{d+1}, δ_d and $8, 2, 1$ hyperbolic tilings with $d = 4, 5, 6$. Last 11 embed into Z^{∞}.

4-dim. regular tilings and honeycombs

	α_4	γ_4	β_4	24-	120-	600-	δ_3
α_4	α_5*		β_5*			3335	
β_4				$De(D_4)$			
γ_4	γ_5*		δ_4*			$4335*$	
24-		$Vo(D_4)$					3434
600-							
120-	5333		5334			5335	
δ_3				$4343*$			

Tilings 4335 and (non-compact) 4343 of hyperbolic 5-space embed into Z^{∞}.

5-dim. regular tilings and honeycombs

	α_5	γ_5	β_5	$Vo(D_4)$	$De(D_4)$	δ_4
α_5	α_6*		β_6*			
β_5					33343	
γ_5	γ_6*		δ_{5*}			
$De(D_4)$				33433		
$Vo(D_4)$		34333				34334
δ_4					43343*	

Four infinite series δ_d, γ_d, α_d and β_d embed into Z^d, H_d, $\frac{1}{2}H_{d+1}$ and (with scale $2t$ for $t = \lceil \frac{d}{4} \rceil$) H_{4t}, respectively.
Existence of Hadamard matrices and finite projective planes have equivalents in terms of variety of embeddings of β_d and α_d.

Archimedean polytopes

- An Archimedean d-polytope is a d-polytope, whose symmetry group acts transitively on its set of vertices and whose facets are Archimedean $(d-1)$-polytopes.
- They are classified in dimension 3 (Kepler: 5 (regular)+ 13 + $Prisms$ + $AntiPrisms$) and 4 (Conway and Guy).
- $\mathcal{K}(S)$ is an Archimedean polytope if $\mathcal{K}$ is a regular one.
- Since $\mathcal{K}(S) = \mathcal{K}^*(\{d-1-s : s \in S\})$, it suffices consider, for any non-empty subset S of $\{0, \dots, d-1\}$, only $\alpha_d(S)$, $\beta_d(S)$ and $Ico(S)$, 24-cell(S), 600-cell(S).
- A complex X embeds into H_m or $\frac{1}{2}H_m$ if its skeleton embeds into hypercube H_m with scale 1 or 2.
- We also will consider Wythoffians $\mathcal{K}(\mathcal{S})$, where $\mathcal{K}$ is an infinite regular polytope, i.e., regular tilings of $\mathbb{R}^m$.

Embeddable Arch. Wythoffians for $d = 3$

Embeddable Wythoffian	n	embedding
Tetrahedron$= \alpha_3(\{0\}) = \alpha_3(\{2\})$	4	$= J(4,1); = \frac{1}{2}H_3$
Octahedron$= \beta_3(\{0\}) = \alpha_3(\{1\})$	6	$= J(4,2)$
Cube$= \beta_3(\{2\}) = \beta_3(\{0\})^*$	8	$= H_3$
Icosahedron$= Ico(\{0\})$	12	$\frac{1}{2}H_6$
Dodecahedron$= Ico(\{2\}) = Ico(\{0\})^*$	20	$\frac{1}{2}H_{10}$
tr Icosidodecahedron$= Ico(\{0,1,2\})$	120	H_{15}
Rhombicosidodecahedron$= Ico(\{0,2\})$	60	$\frac{1}{2}H_{16}$
tr Cuboctahedron$= \beta_3(\{0,1,2\})$	48	H_9
Rhombicuboctahedron$= \beta_3(\{0,2\})$	24	$J(10,5)$
(tr Tetrahedron)$^* = \alpha_3(\{0,1\})^* = \alpha_3(\{1,2\})^*$	8	$\frac{1}{2}H_7$

Embeddable Wythoffian	n	embedding
(tr Icosahedron)$^* = Ico(\{0,1\})^*$	32	$\frac{1}{2}H_{10}$
(tr Dodecahedron)$^* = Ico(\{1,2\})^*$	32	$\frac{1}{2}H_{26}$
(Icosidodecahedron)$^* = Ico(\{1\})^*$	32	H_6
tr Octahedron$= \beta_3(\{0,1\}) = \alpha_3(\{0,1,2\})$	24	H_6
(tr Cube)$^* = \beta_3(\{1,2\})^*$	14	$J(12,6)$
(Cuboctahedron)$^* = \beta_3(\{1\})^* = \alpha_3(\{0,2\})^*$	14	H_4

Remaining semi-regular polyhedra: snub Cube, snub Dodecahedron, m-prisms and m-antiprisms for any $m \geq 3$. They embed into $\frac{1}{2}H_m$ for $m = 9, 15, m+2, m+1$, resp. Moreover, for even $m \geq 4$, m-prism embeds into $H_{\frac{m+2}{2}}$ and (m-1)-antiprism embeds into $J(m, \frac{m}{2})$.

Embeddable Arch. Wythoffians for $d = 4$

Embeddable Wythoffian	n	embedding
$\alpha_4 = \alpha_4(\{0\}) = \alpha_4(\{3\})$	5	$= J(5,1)$
$\beta_4 = \beta_4(\{0\})$	8	$= \frac{1}{2}H_4$
$\gamma_4 = \beta_4(\{3\}) = \beta_4(\{0\})^*$	16	$= H_4$
$\alpha_4(\{1\}) = \alpha_4(\{2\})$=$1_{21}$	10	$= J(5,2)$
$\alpha_4(\{0,3\})^*$	30	H_5
$\beta_4(\{0,3\})$	64	$\frac{1}{2}H_{12}$
$\alpha_4(\{0,1,2,3\})$	120	H_{10}
$\beta_4(\{0,1,2\}) = 24-cell(\{0,1\}) = 24-cell(\{2,3\})$	192	H_{12}
$\beta_4(\{0,1,2,3\})$	384	H_{16}
$24-cell(\{0,1,2,3\})$	1152	H_{24}
$600-cell(\{0,1,2,3\})$	14400	H_{60}

First general results

We say that a complex X embeds into H_m (and denote it by $X \to H_m$) if its skeleton embeds into hypercube H_m.

1 Trivial: $\beta_d(\{d-1\}) = \beta_d(\{0\})^* = \gamma_d$ is the hypercube graph H_d.
  $\beta_d(\{0\}) = \beta_d$ embeds in H_{4t} with scale $2t$, $t = \lceil \frac{d}{4} \rceil$.

2 Easy: if $k \in \{0, \dots, d-1\}$, then $\alpha_d(\{k\})$ is $J(d+1, k+1)$.

3 Theorem: $\alpha_d(\{0, d-1\})^*$ is H_{d+1} with two antipodal vertices removed. It embeds into H_{d+1}.
  It is the zonotopal Voronoi polytope of the root lattice A_d. Moreover, the tiling $Vo(A_d)$ embeds into Z^{d+1}.

Embedding of Arch. order complexes

4 Theorem: $\alpha_d(\{0,\dots,d-1\})$ embeds into $H_{\binom{d+1}{2}}$.
It is the zonotopal Voronoi polytope (called permutahedron) of the dual root lattice A_d^*.
Moreover, $Vo(A_d^*)$ embeds into $Z^{\binom{d+1}{2}}$.

5 Theorem: $\beta_d(\{0,\dots,d-1\})$ embeds into H_{d^2}.
It is a zonotope, but not the Voronoi polytope of a lattice.

6 Computations: embeddings of the skeletons of $24-cell(\{0,1,2,3\})$ into H_{20} and of $600-cell(\{0,1,2,3\})$ into H_{60}, were found by computer.

So (since $Ico(\{0,1,2\})$ embeds into H_{15}), all Arch. order complexes embed into an H_m (moreover, are zonotopes).

All Arch. order complexes are zonotopes

$\mathcal{K}(\{0,\dots,d\text{-}1\}) = \mathcal{K}^*(\{0,\dots,d\text{-}1\})$	G	n	embedding
$\alpha_d(\{0,\dots,d\text{-}1\}) = Vor(A_d^*)$	A_d	$(d+1)!$	$H_{\binom{d+1}{2}}$
$\beta_d(\{0,\dots,d\text{-}1\})$ (not Voronoi)	B_d	$2^d d!$	H_{d^2}
$D_d(\{0,1,\dots,d\text{-}1\})$	D_d	$2^{d-1}d!$	$H_{d(d-1)}$
$I_2(p)(\{0,1\})$	$I_2(p)$	2p	H_p
$Ico(\{0,1,2\})$ = tr Icosidodecahedron	H_3	120	H_{15}
24-cell$(\{0,1,2,3\})$	F_4	1152	H_{24}
600-cell$(\{0,1,2,3\})$	H_4	14400	H_{60}
$E_6(\{0,1,\dots,5\})$	E_6	51840	H_{36}
$E_7(\{0,1,\dots,6\})$	E_7	2903040	H_{63}
$E_8(\{0,1,\dots,7\})$	E_8	696729600	H_{120}

Other Wythoff Arch. embeddings

7 Theorem: $\beta_d(\{0,\ldots,d-2\})$ embeds into $H_{d(d-1)}$.
It is a zonotope, but for $d>3$ it is not a Voronoi polytope of a lattice.

8 Theorem: $\beta_d(\{0,d-1\})$ is an ℓ_1-graph for all d.
But for $d>4$, it does not embed into a $\frac{1}{2}H_m$, i.e. embeds into an H_m with some even scale ≥ 4.

Conjecture: If Γ is the skeleton of (non-regular) Wythoffian $P(S)$ or of its dual, where P is a regular polytope, and Γ embeds into a $\frac{1}{2}H_m$, then Γ belongs to either above Tables for dimension 3, 4, or to one of 6 above infinite series.

l_1-Wythoffians of regular d-polytopes

Conjecture: all such non-regular ones are 9 sporadic ones (600-cell$(\{0,1,2,3\})$, 24-cell$(\{0,1,2,3\})$, $Ico(\{0,1,2\})$; $Ico(\{0,2\})$, $Ico(\{1\})^*$, $Ico(\{0,1\})^*$, $Ico(\{1,2\})^*$, $\beta_3(\{1,2\}^*$, $\alpha_3(\{0,1\})^*$) and 6 following infinite series for $d \geq 2$.

1. $\alpha_d(\{k\}) = J(d+1, k+1)$ for $k = 1, \dots, d-2$.
2. $\alpha_d(\{0, d-1\})^* = Vor(A_d) \to H_{d+1}$ (all but 2 antipods).
3. $\alpha_d(\{0, \dots, d-1\}) = Vor(A_d^*) \to H_{\binom{d+1}{2}}$ (permutahedron).

   Moreover, $Vo(A_d) \to Z^{d+1}$ and $Vo(A_d^*) \to Z^{\binom{d+1}{2}}$.
4. $\beta_d(\{0, \dots, d-1\}) \to H_{d^2}$ (zonotope, not Voronoi).
5. $\beta_d(\{0, \dots, d-2\}) \to H_{d(d-1)}$ (idem, for $d \geq 4$).
6. $\beta_d(\{0, d-1\}) \to H_m$ with scale $2t \geq 2\lceil \frac{d}{4} \rceil$.

IV. Some extensions of Wythoff construction and embedding

Cayley graph construction

- If a group G is generated by g_1,. . . ,g_t, then its Cayley graph is the graph with vertex-set G and edge-set

$$(g, gg_i) \text{ for } g \in G \text{ and } 1 \le i \le t;$$

G is vertex-transitive; its path-distance is length of xy^{-1}.

- If P is a regular d-polytope, then its symmetry group is a Coxeter group with canonical generators g_0,. . . , g_{d-1} (all g_i^2=1=$(g_i g_j)^{m(i,j)}$ for $m(i,j) \ge 2$) and its order complex is

$$P(\{0, \dots, d-1\}) = Cayley(G, g_0, \dots, g_{d-1}).$$

- $Cayley(G, g_0, \dots, g_{n-1})$ embeds into an $H_{|T|}$ (moreover, a zonotope) for any finite Coxeter group G.
  (T the set of elements, which are conjugate to some g_i)

Embeddings for tilings

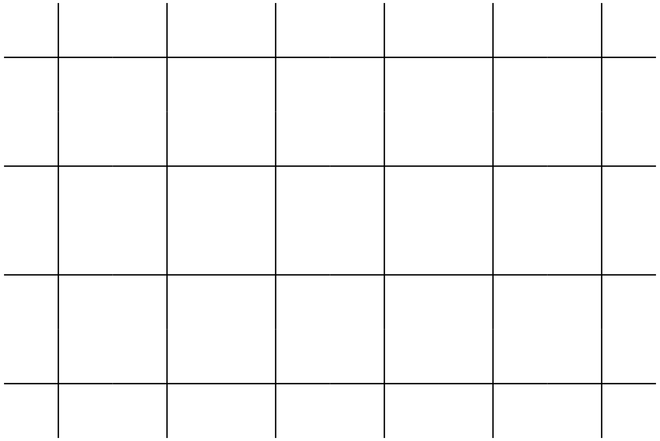
- Z has the natural l_1-metric $d(x, y) = |x - y|$.
- Z is embeddable into ∞-dimensional hypercube $H_{|Z|}$ by

$$x \mapsto (\dots, 0, 0, 1, \dots, 1, \dots).$$

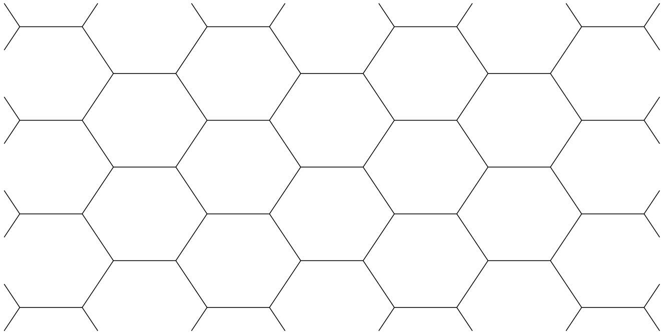
- Any graph (possibly, infinite), which embeds into Z^m, is embeddable into Z^∞.
- ➠ The hypermetric (including 5-gonal) inequality is again a necessary condition.
- For skeletons of infinite tilings, we consider (up to a scale) embedding into Z^m, $m \le \infty$.

There are 3 regular and 8 Archimedean (i.e. semi-regular) tilings of Euclidean plane.

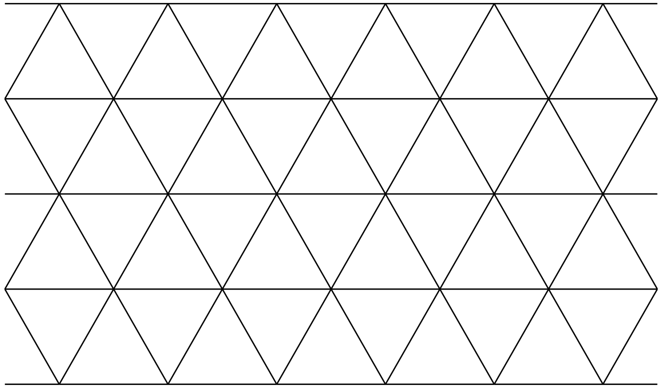
Three regular plane tilings



$44 = \delta_2 = De(Z^2) = Vo(Z^2)$

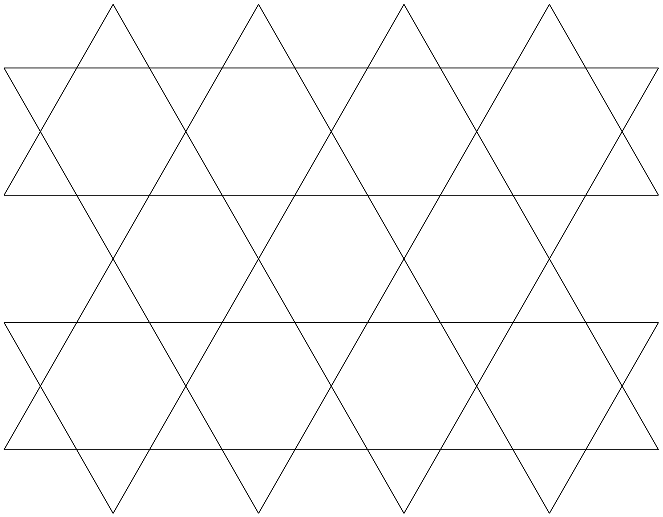


$63 = Vo(A_2) \longrightarrow Z^3$

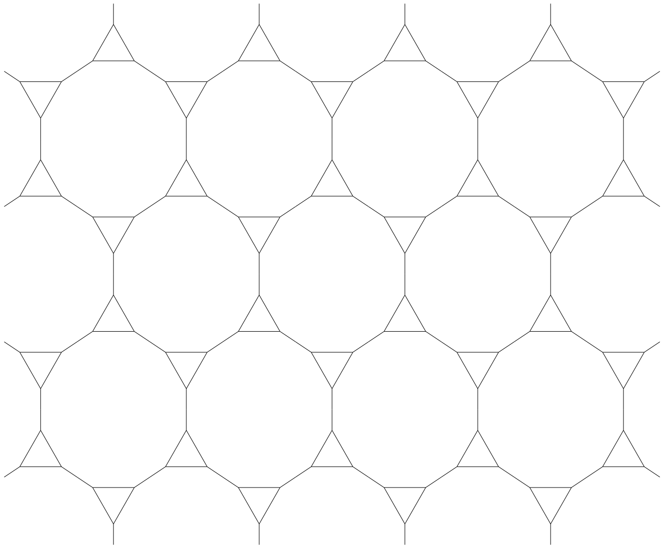


$36 = De(A_2) \longrightarrow \frac{1}{2}Z^3$

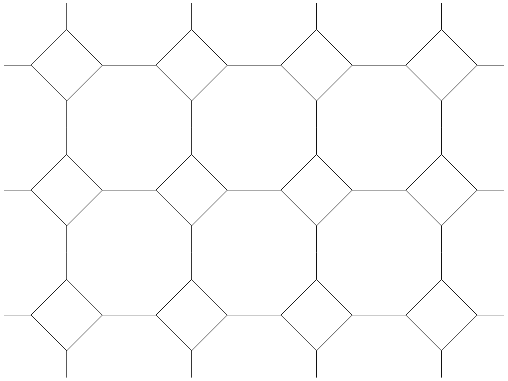
Eight Archimedean plane tilings



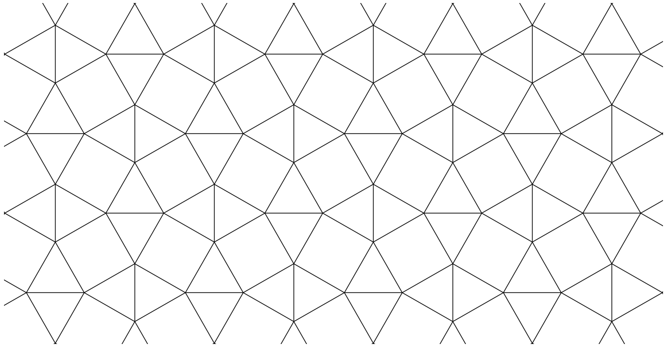
$(3.6.3.6)$=$36(\{1\})$;
dual $\rightarrow Z^3$



(3.12^2)=$36(\{1,2\})$;
dual $\rightarrow \frac{1}{2}Z^\infty$

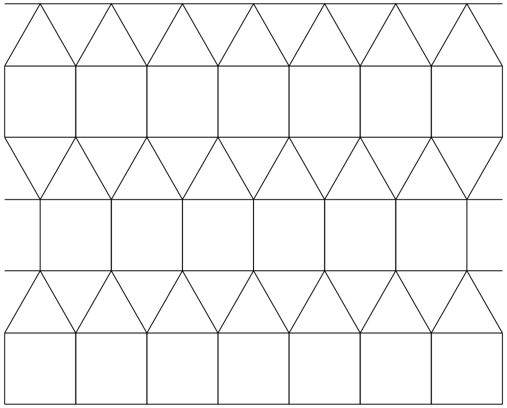


(4.8^2)=$44(\{0,1,2\}) \rightarrow Z^4$

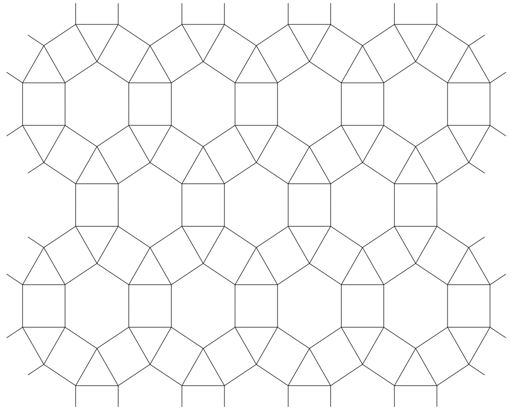


$(3^2.4.3.4) \rightarrow \frac{1}{2}Z^4$

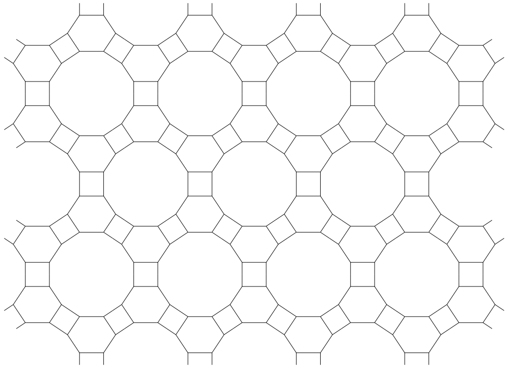
Eight Archimedean plane tilings



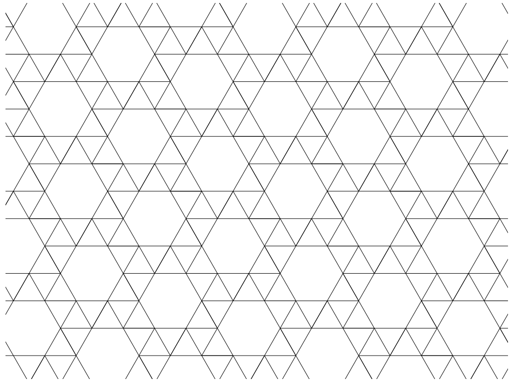
$(3^3.4^2) \to \frac{1}{2}Z^3$



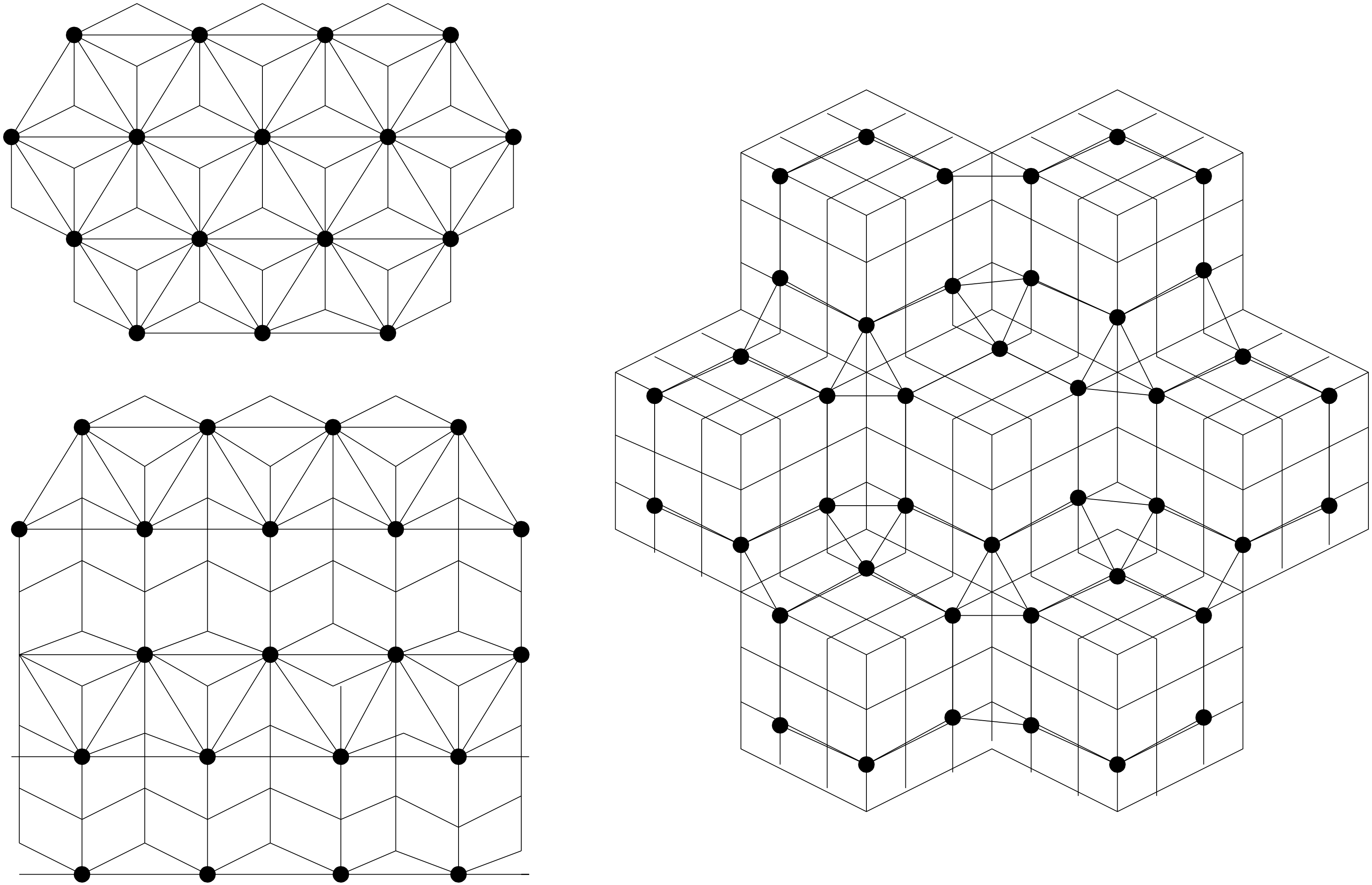
$(3.4.6.4) = 36(\{0,2\}) \to \frac{1}{2}Z^3$



$(4.6.12)$=$36(\{0,1,2\}) \to Z^6$



$(3^4.6) \to \frac{1}{2}Z^6$



Mosaics 36, $(3.4.6.4)$ and $(3^3.4^2)$ embed into $\frac{1}{2}Z^3$

Emb. Wythoffians of reg. plane tilings

Wythoffian	embedding
$\delta_2 = \delta_2(\{0\}) = \delta_2(\{1\}) = \delta_2(\{2\}) = \delta_2(\{0,2\})$	Z^2
$36 = 36(\{0\})$	$\frac{1}{2}Z^3$
$63 = 36(\{2\}) = 36(\{0,1\})$	Z^3
$(4.8^2) = \delta_2(\{0,1\}) = \delta_2(\{1,2\} = \delta_2(\{0,1,2\}$	Z^4
$(4.6.12) = 36(\{0,1,2\})$	Z^6
$(3.4.6.4) = 36(\{0,2\})$	$\frac{1}{2}Z^3$
$(3.6.3.6)^* = (36(\{1\}))^*$	Z^3
$(3.12^2)^* = (36(\{1,2\}))^*$	$\frac{1}{2}Z^\infty$

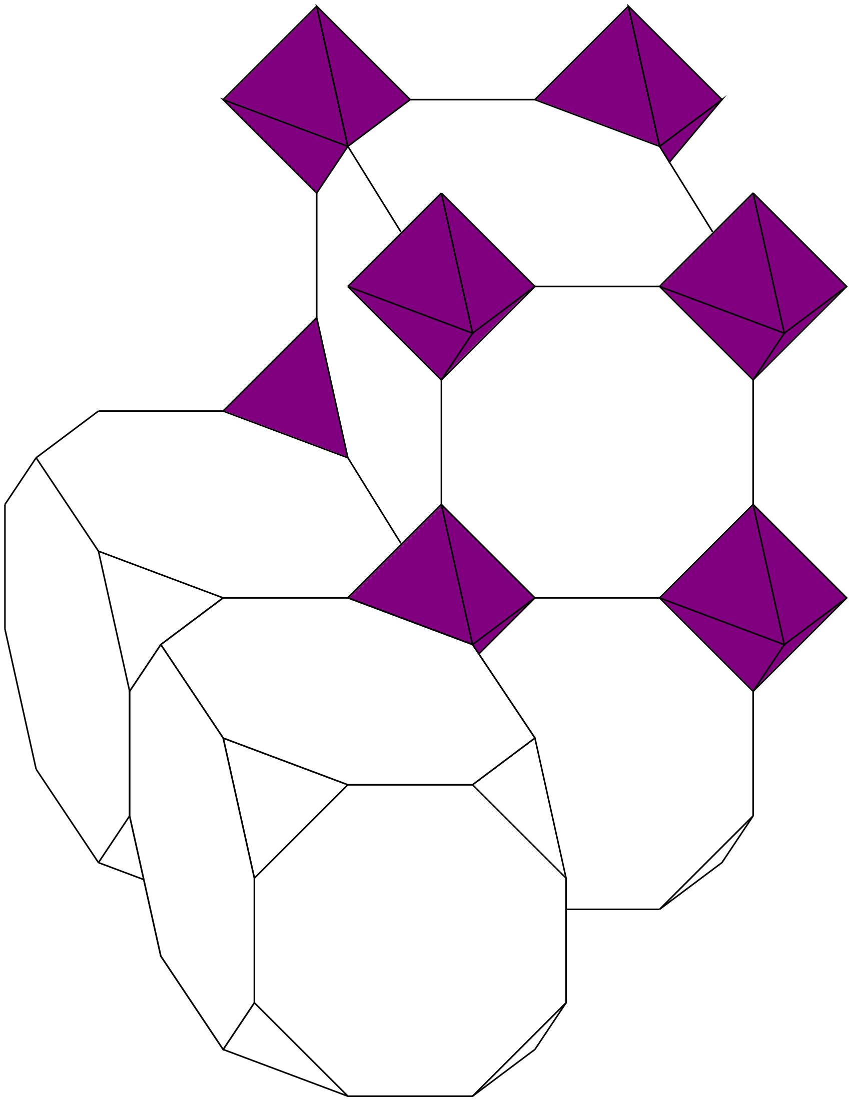
Other semi-regular plane tilings: $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$; see scale 2 embedding of 36, $(3.4.6.4)$ and $(3^3.4^2)$ into Z^3.

Wythoffians of reg. 3-space tilings

Wythoffian	Nr.	embbedding?
$\delta_3 = \delta_3(\{0\}) = \delta_3(\{3\}) = \delta_3(\{0,3\})$	1	Z^3
$\delta_3(\{1,2\}) = Vo(A_3^*)$	2	Z^6
$\delta_3(\{0,1,2\}) = \delta_3(\{1,2,3\})$=zeolit Linde	16	Z^9
$\delta_3(\{0,1,2,3\})$=zeolit ρ	9	Z^9
$\delta_3(\{1\}) = \delta_3(\{2\}) = De(J - complex)$	8	non 5-gonal
$\delta_3(\{0,1\}) = \delta_3(\{2,3\})$=boride CaB_6	7	non 5-gonal
$\delta_3(\{0,2\}) = \delta_3(\{1,3\})$	18	non 5-gonal
$\delta_3(\{0,1,3\}) = \delta_3(\{0,2,3\})$	23	non 5-gonal

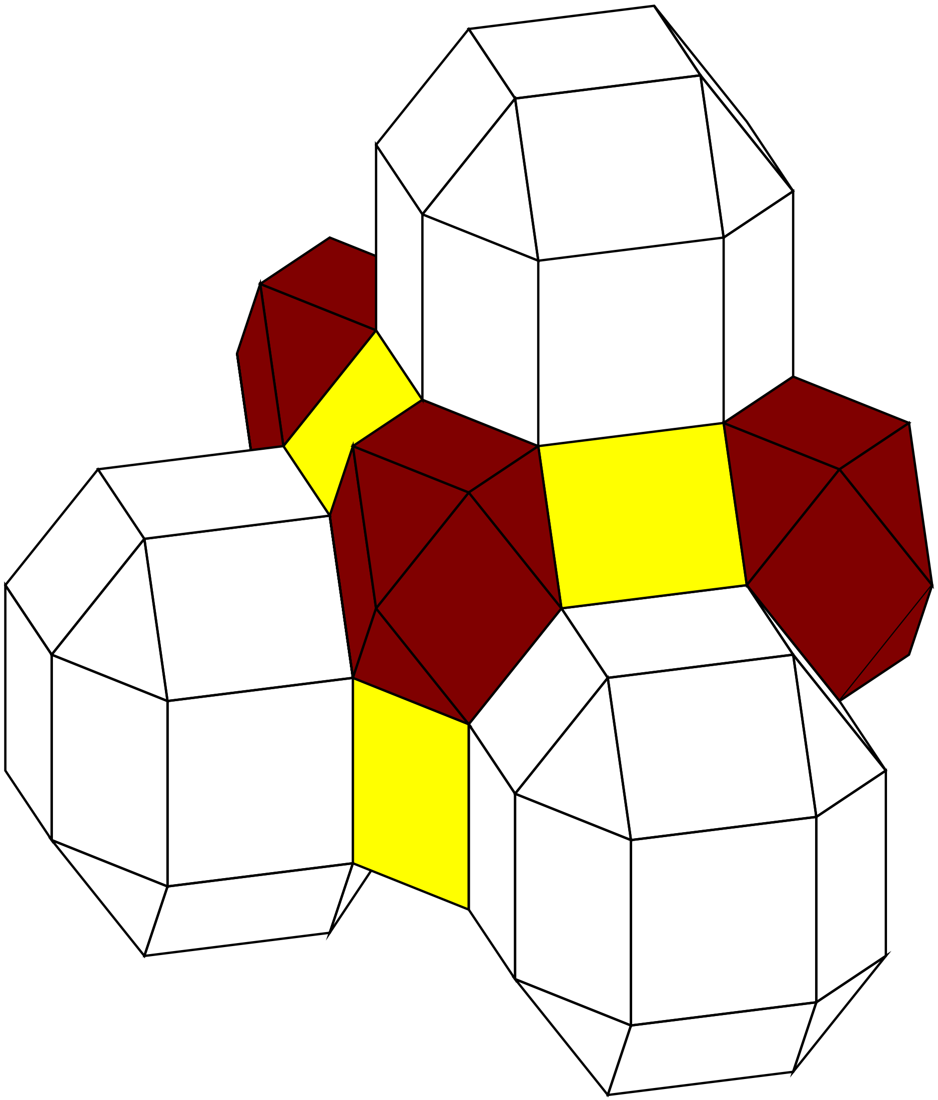
There are 28 vertex-transitive tilings of 3-space by regular and semi-regular polyhedra (Andreini, Johnson, Grunbaum, Deza–Shtogrin).

Exp.: not 5-gonal $\delta_3(\{0,1\}) = \delta_3(\{2,3\})$



Nr. 7 (of 28), tiled 1:4 by β_3 and tr. γ_3; boride CaB_6

Exp.: not 5-gonal $\delta_3(\{0,2\}) = \delta(\{1,3\})$



Nr. 18 (of 28), tiled 2:1:2 by γ_3, Cbt and $Rcbt$

Some Wyth. of reg. d-space tilings, $d \geq 4$

Wythoffian	tiles	embbedding?
$\delta_d = \delta_d(\{0\}) = \delta_d(\{d\}) = \delta_d(\{0, d\})$	γ_d	Z^d
$\delta_d(\{0, 1\})$=tr δ_d	β_d, tr γ_d	non 5-gonal
$Vo(D_4) = Vo(D_4)(\{0\})$	$24 - cell$	non 5-gonal
$Vo(D_4)^* = Vo(D_4)(\{4\})$	β_4	non 5-gonal
$Vo(D_4)(\{1\}) = Med(Vo(D_4))$	γ_4, $Med(24 - cell)$	non 5-gonal
$Vo(D_4)(\{0, 1\})$=tr $Vo(D_4)$	γ_4, tr $24 - cell$	Z^{12}

Conjecture (holds for $d \leq 3$):

$\delta_d(\{0, \ldots, d\})$ and $\delta_d(\{0, \ldots, d-1\})$ embed into Z^{d^2}.
Remind that $\beta_d(\{0, \ldots, d-1\})$ embeds into H_{d^2}.